Homework Set 5

Please, try to do all of the following problems. Solutions to three of them are due on Monday February 20.

Problem 1. Let X be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Show that if $\mathcal{G}$ is a quasicoherent (or coherent) sheaf on X, then so is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$.

Problem 2. Let $(X,\mathcal{O}_X)$ be a ringed space and $\mathcal{E}$ a locally free sheaf on X of finite rank. The *dual* of $\mathcal{E}$ is defined by $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X)$.

(i) Show that there is a canonical isomorphism $(\mathcal{E}^\vee)^\vee \simeq \mathcal{E}$.
(ii) For every $\mathcal{O}_X$-module $\mathcal{F}$ there is a canonical isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \simeq \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Problem 3. Recall that the *support* of a sheaf $\mathcal{F}$ on a topological space X is the set

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Show that if M is a finitely generated module over a ring R, then the support of $\widetilde{M}$ is $V(\mathrm{Ann}_R(M))$. Deduce that the support of a coherent sheaf on a Noetherian scheme X is closed.

Problem 4. Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X.

(i) Show that if for some $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank r, then the same property holds for all points in a neighborhood of x.
(ii) Show that $\mathcal{F}$ is locally free of rank r if and only if for every $x \in X$, the stalk $\mathcal{F}_x$ is a locally free module of rank r over $\mathcal{O}_{X,x}$.
(iii) Show that $\mathcal{F}$ is locally free of rank one on X if and only if there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \mathcal{O}_X$ (this is why rank one locally free sheaves are called *invertible*).

Problem 5. Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X. Define the function $\phi\colon X \to \mathbb{N}$ by $\phi(x) = \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)$, where $k(x)$ is the residue field of X at x. Use Nakayama's Lemma to prove the following:

(i) For every m, the set $\{x \in X \mid \phi(x) \geq m\}$ is closed in X.
(ii) If $\mathcal{F}$ is locally free and X is connected, then ϕ is constant on X.
(iii) Conversely, show that if X is reduced and ϕ is constant, then $\mathcal{F}$ is locally free.