Homework Set 3

Please, try to do all of the following problems. Solutions to three of them are due on Monday February 6.

Problem 1. Let $f\colon X \to Y$ be a morphism of schemes, and $Z \hookrightarrow Y$ a closed subscheme. Show that there is a unique closed subscheme $f^{-1}(Z) \hookrightarrow X$ (called the *inverse image of* Z) that satisfies the following property: for every morphism of schemes $g\colon W \to X$, $f \circ g$ factors through Z (i.e. $f \circ g$ factors as $W \xrightarrow{h} Z \hookrightarrow Y$ for some h) if and only if g factors through $f^{-1}(Z)$. Show that the support of $f^{-1}(Z)$ is the inverse image of the support of Z.

Problem 2. Show that if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

is a fiber product diagram and f is separated, then so is f'.

Problem 3. Let $\mathcal{P}$ be a property of morphisms of schemes such that

(a) A closed immersion has $\mathcal{P}$.
(b) $\mathcal{P}$ is stable under compositions.
(c) $\mathcal{P}$ is stable under base extensions.

Show that the following hold:

(i) If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms such that $g \circ f$ has $\mathcal{P}$ and g is separated, then f has $\mathcal{P}$.
(ii) If $f\colon X \to Y$ has $\mathcal{P}$, then $f_{\mathrm{red}}\colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has $\mathcal{P}$.
This applies if $\mathcal{P}$ is one of the following properties: *closed immersion*, *finite*, *affine*, *finite type*, *separated*, *proper*.

Problem 4. Let $f\colon X \to Y$ be a quasicompact morphism of schemes. Show that if $f(X)$ denotes the closed subscheme of Y that is the scheme-theoretic image of f, then for every affine open subset V of Y, the ideal corresponding to the closed subscheme $V \cap f(X)$ of V is $\ker(\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)))$.

Problem 5. A morphism $f\colon X \to Y$ is a *locally closed immersion* if it factors as $X \xrightarrow{g} U \xrightarrow{h} Y$, where g is a closed immersion and h is an open immersion.

(i) Show that a composition $X \xrightarrow{i} W \xrightarrow{j} Y$, with i an open immersion and j a closed immersion is a locally closed immersion.
(ii) Deduce that the class of locally closed immersions is closed under compositions.