Homework Set 1

Please, try to do all of the following problems. Solutions to three of them are due on Monday January 30.

Problem 1. Show that if $f\colon X \to Y$ is a morphism of finite type, and $V \subseteq Y$ and $U \subseteq f^{-1}(V)$ are affine open subsets, then $\mathcal{O}_X(U)$ is an algebra of finite type over $\mathcal{O}_Y(V)$.

Problem 2. Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms that are locally of finite type, then $g \circ f$ is locally of finite type. If

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

is a fiber product and f is locally of finite type, then f' is also locally of finite type.

Problem 3. Let X be an integral scheme. Show that the local ring $\mathcal{O}_{X,\eta}$ at the generic point η of X is a field (it is called the *function field* of X and it is denoted by $K(X)$). Moreover, if $U \subseteq X$ is an open affine subset, then $K(X)$ is the quotient field of $\mathcal{O}_X(U)$.

Problem 4. Show that a finite morphism is *closed*, i.e. the image of a closed subset is closed.

Problem 5. Let $f\colon X \to Y$ be a morphism of schemes. Show that there is a unique closed subscheme Z of X such that f factors through Z and which is the smallest with this property: if Z' is a closed subscheme of Y such that f factors through Z', then Z is a closed subscheme of Z'. Z is called the *scheme-theoretic image* of f. Show that if X is reduced, then Z is given by the reduced structure on $\overline{f(X)}_{\rm red}$.