Homework Set 11

Please, try to do all of the following problems. Solutions to five of them are due Friday April 21.

Problem 1.

1) Let X be the union of two planes meeting at a point, each of which is mapped isomorphically to a plane Y:
$$f\colon X = \text{Spec } k[x,y,z,w]/(z,w)\cap(x+z,y+w) \to Y = \text{Spec } k[x,y].$$
Show that f is not flat
2) Let $f\colon X = \text{Spec } k[x,y,z,w]/(z^2, zw, w^2, xz-yw) \to Y = \text{Spec } k[x,y]$. Show that $X_{\text{red}} \simeq Y$, that every associated point of X is the generic point of an irreducible component (i.e. X has no embedded points), but that f is not flat.

Problem 2. Let A be a Noetherian ring and consider $\pi\colon X = \mathbb{P}^n_A \to Y = \text{Spec}(A)$. Show that there is an exact sequence of sheaves on X
$$0 \to \mathcal{O}_X \xrightarrow{i} \mathcal{O}_X(1)^{\oplus(n+1)} \xrightarrow{p} T_{X/Y} \to 0, \tag{1}$$
as follows:

1) Show that a global section of $T_{X/Y}(1)$ can be identified with a derivation $D\colon \mathcal{O}_X \to \mathcal{O}_X(1)$ over A. Deduce that for every i, the derivation ∂_{X_i} on the ring $A[X_0,\ldots,X_n]$ induces a global section of $T_{X/Y}(-1)$.
2) Define i to correspond to the global section $(X_0,\ldots,X_n)$ of $\mathcal{O}_X(1)^{\oplus(n+1)}$. Show that if p is obtained by tensoring with $\mathcal{O}_X(1)$ the morphism $\mathcal{O}_X^{\oplus(n+1)} \to T_{X/Y}(-1)$ defined by $(\partial_{X_0},\ldots,\partial_{X_n})$, then the sequence (1) is exact.

Problem 3. Show that if $f\colon X_i \to Y$ are morphisms of schemes for $i=1, 2$, and if $X = X_1 \times_Y X_2$ with projections $p_i\colon X \to X_i$, then we have
$$\Omega_{X/Y} \simeq (p_1)^*\Omega_{X_1/Y} \oplus (p_2)^*\Omega_{X_2/Y}.$$

Problem 4. Let Y be a nonsingular cubic curve in $\mathbb{P}^2_k$, where k is an algebraically closed field. Show that if $S = Y \times Y$, then $p_a(S) = -1$ but $p_g(S) = 1$. This gives an example of a nonsingular variety for which the arithmetic genus and the geometric genus are not equal.

Problem 5. Compute the geometric genus of a smooth hypersurface of degree d in $\mathbb{P}^n_k$, where k is an algebraically closed field.

Problem 6. We construct the parameter space for hypersurfaces of degree d in $\mathbb{P}^n_k$, where k is an algebraically closed field.

1) Let S_d be the vector space of homogeneous polynomials of degree d in $S = k[X_0, \ldots, X_n]$. Show that the set of hypersurfaces in $\mathbb{P}^n_k$ is in bijection with the closed points of $Z := \mathbb{P}(S_d^*) \simeq \mathbb{P}^N_k$, where $N = \binom{d+n}{n} - 1$.
2) Show that there is a Cartier divisor $\mathcal{Z}$ in $Z \times \mathbb{P}^n_k$ such that the fiber of $\mathcal{Z}$ over a point in Z corresponding to the hypersurface H is isomorphic to H.
3) Show that $\mathcal{Z}$ is flat over Z.

Problem 7. Let k be an algebraically closed field and $X = \mathbb{P}^n_k$.

1) Show that X has no nonzero global vector fields.
2) Show that if H is a smooth hypersurface in X of degree d, then we have an exact sequence
$$0 \to T_H \to T_X|_H \to \mathcal{O}_H(d) \to 0.$$
3) Compute $H^0(H, T_H)$ and $H^1(H, T_H)$.

Problem 8. Let $X \subset \mathbb{P}^n_k$ be a nonsingular variety of codimension r that is a complete intersection, i.e. there are hypersurfaces $H_1, \ldots, H_r$ such that $X = H_1 \cap \ldots \cap H_r$.

1) Show that for every point $p \in X$ and every i the intersection $H_1 \cap \ldots \cap H_i$ is nonsingular at p.
2) Show that if each H_i has degree d_i, then $\omega_X \simeq \mathcal{O}(d_1 + \ldots + d_r - n - 1)$.