Homework Set 10

Please, try to do all of the following problems. Solutions to three of them are due on Monday April 10.

Problem 1. Let X be a Noetherian scheme.

1) Show that if $\mathcal{E}$ and $\mathcal{F}$ are globally generated coherent sheaves on X, then $\mathcal{E} \otimes \mathcal{F}$ is globally generated.
2) Show that if L and M are invertible sheaves on X, with L ample and M globally generated, then $L \otimes M$ is ample.
3) Show that if L and M are ample invertible sheaves on X, then $L \otimes M$ is ample.

Problem 2. Let A be a Noetherian ring and X a proper scheme over $\mathrm{Spec}(A)$. Suppose that L is an invertible sheaf on X.

1) Show that L is ample if and only if its restriction $L|_{X_{\mathrm{red}}}$ to X_{red} is ample.
2) Show that if $X_1, \ldots, X_r$ are the irreducible components of X (with the reduced scheme structure), then L is ample if and only if $L|_{X_i}$ is ample for every i.

Problem 3. Let X be a scheme of finite type over an algebraically closed field k. Suppose that $f\colon X \to \mathbb{P}^n$ is defined by the invertible sheaf L on X and by the sections $s_0, \ldots, s_n \in \Gamma(X, L)$.

1) A closed subscheme V of $\mathbb{P}^n$ is called *nondegenerate* if there is no hyperplane H in $\mathbb{P}^n$ such that V is a subscheme of H. Show that the scheme-theoretic image of f is nondegenerate if and only if the sections $s_0, \ldots, s_n$ are linearly independent.
2) A closed subscheme V of $\mathbb{P}^n$ is called *linearly normal* if the canonical morphism
$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(V, \mathcal{O}(1)|_V)$$
is surjective. Assuming that f is a closed immersion, show that X is linearly normal in $\mathbb{P}^n$ if and only if $s_0, \ldots, s_n$ span $\Gamma(X, L)$.
3) Show that a nondegenerate closed subscheme V of $\mathbb{P}^n$ is not linearly normal if and only if there is a nondegenerate closed subscheme W of $\mathbb{P}^{n+1}$ and a point Q in $\mathbb{P}^{n+1} \smallsetminus W$ such that the projection from Q induces an isomorphism $W \simeq V$.

Problem 4. Let k be an algebraically closed field and $X \subset \mathbb{P}^n_k$ a closed subscheme defined by the ideal sheaf $\mathcal{I}_X$.

1) Show that X is nondegenerate if and only if $H^0(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$, and X is linearly normal if and only if $H^1(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$.
2) Show that if X is integral and nondegenerate and H is a hyperplane in $\mathbb{P}^n$, then $X \cap H$ is nondegenerate in $H \simeq \mathbb{P}^{n-1}$.

Problem 5. Let X be a nonsingular projective curve over an algebraically closed field k. Show that if L is an invertible sheaf on X, then L is very ample if and only if for every points P and Q on X (not necessarily distinct), we have

$$h^0(X, L(-P-Q)) = h^0(X, L) - 2.$$