Homework Set 1

Please, try to do all of the following problems. Solutions to two of them are due on Friday January 20.

Problem 1. The goal of this problem is to describe the *open subschemes* of a scheme.

i) Show that if $(X, \mathcal{O}_X)$ is a locally ringed space, then for every open subset $U \subseteq X$ we have a canonical locally ringed space $(U, \mathcal{O}_U)$ supported on U.
ii) Show that if $X = \operatorname{Spec} A$ and if
$$U = D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \not\in \mathfrak{p}\}$$
for some f in A, then U with the above ringed space structure is isomorphic to $\operatorname{Spec} A_f$.
iii) Deduce that if X is a scheme and $U \subseteq X$ is open, then $(U, \mathcal{O}_U)$ is a scheme such that the natural the set-theoretic inclusion $U \hookrightarrow X$ can be extended to a morphism of schemes i_U.
iv) Show that if U is as above and if $f \colon Y \to X$ is a morphism of schemes, then f factors as $i_U \circ g$ for some morphism of schemes $g \colon Y \to U$ if and only if set-theoretically we have $\operatorname{Im}(f) \subseteq U$ (in this case g is uniquely determined).

Remark. A morphism of schemes $f \colon Y \to X$ is an *open immersion* if it factors as $i_u \circ g$, for some U as above and an isomorphism of schemes $g \colon Y \to U$.

Problem 2. Let $f \colon Y \to X$ be a morphism of schemes. Show that if there is an open cover $X = \bigcup_i U_i$ such that the induced morphisms $f^{-1}(U_i) \to U_i$ are isomorphisms, then f is an isomorphism (this says that the notion of isomorphism is local on the base).

Problem 3. Recall that a topological space Y is *irreducible* if whenever $Y = A \cup B$, with A and B closed, we have $A = Y$ or $B = Y$.

i) Show that for every topological space Y, if y is a point in Y, then the set $\overline{\{y\}}$ is irreducible (with the induced topology).
ii) Show that if X is a scheme and $Z \subseteq X$ is an irreducible closed subset, then Z has a unique *generic point*, i.e. a point x such that $Z = \overline{\{x\}}$.
iii) Show that if $f \colon X \to Y$ is a morphism of schemes and if x is the generic point of the closed subset $Z \subseteq X$, then $f(x)$ is the generic point of $\overline{f(Z)}$.

Remark. Generic points provide a convenient tool. The general idea is that the behavior at the generic point of Z translates in the behavior on some open subset of Z. Existence of generic points is one of the advantages of working with schemes.

Problem 4. Describe the closed points of $X = \operatorname{Spec} \mathbb{R}[x_1, \ldots, x_n]$. What are the $\mathbb{C}$-valued and the $\mathbb{R}$-valued points of X ?

Problem 5. Let $f\colon X \to Y$ be a morphism of schemes, y a point in Y and $i\colon X_y \to X$ the fiber over y, i.e. we have a fiber product diagram

$$\begin{array}{ccc} X_y & \xrightarrow{j} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k(y) & \longrightarrow & Y \end{array}$$

Show that at the level of topological spaces, j gives a homeomorphism onto its image, the set $\{x \in X \mid f(x) = y\}$.