ERRATUM TO THE PAPER: ASYMPTOTIC INVARIANTS OF BASE LOCI

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Abstract. This note points out a gap in the proof of one of the technical results in the paper Asymptotic Invariants of Base Loci, that appeared in Ann. Inst. Fourier (Grenoble) 56 (2006), 1701–1734. We provide a correct proof of this result.

1. The setup

We work over an algebraically closed field \( k \) and let \( X \) be a variety over \( k \) (that is, a scheme of finite type over \( k \) that is irreducible and reduced). Let \( N \) be a finitely generated, free abelian group and \( S \subseteq N \) a finitely generated, saturated subsemigroup. We denote by \( C \) the cone generated by \( S \) in \( N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R} \), so \( C \) is a rational polyhedral convex cone and \( S = C \cap N \). For standard facts of convex geometry, we refer to [Ewa96] and [Zie95].

An \( S \)-graded system of ideals on \( X \) is a family \( a_\bullet = (a_m)_{m \in S} \) of coherent ideals \( a_m \subseteq \mathcal{O}_X \) for \( m \in S \) such that \( a_0 = \mathcal{O}_X \) and \( a_m \cdot a_{m'} \subseteq a_{m+m'} \) for every \( m, m' \in S \). The Rees algebra of \( a_\bullet \) is the quasi-coherent sheaf of \( S \)-graded \( \mathcal{O}_X \)-algebras

\[
R(a_\bullet) = \bigoplus_{m \in S} a_m.
\]

We say that \( a_\bullet \) is finitely generated if \( R(a_\bullet) \) is a finitely generated \( \mathcal{O}_X \)-algebra. For a coherent ideal \( b \) on \( X \), we denote by \( \overline{b} \) the integral closure of \( b \) (see [HS06] for the definition and basic properties of integral closure of ideals).

The following result is Proposition 4.7 in [ELM+06].

Proposition 1.1. If \( a_\bullet \) is a finitely generated \( S \)-graded system of ideals on the variety \( X \), then there is a smooth fan \( \Delta \) with support \( C \) such that for every smooth fan \( \Delta' \) refining \( \Delta \), there is a positive integer \( d \) with the following property: if \( \sigma \) is a cone in \( \Delta' \) and \( e_1, \ldots, e_s \) are the generators of \( S_\sigma = \sigma \cap N \), then

\[
\frac{a_d \sum_{i \mid p_i e_i}}{i} = \prod_{i} a_{p_i}^{e_i} \quad \text{for all} \quad p_1, \ldots, p_s \in \mathbb{Z}_{\geq 0}.
\]

The argument in [ELM+06] proceeds by induction on the dimension of \( C \). A key claim is that one can choose a smooth fan \( \Delta \) with support \( C \) such that the degrees corresponding to a finite system of generators of \( R(a_\bullet) \) lie on the rays of \( \Delta \) and such that the equality (1) holds on each cone of \( \Delta \) of dimension \( \dim(C) - 1 \). However, it is not clear that this can be achieved when \( \dim(C) \geq 3 \): given any fan \( \Delta \) with support \( C \), we can apply the inductive hypothesis to get suitable refinements for the cones in \( \Delta \) of dimension \( \dim(C) - 1 \), but we then need to further refine \( \Delta \), leading to new cones of dimension \( \dim(C) - 1 \). It is not clear that this process terminates.
2. THE CORRECTED PROOF

In what follows we provide a different proof of Proposition 1.1. The key ingredient is the following general lemma. While the statement is familiar to the experts in convex geometry, we provide a proof since we could not find a reference in the literature.

**Lemma 2.1.** Let \( N \) be a finitely generated, free abelian group and \( C \) the convex cone in \( N_{\mathbb{R}} \) generated by \( v_1, \ldots, v_r \in N_Q = N \otimes_{\mathbb{Z}} \mathbb{Q} \). Given \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r_{\geq 0} \), we consider the function \( \varphi_\alpha: C \cap N_Q \to \mathbb{R}_{\geq 0} \) given by

\[
\varphi_\alpha(v) = \inf \{ \lambda_1 \alpha_1 + \cdots + \lambda_r \alpha_r \mid \lambda_1, \ldots, \lambda_r \in \mathbb{Q}_{\geq 0}, \lambda_1 v_1 + \cdots + \lambda_r v_r = v \}.
\]

For every \( \alpha \), the infimum in (2) is a minimum and \( \varphi_\alpha \) is a convex, piecewise linear function. Moreover, there is a fan \( \Delta \), with support \( C \), such that each \( \varphi_\alpha \) is linear on every cone of \( \Delta \).

Before giving the proof of the lemma, we recall one well-known fact. For \( u = (u_1, \ldots, u_n) \), \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \), we put \( \langle u, v \rangle = \sum_{i=1}^n u_i v_i \). We use the same notation for the corresponding pairing of vectors in \( \mathbb{R}^r \).

**Remark 2.2.** Recall that a (rational) polyhedron in \( \mathbb{R}^n \) is a subset defined by finitely many affine linear inequalities (defined over \( \mathbb{Q} \)). A (rational) polytope in \( \mathbb{R}^n \) is a bounded (rational) polyhedron, or equivalently, the convex hull of finitely many points (in \( \mathbb{Q}^n \)); see [Zie95, Theorem 1.1]. Any polyhedron \( P \) in \( \mathbb{R}^n \) can be written as \( P_0 + C \), where \( P_0 \) is a polytope and \( C \) is a polyhedral convex cone (see [Zie95, Theorem 1.2]); moreover, if \( P \) is rational, then \( P_0 \) and \( C \) can be taken rational as well. Suppose now that \( \ell \) is a linear function on \( \mathbb{R}^n \) given by \( \ell(v) = \langle u, v \rangle \) for some \( u \in \mathbb{R}^n \). It is clear that \( \ell \) is bounded below on \( P \) if and only if \( \ell \geq 0 \) on \( C \), in which case we have

\[
\inf_{v \in P} \ell(v) = \min_{v \in P_0} \ell(v) = \min_{v \in P_0} \{ \ell(w_1), \ldots, \ell(w_s) \},
\]

where \( w_1, \ldots, w_s \) are the vertices (that is, the 0-dimensional faces) of \( P_0 \). Note that if \( P_0 \) is a rational polytope, then \( w_i \in \mathbb{Q}^n \) for all \( i \), hence if \( P \) is a rational polyhedron, we have

\[
\min_{v \in P} \ell(v) = \min_{v \in P_0 \cap \mathbb{Q}^n} \ell(v).
\]

**Proof of Lemma 2.1.** Let us choose an isomorphism \( N \cong \mathbb{Z}^n \) that allows us to identify \( N_Q \) and \( N_{\mathbb{R}} \) with \( \mathbb{Q}^n \) and \( \mathbb{R}^n \), respectively. We can thus write \( v_i = (v_{i1}, \ldots, v_{in}) \) for \( 1 \leq i \leq r \), with \( v_{ij} \in \mathbb{Q} \) for all \( i \) and \( j \). For \( \alpha \in \mathbb{R}^r_{\geq 0} \), let us denote by \( \tilde{\varphi}_\alpha \) the map \( C \to \mathbb{R} \) given by

\[
\tilde{\varphi}_\alpha(v) = \inf \{ \langle \alpha, \lambda \rangle \mid \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geq 0}, \lambda_1 v_1 + \cdots + \lambda_r v_r = v \}.
\]

If \( v = (b_1, \ldots, b_n) \in \mathbb{R}^n \) and \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r \), the conditions \( \lambda_1, \ldots, \lambda_r \geq 0 \) and \( v = \sum_{i=1}^r \lambda_i v_i \) are equivalent to \( \lambda \in P(v) \), where \( P(v) \) is the polyhedron in \( \mathbb{R}^r \) given by

\[
\sum_{i=1}^r v_{ij} \lambda_i = b_j \text{ for } 1 \leq j \leq n \quad \text{and} \quad \lambda_i \geq 0 \text{ for } 1 \leq i \leq r.
\]

For every \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r_{\geq 0} \), we have \( \langle \alpha, \lambda \rangle \geq 0 \) for all \( \lambda \in P(v) \). We thus conclude using Remark 2.2 that \( \varphi_\alpha(v) = \tilde{\varphi}_\alpha(v) \) for every \( v \in C \cap \mathbb{Q}^n \) and the infimum in the definition of \( \varphi_\alpha(v) \) is a minimum.

The fact that each \( \tilde{\varphi}_\alpha \) is a convex function follows easily from the definition. Indeed, since we clearly have \( \tilde{\varphi}_\alpha(tv) = t \cdot \tilde{\varphi}_\alpha(v) \) for all \( v \in C \) and \( t \geq 0 \), convexity is equivalent to the fact
that

\[ \tilde{\varphi}_\alpha(v + v') \leq \tilde{\varphi}_\alpha(v) + \tilde{\varphi}_\alpha(v') \quad \text{for all } v, v' \in C. \]

This follows from the fact that if \( v = \sum_{i=1}^r \lambda_i v_i \) and \( v' = \sum_{i=1}^r \lambda'_i v_i \), with \( \lambda_i, \lambda'_i \in \mathbb{R}_{\geq 0} \) for all \( i \), are such that \( \sum_{i=1}^r \lambda_i \alpha_i = \tilde{\varphi}_\alpha(v) \) and \( \sum_{i=1}^r \lambda'_i \alpha_i = \tilde{\varphi}_\alpha(v') \), then \( v + v' = \sum_{i=1}^r (\lambda_i + \lambda'_i) v_i \), hence

\[ \tilde{\varphi}_\alpha(v + v') \leq \sum_{i=1}^r \lambda_i \alpha_i + \sum_{i=1}^r \lambda'_i \alpha_i = \tilde{\varphi}_\alpha(v) + \tilde{\varphi}_\alpha(v'). \]

It is a consequence of the Duality Theorem in Linear Programming (see [Bar02, Theorem IV.8.2]) that for every \( v = (b_1, \ldots, b_n) \in C \), we have

\[ \tilde{\varphi}_\alpha(v) = \max_{\gamma \in Q(\alpha)} \langle v, \gamma \rangle, \]

where \( Q(\alpha) \) is the polyhedron in \( \mathbb{R}^n \) consisting of those \( \gamma = (\gamma_1, \ldots, \gamma_n) \) with \( \sum_{j=1}^n v_i \gamma_j \leq \alpha_i \) for \( 1 \leq i \leq r \).

In order to complete the proof of the lemma, it is enough to show that there is a fan \( \Delta \) (consisting of strongly convex, rational polyhedral convex cones) with support \( C \), such that \( \tilde{\varphi}_\alpha \) is a linear function on every cone of \( \Delta \) for all \( \alpha \in \mathbb{R}^r_{\geq 0} \). Note that if \( v \in C \) is fixed and we write \( P(v) = P_0(v) + C_0(v) \), for a polytope \( P_0(v) \) and a polyhedral convex cone \( C_0(v) \), then it follows from Remark 2.2 that

\[ \tilde{\varphi}_\alpha(v) = \min \{ \langle \alpha, w_1 \rangle, \ldots, \langle \alpha, w_s \rangle \}, \]

where \( w_1, \ldots, w_s \) are the vertices of \( P_0(v) \). We thus conclude that the function \( \alpha \mapsto \tilde{\varphi}_\alpha(v) \) is continuous on \( \mathbb{R}^r_{\geq 0} \). In particular, it is enough to find a fan \( \Delta \) as above such that \( \tilde{\varphi}_\alpha \) is linear on the cones of \( \Delta \) for all \( \alpha \in \mathbb{R}^r_{\geq 0} \).

Note now that if \( \alpha \in \mathbb{R}^r_{\geq 0} \), then 0 lies in the interior of \( Q(\alpha) \). We consider the normal fan of \( Q(\alpha) \) (see [Zie95, Example 7.3]). Its cones are of the form

\[ \sigma_F = \{ w \in \mathbb{R}^n \mid \langle w, u' \rangle \geq \langle w, u \rangle \text{ for all } u \in Q(\alpha), u' \in F \}, \]

where \( F \) runs over the faces of \( Q(\alpha) \). It is clear that \( \tilde{\varphi}_\alpha \) is linear on each cone of \( \sigma_F \); on \( \sigma_F \) it is given by \( \langle \cdot, u' \rangle \) for every \( u' \in F \).

The support of \( \Delta(\alpha) \) consists precisely of those \( w \in \mathbb{R}^n \) such that the function \( \langle w, \cdot \rangle \) is bounded above on \( Q(\alpha) \); equivalently, if we write \( Q(\alpha) = Q_0(\alpha) + T(\alpha) \), where \( Q_0(\alpha) \) is a polytope and \( T(\alpha) \) is a polyhedral convex cone, then \( -w \) lies in the dual \( T(\alpha)^\vee \) of \( T(\alpha) \). Note that by definition of \( Q(\alpha) \), the cone \( T(\alpha) \) is defined by \( \sum_{j=1}^n v_{ij} \gamma_j \leq 0 \) for \( 1 \leq i \leq r \) (see [Zie95, Proposition 1.12]), hence it is the dual of \( -C \). We thus conclude that the support of \( \Delta(\alpha) \) is \( C \).

Note also that every facet of \( Q(\alpha) \) is of the form

\[ Q(\alpha) \cap \{ \gamma \mid \langle v_i, \gamma \rangle = \alpha_i \} \]

for some (nonzero) \( v_i \), hence the corresponding ray of \( \Delta(\alpha) \) is \( \mathbb{R}_{\geq 0} v_i \). We deduce that when we vary \( \alpha \), the rays of \( \Delta(\alpha) \) belong to a finite set, hence we have finitely many such fans. If we let \( \Delta \) be any common refinement of all such \( \Delta(\alpha) \), we conclude that the support of \( \Delta \) is \( C \) and \( \tilde{\varphi}_\alpha \) is linear on the cones of \( \Delta \) for all \( \alpha \in \mathbb{R}^r_{\geq 0} \). This completes the proof of the lemma. \( \square \)

\[^1\text{We consider the version of the normal fan whose rays are the outer normals to the facets of the polyhedron.} \]
Remark 2.3. We now explain a more elementary argument for the existence of the fan $\Delta$ in Lemma 2.1. This avoids the use of the Duality Theorem in Linear Programming and also makes the choice of fan $\Delta$ more explicit. First, arguing as in the proof of Carathéodory’s theorem (see [Zie95, Proposition 1.15]), we show the following

Claim: in the definition of $\tilde{\varphi}_\alpha(v)$ it is enough to only consider those $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geq 0}$ with the property that the $v_i$ with $i \in J(\lambda) := \{ i \mid \lambda_i \neq 0 \}$ are linearly independent.

In order to see this, it is enough to show that if the $v_i$ with $i \in J(\lambda)$ are linearly dependent, then we can find $\lambda' = (\lambda_1', \ldots, \lambda_r') \in \mathbb{R}^r_{\geq 0}$ such that $\sum_{i=1}^r \lambda_i' v_i = \sum_{i=1}^r \lambda_i v_i$ and we have $\sum_{i \in J(\lambda)} \lambda_i' \alpha_i \leq \sum_{i \in J(\lambda)} \lambda_i \alpha_i$ and $J(\lambda') \subseteq J(\lambda)$. Note that, by assumption, we have a relation $\sum_{i \in J(\lambda)} b_i \lambda_i = 0$ such that $J := \{ i \in J(\lambda) \mid b_i \neq 0 \}$ is nonempty. After possibly multiplying this relation with $-1$, we may and will assume that $\sum_{i \in J} b_i \alpha_i \geq 0$ and $b_i > 0$ for some $i \in J$ (we use here the fact that $\alpha_i \geq 0$ for all $i$). Let $j \in J$ be such that

$$\frac{\lambda_j}{b_j} = \min \left\{ \frac{\lambda_i}{b_i} \mid i \in J, b_i > 0 \right\}.$$ 

In this case, it is straightforward to see that if $\lambda_i' = \lambda_i - \frac{\lambda_j}{b_j} b_i$ for all $i \in J$ and $\lambda_i' = 0$ for $i \not\in J$, then $\lambda' \in \mathbb{R}^r_{\geq 0}$ and we have $J(\lambda') \subseteq J(\lambda) \setminus \{ j \}$ and

$$v = \sum_{i=1}^r \lambda_i' v_i \quad \text{and} \quad \sum_{i=1}^r \lambda_i' \alpha_i \leq \sum_{i=1}^r \lambda_i \alpha_i.$$

This proves the claim.

Let $\Lambda$ be the set of those $J \subseteq \{ 1, \ldots, r \}$ such that the $v_i$ with $i \in J$ are linearly independent. For every $J \in \Lambda$, let $\sigma_J$ be the convex cone in $N_\mathbb{R}$ generated by the $v_i$ with $i \in J$. It is a consequence of Carathéodory’s theorem that $C = \bigcup_{J \in \Lambda} \sigma_J$. Consider a fan $\Delta$ with support $C$ such that every cone $\sigma_J$, for $J \in \Lambda$, is a union of cones in $\Delta$. We now show that for every $\alpha \in \mathbb{R}^r_{\geq 0}$ and every $\tau \in \Delta$, the restriction $\tilde{\varphi}_\alpha|_\tau$ is a linear function. Note first that if $J \in \Lambda$ and for some $v \in \sigma_J$ we write $v = \sum_{i \in J} \lambda_i v_i$, then each $\lambda_i$ is given by a linear function of $v$; therefore $\sum_{i \in J} \lambda_i \alpha_i$ is given by a linear function $\ell_J$ of $v$. We next note that if $w$ lies in the relative interior $\text{Relint}(\tau)$ of $\tau$ and $J \in \Lambda$, then $w \in \sigma_J$ if and only if $\tau \subseteq \sigma_J$. Indeed, by construction of $\Delta$, we have $\sigma_1, \ldots, \sigma_d \in \Delta$ such that $\sigma_J = \bigcup_{j=1}^d \sigma_j$, hence $\sigma_J \cap \tau = \bigcup_{j=1}^d (\sigma_j \cap \tau)$. Since $\Delta$ is a fan, each $\sigma_j \cap \tau$ is a face of $\tau$, so the union contains a point in $\text{Relint}(\tau)$ if and only if $\tau \subseteq \sigma_j$ for some $j$, in which case $\tau \subseteq \sigma_J$. Our claim thus implies that

$$\tilde{\varphi}_\alpha(v) = \min \{ \ell_J(v) \mid \tau \subseteq \sigma_J \} \quad \text{for all} \quad v \in \text{Relint}(\tau).$$

It is well-known (and easy to see) that (4) implies that $-\tilde{\varphi}_\alpha$ is convex on $\text{Relint}(\tau)$. Since $\tilde{\varphi}_\alpha$ is a convex function on $C$ (the easy argument for this was given in the proof of Lemma 2.1), it follows that we have a linear function $\ell$ on $N_\mathbb{R}$ such that $\tilde{\varphi}_\alpha = \ell$ on $\text{Relint}(\tau)$. Given any $v \in \tau$, if $v' \in \text{Relint}(\tau)$, then $v + v' \in \text{Relint}(\tau)$, and the convexity of $\tilde{\varphi}_\alpha$ implies that

$$\ell(v + v') = \tilde{\varphi}_\alpha(v + v') \leq \tilde{\varphi}_\alpha(v) + \tilde{\varphi}_\alpha(v') = \tilde{\varphi}_\alpha(v) + \ell(v').$$

Therefore we have $\ell \leq \tilde{\varphi}_\alpha$ on $\tau$. On the other hand, it is an immediate consequence of the definition of $\tilde{\varphi}_\alpha$ that if $\{ w_m \}_{m \geq 1}$ is a sequence of vectors in $C$ with $\lim_{m \to \infty} w_m = v$, then $\tilde{\varphi}_\alpha(v) \leq \liminf_{m \to \infty} \tilde{\varphi}_\alpha(w_m)$. By taking $w_m \in \text{Relint}(\tau)$, we see that $\tilde{\varphi}_\alpha \leq \ell$ on $\tau$. We thus conclude that $\tilde{\varphi}_\alpha|_\tau$ is a linear function.

We can now give the proof of the result from [ELM+06].
Proof of Proposition 1.1. Recall first that every fan admits a smooth refinement (which has the same support), see [Ewa96, Theorem 8.5]. Furthermore, it is clear that if $\Delta$ is a smooth fan whose cones satisfy (1), then any smooth refinement of $\Delta$ satisfies the same property.

Let

$$T = \{ m \in S \mid a_m \neq 0 \} \quad \text{and} \quad S_+ = \{ m \in S \mid \ell m \in T \text{ for some } \ell \in \mathbb{Z}_{>0} \}.$$  

Since $R(a_\bullet)$ is finitely generated, we can choose $m_1, \ldots, m_r \in S$ such that over suitable subsets in a finite affine open cover of $X$, $R(a_\bullet)$ is generated over $\mathcal{O}_X$ by elements in degrees in $\{ m_1, \ldots, m_r \}$. We may and will assume that $m_i \in T$ for all $i$, so $T$ is generated by $m_1, \ldots, m_r$. Therefore the saturation $S_+$ of $T$ is finitely generated. Note that if $\Delta_0$ satisfies the condition in the proposition for $(a_m)_{m \in S_+}$, then we may take $\Delta$ to be any smooth fan with support $C$ with the property that every cone of $\Delta_0$ is a union of cones of $\Delta$. Indeed, the condition (1) holds trivially on the cones not contained in the support of $\Delta_0$. We thus may and will assume that $S = S_+$.

Since $R(a_\bullet)$ is finitely generated, for every $m \in S$, the $\mathcal{O}_X$-algebra $\bigoplus_{\ell \geq 0} a_{\ell m}$ is finitely generated (see [ELM+06, Lemma 4.8]). In this case it follows from [Bou61, Chap. III, Section 1, Proposition 3] that there is a positive integer $d$ such that $a_{d \ell m} = a_{d_m}^\ell$ for all $\ell \geq 1$. We denote the smallest such $d$ by $d_m$.

In what follows, it is convenient to use the formalism of asymptotic multiplicities, as in [ELM+06]. If $v$ is a discrete valuation of the function field of $X$, having center on $X$, and if $m \in S$, then we put

$$v^\bullet (m) := \inf_{\ell} \frac{v(a_{\ell m})}{\ell} = \lim_{\ell \to \infty} \frac{v(a_{\ell m})}{\ell},$$

where both the infimum and the limit are over those $\ell$ such that $a_{\ell m} \neq 0$. Note that by definition of $d_m$, we have $v^\bullet (m) = \frac{v(a_d \ell m)}{d m}$ for every $m \in S$ and every $\ell \in \mathbb{Z}_{>0}$.

Our choice of $m_1, \ldots, m_r$ implies that for every $m \in S$, we have

$$a_m = \sum_{\ell_1, \ldots, \ell_r} a_{m_1}^{\ell_1} \cdots a_{m_r}^{\ell_r},$$

where the sum is over all $\ell_1, \ldots, \ell_r \in \mathbb{Z}_{>0}$ with $m = \sum_{i=1}^r \ell_i m_i$. We now show that for every $m \in S$, we have

$$v^\bullet (m) = \inf \left\{ \sum_{i=1}^r \lambda_i \cdot v(a_{m_i}) \mid \lambda_1, \ldots, \lambda_r \in \mathbb{Q}_{\geq 0}, m = \sum_{i=1}^r \lambda_i m_i \right\}.$$  

In order to prove "$\leq$", note that given $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}_{\geq 0}$ with $m = \sum_{i=1}^r \lambda_i m_i$, we may choose $\ell \in \mathbb{Z}_{>0}$ such that $\ell \lambda_i \in \mathbb{Z}$ for all $i$. In this case the inclusion $\prod_i a_{m_i}^{\lambda_i} \subseteq a_{\ell m}$ implies

$$v(a_{\ell m}) \leq \sum_{i=1}^r \ell \lambda_i \cdot v(a_{m_i}),$$

and thus

$$v^\bullet (m) = \frac{v(a_{\ell m})}{\ell} \leq \sum_{i=1}^r \lambda_i \cdot v(a_{m_i}).$$

This gives the inequality "$\leq$" in (6). In order to prove the opposite inequality, note that if $\ell \in \mathbb{Z}_{>0}$ is such that $a_{\ell m} \neq 0$, then it follows from (5) that there are $\ell_1, \ldots, \ell_r \in \mathbb{Z}_{\geq 0}$ such
that \( \sum_i \ell_im_i = \ell m \) and
\[
v(a_{\ell m}) \geq \sum_{i=1}^r \ell_i \cdot v(a_{m_i}).
\]
Dividing by \( \ell \) and then letting \( \ell \) vary, we obtain the inequality “\( \geq \)” in (6).

It follows from (6) that we may apply Lemma 2.1 to obtain a fan \( \Delta \) (that we may assume to be smooth), with support \( C \), such that for every valuation \( v \) as above, we have \( v^{\bullet}(m+m') = v^{\bullet}(m) + v^{\bullet}(m') \) whenever \( m, m' \in S \) lie in the same cone of \( \Delta \). Let \( d \) be the least common multiple of the \( d_w \), when \( w \) runs over the primitive ray generators of \( \Delta \). In this case, if \( \sigma \) is a cone in \( \Delta \) with primitive ray generators \( e_1, \ldots, e_s \), then for every \( p_1, \ldots, p_s \in \mathbb{Z}_{\geq 0} \), if \( m = \sum_{i=1}^s p_i e_i \), then
\[
(7) \quad v(a_{dm}) \leq \sum_{i=1}^s p_i \cdot v(a_{de_i}) = \sum_{i=1}^s p_i \cdot v^{\bullet}(de_i) = v(v^{\bullet}(dm)) \leq v(a_{dm}),
\]
where the first inequality follows from the inclusion \( \prod_i a_{de_i}^{p_i} \subseteq a_m \). Therefore all inequalities in (7) are equalities. Since
\[
v(a_{dm}) = v(a_{de_1}^{p_1} \cdots a_{de_s}^{p_s})
\]
for every discrete valuation \( v \) of the function field of \( X \) that has center on \( X \), it follows from [HS06, Proposition 6.8.2] that
\[
a_{dm} = a_{de_1}^{p_1} \cdots a_{de_s}^{p_s}.
\]
This completes the proof. \( \square \)

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**References**


