Lecture 9: Planar graphs

October 6, 2020

Planar graphs: definition

Definitions. A curve in $\mathbf{R}^2$ is the image of a continuous map $\gamma\colon [a,b] \to \mathbf{R}$. The end points of the curve are $\gamma(a)$ and $\gamma(b)$. The curve is simple if it does not cross itself: more precisely, the map γ has the property that if $s < t$ in $[a,b]$ are such that $\gamma(s) = \gamma(t)$, then $s = a$ and $t = b$.
A curve can be closed (if the ends are the same point) or non-closed.

Definition. A planar drawing (or planar representation) of a finite graph G is given by

1) An injective map $f\colon V(G) \to \mathbf{R}^2$, and
2) For every edge $e \in E(G)$ with ends x and y, a simple curve with endpoints $f(x)$ and $f(y)$

such that the following condition holds:

$(\star)$ the curves corresponding to two distinct edges can only intersect at end points.

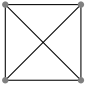
Planar graphs: definition, cont'd

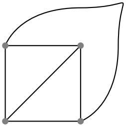
Given a planar drawing of G as before, we will still refer to the images in $\mathbf{R}^2$ of the elements of $V(G)$ as the "vertices of G", but will refer to the curves associated to the edges of G by the"drawn edges" of G.

Definition. A finite graph G is planar if it has some planar drawing.

Important note. The subtlety of this definition is that if we have a diagram of a graph in the plane with "intersecting drawn edges", it might still be possible that the graph has a planar drawing.

Easy example. The following plane diagrams both represent K_4, and the first one does not satisfy condition $(\star)$, while the second one does:





Upshot: proving that a graph is not planar is somewhat subtle.

Planar graphs: examples

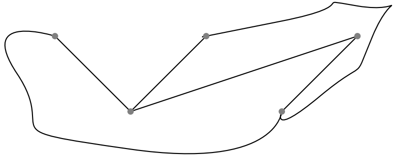
Here are a few examples:

1) Every polygon P_n is a planar graph (clear).

2) In particular, we see that K_3 is planar. We have already seen that K_4 is planar. This stops here: we will see later that K_5 is not planar.

3) Every tree is a planar graph. This follows easily by induction on the number of vertices (exercise).

4) The complete bipartite graph $K_{2,2}$ is isomorphic to P_4, hence it is planar. Also $K_{3,2}$ is planar, as shown by the following drawing:



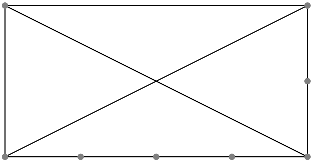
Again, this stops here: we will see later that $K_{3,3}$ is not planar.

Subdivisions and subgraphs

Let G be a finite graph. A graph H is a subdivision of G if it can be obtained from G by successively applying the operation that consists of deleting an edge (the *subdivided* edge) and replacing it with a length two path (whose central point was not in G).

You can think of the above operation as "dividing an edge in two edges".

Example. The graph in the picture below has been obtained by applying the subdivision procedure 4 times to the graph K_4:



Subdivisions and subgraphs, cont'd

Remark. If H is a subdivision of G, then G is planar if and only if H is planar. Of course, it is enough to only consider the case when H is obtained by subdividing one edge. The fact that a planar drawing of G induces a planar drawing of H is clear. The converse follows from the fact that the new vertex in H is only incident to the two new edges (the curve corresponding to the subdivided edge in G is obtained by concatenating the two curves corresponding to the new edges in H).

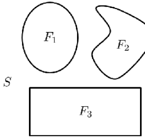
Another useful fact is the following:
Remark. If G is a planar graph, then every subgraph of G is planar.

This is often used via the counterpositive: if a graph G has a subgraph that is not planar, then G is not planar.

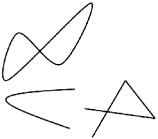
Jordan's Curve Theorem

Caveat: Proving certain properties of planar graphs rigorously requires some delicate topological arguments. In what follows, I will ignore these points. Hopefully, the statements we will use will be clear intuitively, and our focus in this class is on the combinatorial part. But it is important to be aware that complete proofs would require more care.

For example, one basic result that is useful in this context is the Jordan curve theorem. This is intuitively clear, but a complete proof requires delicate arguments that go beyond the scope of our class. A Jordan curve is a simple closed curve in $\mathbf{R}^2$.



Jordan Curves



Not Jordan Curves

Jordan's Curve Theorem, cont'd

Jordan curve theorem. For every Jordan curve, its complement in $\mathbf{R}^2$ has precisely two connected components, one bounded and one unbounded.

The notion of connected components makes sense for every topological space. However, for open subsets of $\mathbf{R}^2$, it can also be defined via "pathwise-connectedness" as follows (leading to a similar definition to the one in the case of graphs).

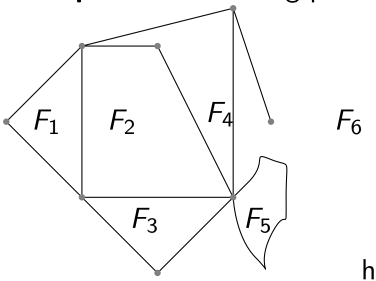
Suppose that U is an open subset of $\mathbf{R}^2$. For two points $x, y \in U$, we put $x \sim y$ if there is a curve contained in U with end points x and y. It is easy to see that this is an equivalence relation (using concatenation of curves). The equivalence classes are the connected components of U.

Euler's formula for planar graphs

We now prove the first important result about planar graphs. We begin with the following

Definition. Given a planar graph G and a planar representation, the faces of G (with respect to this representation) are the connected components of the complement in $\mathbf{R}^2$ for the union of the drawn edges of G.

Example. The following planar graph representation



has 6 faces.

Euler's formula for planar graphs

Theorem 1 (Euler). For every planar drawing of a connected planar graph G, if F is the set of faces of this drawing, then

$$\#F - \#E(G) + \#V(G) = 2.$$

Remark. A consequence of Euler's formula is that the number of faces is independent of the planar drawing of G.

Proof of the theorem. We argue by induction on the number d of edges of G. If $d = 0$, since G is connected, it follows that $\#V(G) = 1$. In this clear it is clear that $\#F = 1$ and the formula in the theorem is clear.

If G is a tree, then we know that

$$\#E(G) = \#V(G) - 1.$$

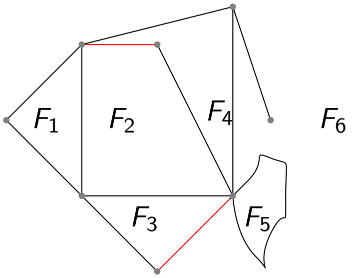
Moreover, it is "easy to see" that in this case $\#F = 1$ and the formula in the theorem follows.

Euler's formula for planar graphs, cont'd

Suppose now that G is not a tree. In this case, G has a subgraph that is isomorphic to some P_n, with $n \geq 1$. Let G' be the subgraph of G with the same vertices, but with one edge deleted that is part of the above polygon. In this case G' is again connected.

It is clear that in this case we have

$$\#V(G') = \#V(G) \quad \text{and} \quad \#E(G') = \#E(G) - 1.$$



One can also show that if F' is the set of faces of G', then we have $\#F' = \#F - 1$ (e.g. check what happens if we delete one of the red edges on the left).

A restriction on the number of edges of a planar graph

By induction, we have

$$\#F - \#E(G) + \#V(G) = \#F' - \#E(G') + \#V(G') = 2.$$

This completes the proof of the theorem.

Remark. We can't use Euler's formula to show that a certain graph is not a planar graph, since in order to consider the number of faces, we need the graph to have a planar representation. However, the following consequence of Euler's formula will provide us with such a criterion.

Corollary 1. If G is a connected planar simple graph, with $n \geq 3$ vertices, then

$$\#E(G) \leq 3n - 6.$$

A restriction on the number of edges of a planar graph, cont'd

Proof of Corollary 1. If $n = 3$, then $\#E(G) \leq 3$, hence the bound holds. Suppose now that $n \geq 4$, in which case $\#E(G) \geq 3$, since G is connected.

Consider now a planar representation of G. Since G is simple, one can check that every face contains ≥ 3 drawn edges in its boundary. Moreover, every edge is in the boundary of ≤ 2 faces. These two facts imply

$$\#F \leq \frac{2}{3}\#E(G).$$

On the other hand, Euler's formula gives

$$\#F = \#E(G) - \#V(G) + 2.$$

Combining these formulas gives

$$\#E(G) - \#V(G) + 2 \leq \frac{2}{3}\#E(G).$$

This immediately gives the formula in the corollary.

A restriction on the number of edges of a planar graph, cont'd

Application. We can use Corollary 1 to show that K_5 is not a planar graph. Indeed, note that

$$\#E(K_5) = \binom{5}{2} = 10 > 3\#V(G) - 6 = 9.$$

Remark. We can't apply the same argument for $K_{3,3}$, since in that case we have

$$\#E(G) = 9 \leq 3\#V(G) - 6 = 12.$$

However, we have the following more general result:
Corollary 2. Let G be a connected simple planar graph. If $g \geq 3$ is such that $\#E(G) \geq g$ and G has no P_n with $n < g$, then

$$\#E(G) \leq \frac{g}{g-2}(\#V(G) - 2).$$

A restriction on the number of edges of a planar graph, cont'd

Proof of Corollary 2. The argument is similar to the one in the proof of the previous corollary. The hypothesis implies that every face has $\geq g$ drawn edges in the boundary; since every edge is in the boundary of ≤ 2 faces, we obtain

$$\#F \leq \frac{2}{g}\#E(G).$$

Using Euler's formula, we obtain

$$g\big(2 - \#V(G) + \#E(G)\big) \leq 2\#E(G).$$

This gives the formula in the corollary.
Application. For the graph $K_{3,3}$ we have

$$\#E(G) = 9 > \frac{4}{4-2}\big(\#V(G) - 2\big) = 8.$$

Note that $K_{3,3}$ has contains no subgraph isomorphic to P_3 since it is a bipartite graph. Corollary 2 thus implies that $K_{3,3}$ is not planar.

So far, we have seen 2 examples of non-planar graphs: K_5 and $K_{3,3}$. The surprising result is that these are the minimal examples of non-planar graphs.

More precisely, we have the following striking result:

Theorem 2 (Kuratowski). A graph G is not planar if and only if it contains a subgraph which is a subdivision of either K_5 or $K_{3,3}$.

Kuratowski: important Polish mathematician and logician, 1896-1980.

He made important contributions to topology and set theory (he was actually the first to prove Zorn's lemma).

Part of a famous group of Polish mathematicians from Lwow (Banach, Borsuk, Ulam, Schauder, Steinhaus). He moved to Warsaw University in 1934, where he played an important role both before and after the war.

Colorings of planar graphs

We next discuss some results concerning proper colorings of planar graphs. From now on, we assume that all graphs are simple.

Proposition 1. Every planar graph can be colored with 6 colors.

Proof. We may and will assume that G is connected and argue by induction on $n = \#V(G)$. Of course, if $n \leq 5$, the assertion is trivial. Suppose now that $n \geq 6$.
The key point is that since $n \geq 3$ and G is a planar connected graph, it follows from Corollary 1 that

$$\#E(G) \leq 3n - 6.$$

Since $\sum_{x \in V(G)} \deg(x) = 2\#E(G) \leq 6n - 12$, it follows that there is $x \in V(G)$ with $\deg(x) \leq 5$.

Colorings of planar graphs, cont'd

Let H be the subgraph of G spanned by $V(G) \smallsetminus \{x\}$. By induction, H admits a proper coloring with 6 colors. Since $\deg(x) \leq 5$, it follows that there is one color not used for the neighbors of x in G. If we color x with that color, we obtain a proper coloring of G with 6 colors. This completes the proof.

With a bit more effort, we see that we can do better:

Theorem 3 (Five color theorem). For every planar graph G, we have $\chi(G) \leq 5$.

Proof. We argue as in the proof of the previous proposition, by induction on $n = \#V(G)$. We may assume that G is connected, $n \geq 5$, and that $\deg(x) \geq 5$ for every $x \in V(G)$ (since otherwise we are done by induction).

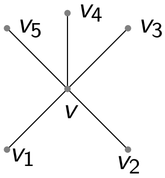
By Corollary 1, we have a vertex v with $\deg(v) = 5$.

Colorings of planar graphs, cont'd

Let H be the subgraph of G spanned by $V(G) \setminus \{v\}$. By induction, we know that H admits a proper coloring with 5 colors.

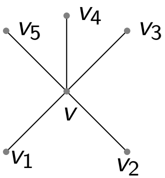
If the neighbors of v don't use all 5 colors, then we are done: by coloring v with the missing color, we get a proper coloring of G.

We consider a planar drawing of G and let $v_1, \ldots, v_5$ be the neighbors of v, numbered in anticlockwise order, as in the picture below:



Let c_i be the color of v_i. We first try to replace the color of v_1 by c_3.

Colorings of planar graphs, cont'd

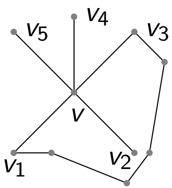


Let G' be the subgraph of H spanned by the vertices colored c_1 and c_3.

Case 1. If v_1 and v_3 lie in different connected components of G', let's swap the colors c_1 and c_3 in the connected component of G' containing v_1. We then end up with a proper coloring of H only using 4 colors for the neighbors of v, in which case we are done.

Case 2. If v_1 and v_3 lie in the same component of G', consider the path in $\mathbf{R}^2$ formed by the corresponding drawn edges.

Colorings of planar graphs, cont'd



This path, together with the drawn edges $\{v, v_3\}$ and $\{v, v_1\}$ form a Jordan curve such that v_2 and v_4 lie in different connected components of the complement.

This implies that v_2 and v_4 do not lie in the same connected component of the subgraph of H spanned by the vertices colored with c_2 or c_4. Then we can apply the argument in Case 1 on the previous slide to conclude that we can get a proper coloring of H using only 4 colors for the neighbors of v. This implies that G has a proper coloring with 5 colors.

Colorings of planar graphs, cont'd

In fact, the result in Theorem 3 is not optimal either:

Theorem 4 (Four color theorem). If G is a planar graph, then $\chi(G) \leq 4$.

Remark. Given a map drawn on the plane, we get a graph whose vertices correspond to the countries on the map, with edges joining two countries that have a common border. It is easy to see that this is a planar graph. The above theorem implies that we can use 4 colors to color each country on the map such that no two countries that share a border get the same color.

As I mentioned in the first lecture, this was a famous open problem for over 100 years and was solved in 1976 by two mathematicians from University of Illinois, Urbana-Champaign, Kenneth Appel and Wolfgang Haken, by making use of computers.