Lecture 8: Hall's marriage theorem and systems of distinct representatives

October 1, 2020

Matchings

We will discuss an important result of P. Hall. We will first formulate and prove a graph-theoretic formulation and then will give a proof in a set-theoretic formulation that also comes with precise estimates.

Recall that we say that $G = G(X, Y)$ is a bipartite graph if G is a simple graph and X and Y are subsets of $V(G)$ such that every edge in G has one end in X and one end in Y.

Terminology. 1) A matching in a graph G is a subset $\Lambda \subseteq E(G)$ such that every vertex in $V(G)$ is incident to at most one edge in Λ.

2) A *complete matching* from X to Y in a bipartite graph $G(X, Y)$ is a matching Λ such that every vertex in X is incident to some edge in Λ.

The statement of Hall's theorem

Given a bipartite graph $G(X, Y)$, we are interested in when there is a complete matching from X to Y. It is very easy to give a necessary condition, as follows. For a vertex $a \in V(G)$ we denote $\Gamma(a)$ the set of neighbors of a and for a subset $A \subseteq V(G)$, we put $\Gamma(A) = \bigcup_{a \in A} \Gamma(a)$.

It is clear that if we have a complete matching $\Lambda \subseteq E(G)$ from X to Y, then for every $A \subseteq X$, we have a subset $\Lambda_0 \subseteq \Lambda$ consisting of those elements of Λ incident to the vertices in A. Note that $\#\Lambda_0 = \#A$. On the other hand, the set of vertices in Y incident to the edges in Λ_0 is a subset of $\Gamma(A)$ with $\#A$ elements. Hence $\#\Gamma(A) \geq \#A$.

The surprising fact is that this condition is also sufficient. This is the content of Hall's marriage theorem.

The statement of Hall's theorem, cont'd

Theorem 1 (Hall). Given a bipartite graph $G(X, Y)$, there is a complete matching from X to Y if and only if for every $A \subseteq X$, we have

$$\#\Gamma(A) \geq \#A.$$

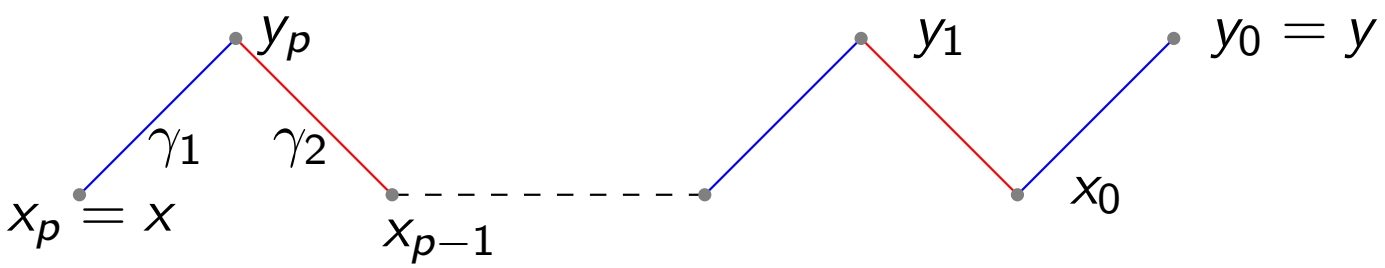
Reason for the name: suppose that we have two sets, X consisting of women and Y consisting of men (or viceversa). We link a woman in X and a man in Y by an edge if they like each other. A complete matching from X to Y is given by a way of assigning marriage partners to all the women in X in such a way that everybody ends up with somebody they like.

Proof of Hall's theorem. We have already seen that the condition in the theorem is necessary. We now focus on sufficiency. Let $n = \#X$. In order to prove this implication it is enough to show that if Λ is a matching with m edges, where $0 \leq m < n$, then we can find another matching with $m + 1$ edges. After several steps, we get a matching with n edges, which is a complete matching.

Proof of Hall's theorem

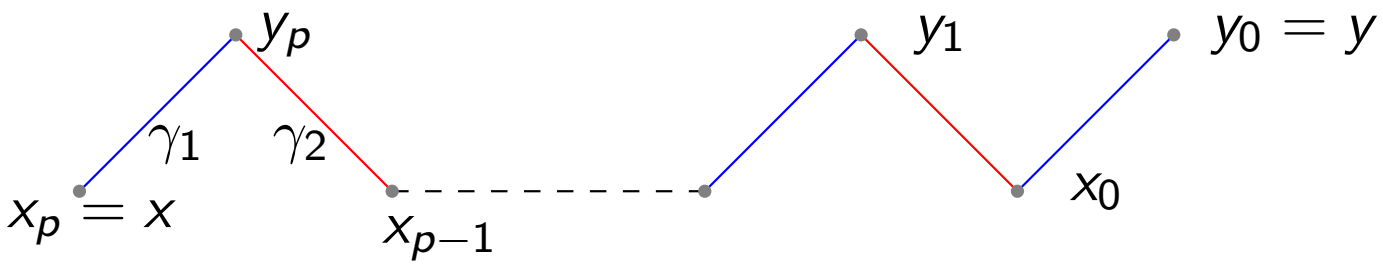
In what follows, we will refer to the edges in Λ as red and to those in $E(G) \setminus \Lambda$ as blue. Since $m < n$, it follows that there is a vertex x in X that isn't incident to any red edges.

Key claim. There is a simple path γ starting with x and a blue edge, alternating blue and red edges, and ending with a blue edge and a vertex y that is not incident to any red edge.



Given such a path, we take a new subset of $E(G)$ by flipping the colors of the edges of γ and leaving all other edges unchanged. We take Λ' to consist of the new red edges.

Proof of Hall's theorem, cont'd



Note that we replace p old red edges with $p+1$ new ones, hence $\#\Lambda' = m+1$. Moreover, Λ' is a matching, too: indeed, suppose that we have two edges in Λ' that are incident to the same vertex v. Clearly, v has to be one of $x_0, y_0, \ldots, x_p, y_p$.

However, v can't be x_p: by the original assumption on x, the only edge in Λ' that is incident to x_p is γ_1.

Also, v can't be any x_i with $0 \leq i < p$: the only edge in Λ incident to x_i is $\{y_{i+1}, x_i\}$ and thus the only edge in Λ' incident to x_i is $\{x_i, y_i\}$. Similarly, v can't be any y_i, with $1 \leq i \leq p$.

Finally, v can't be y_0: by construction the only edge in Λ' incident to y is the last edge in γ. We thus have a matching with $m+1$ edges.

Proof of Hall's theorem, cont'd

We proceed to construct the path γ in the key claim. We first construct recursively a sequence of vertices $z_0, \ldots, z_q \in X$ and $w_1, \ldots, w_{q+1} \in Y$, as follows.
We start by taking $z_0 = x$. Since $\#\Gamma(z_0) \geq 1$ by hypothesis, we have one edge in G (necessarily blue, by the assumption on x), joining z_0 with another vertex $w_1 \in Y$. If w_1 is not incident to any red edges, then we stop.

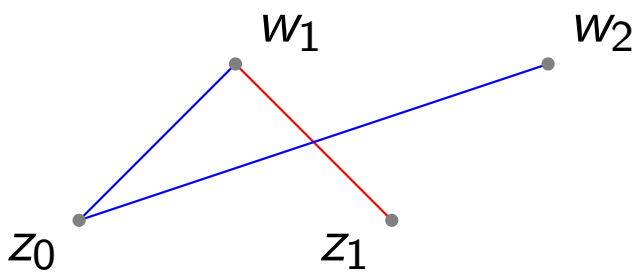
Otherwise, say that w_1 is incident to a red edge $\{w_1, z_1\}$. Of course, we have $z_1 \neq z_0$. By hypothesis, we have

$$\#\Gamma\big(\{z_0, z_1\}\big) \geq 2,$$

hence there is a vertex $w_2 \neq w_1$ in Y that is joined to either z_0 and z_1 by an edge (note that this automatically has to be blue).

Proof of Hall's theorem, cont'd

We thus are in one of the following two situations:



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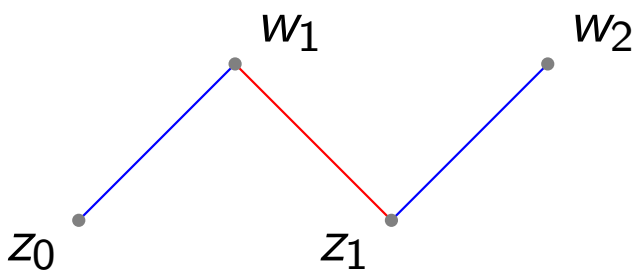
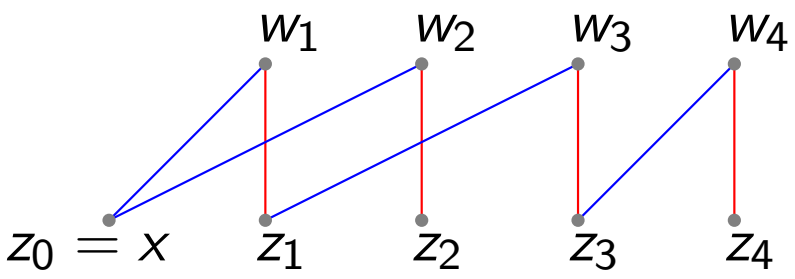


Suppose now that we have constructed distinct vertices $z_0, z_1, \ldots, z_k \in X$ and distinct vertices $w_1, \ldots, w_k \in Y$ such that for $1 \leq i \leq k$, the vertices z_i and w_i are connected by a red edge and w_i is connected to one of $z_0, \ldots, z_{i-1}$ by a blue edge. Here is an example with $k = 4$:

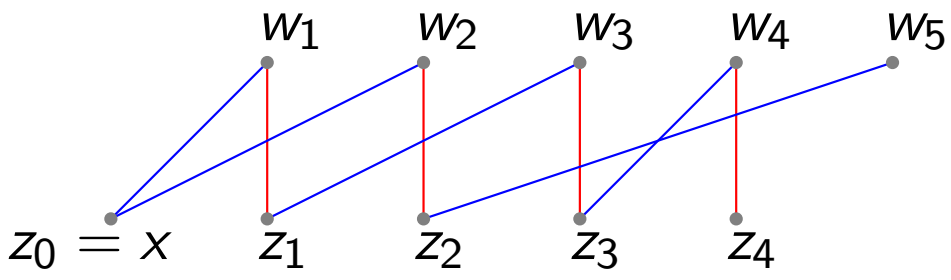


Proof of Hall's theorem, cont'd

By hypothesis, we have

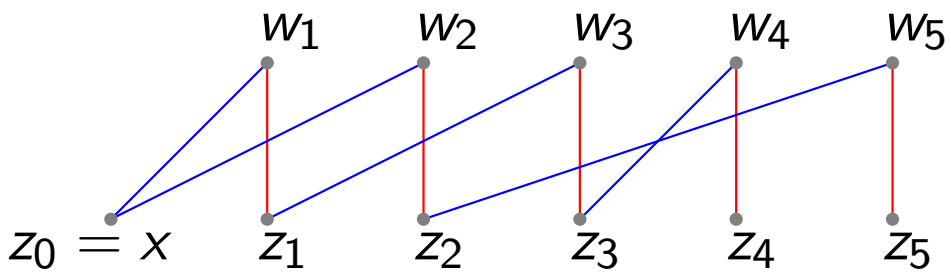
$$\#\Gamma\big(\{z_0, \ldots, z_k\}\big) \geq k + 1,$$

hence we can find a vertex w_{k+1} in Y, different from $w_1, \ldots, w_k$, that is joined to one of $z_0, \ldots, z_k$ by an edge (this has to be blue, since for every $i \leq k$, the vertex z_i is already joined to w_i by a red edge). If w_{k+1} is not incident to any red edges, we stop the precess here. For example, the example on the previous slide might lead to:



Proof of Hall's theorem, cont'd

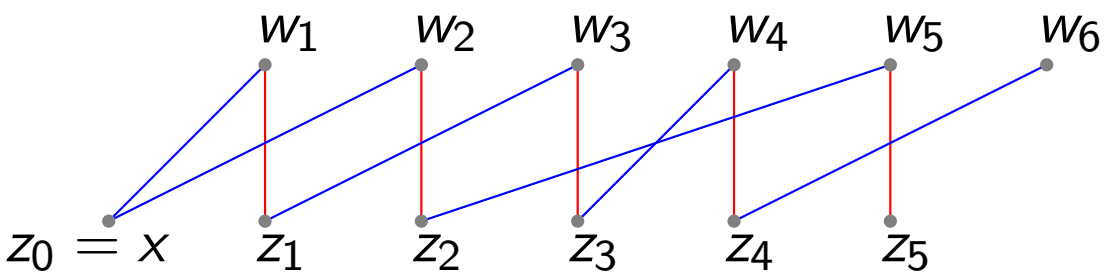
On the other hand, if there is a vertex $z_{k+1} \in X$ joined to w_{k+1} via a red edge (note that $z_{k+1} \neq z_i$ if $i \leq k$ since z_i is joined by a red edge with w_i), we add it, together with this red edge as well. We have constructed w_{k+1} and z_{k+1} and repeat the process.



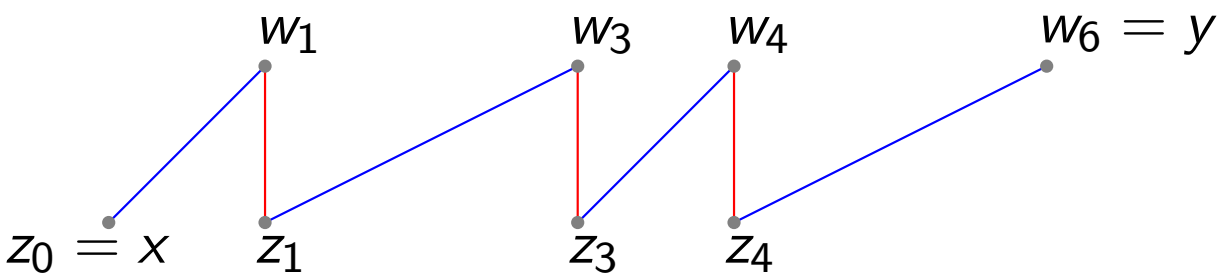
Proof of Hall's theorem, cont'd

Suppose that this process ends after we constructed $w_{q+1} = y$, hence by construction y is not incident to any red edge. We proceed by constructing γ backwards.

For example, out of the following vertices and edges:



we would obtain the simple path



Proof of Hall's theorem, cont'd

Let's formalize now the general case.

We put $y_0 = y$. By construction, y_0 is connected by a blue edge to some z_{i_1}, with $i_1 < q + 1$. We put $x_0 = z_{i_1}$ and if $i_1 = 0$, we stop; otherwise, we put $y_1 = w_{i_1}$, so that x_0 and y_1 are connected by a red edge.

We can now find $i_2 < i_1$ such that y_1 is joined to z_{i_2} by a blue edge. We put $x_1 = z_{i_2}$ and if $i_2 = 0$ we stop; otherwise we put $y_2 = w_{i_2}$, so that x_1 and y_2 are connected by a red edge. We continue this process until we obtain $x_p = z_{i_p}$, with $i_p = 0$. Therefore we have $x_p = x$ and by combining the edges we constructed we obtain the simple path γ, as in the key claim. This completes the proof of the theorem.

A set-theoretic variant of the marriage problem

Next goal: discuss a version of Hall's marriage theorem in the context of set theory. In particular, we will obtain a new proof of the theorem. While non-constructive, this has the advantage that it gives a lower bound for the number of complete matchings.

Setup. Suppose that S is a finite set and $A_0, \ldots, A_{n-1}$ are subsets of S. A system of distinct representatives (or SDR, for short) is given by n distinct elements $a_0, \ldots, a_{n-1} \in S$ such that $a_i \in A_i$ for $0 \leq i \leq n-1$.

Terminology. We say that the sets $A_0, \ldots, A_{n-1}$ satisfy property (H) if for every k, the union of any k of these sets has $\geq k$ elements. In this case, a set of k such subsets, with $0 < k < n$, whose union has precisely k elements is called a critical block.
Note that if there is at least one SDR, then $A_0, \ldots, A_{n-1}$ satisfy condition (H). Our goal will be to give a lower bound for the number of SDRs when this condition is satisfied (in particular, we will see that this number is strictly positive).

The equivalence of the two formulations

Suppose that $G = G(X, Y)$ is a bipartite graph and say $X = \{x_0, \ldots, x_{n-1}\}$. For every i, with $0 \leq i \leq n-1$, let $A_i = \Gamma(x_i) \subseteq Y$. An SDR for $A_0, \ldots, A_{n-1}$ consists precisely of a complete matching in G from X to Y. Moreover, G satisfies the condition in Hall's marriage theorem if and only if $A_0, \ldots, A_{n-1}$ satisfy condition (H).

Conversely, suppose that we have subsets $A_0, \ldots, A_{n-1}$ of a finite set S. We define the bipartite graph $G = G(X, Y)$, where $X = \{0, 1, \ldots, n-1\}$ and $Y = S$, such that $i \in X$ and $a \in S$ are joined by an edge precisely when $a \in A_i$. Giving and SDR for $A_0, \ldots, A_{n-1}$ is equivalent to giving a complete matching in G from X to Y. Moreover, $A_0, \ldots, A_{n-1}$ satisfy condition (H) if and only if G satisfies the condition in Hall's marriage theorem. Hence by the theorem, $A_0, \ldots, A_{n-1}$ satisfy condition (H) if and only if they have an SDR.

König's theorem

Before discussing the lower bound for the number of SDRs, we discuss some applications of Hall's marriage theorem in its set-theoretical formulation.

Theorem 2 (König). Let $A = (a_{i,j})$ be a $p \times q$ matrix, with $a_{ij} \in \{0, 1\}$ for all i and j. In this case the minimum number m of lines of A that contain all 1's is equal to the maximum number M of 1's, no two on the same line.

In the above: a line of A is either a row or a column of A.

Proof of the theorem. Given a set B of 1's, no two of them on the same line of A, and a set C of lines of A containing all 1's in A, we get an injective map from B to C, hence $\#B \leq \#C$. We thus have $M \leq m$.

We next need to prove the opposite inequality.

König's theorem, cont'd

Note that we may permute the rows and the columns of A, hence we may assume that m is achieved by taking the first r rows and the first s columns of A (hence $m = r + s$).

For $1 \leq i \leq r$, we put $A_i = \{j > s \mid a_{ij} = 1\}$.

Note that $A_1, \ldots, A_r$ satisfy condition (H): if the union of k of these sets contains $< k$ elements, we can replace the corresponding k rows by $< k$ columns of index $> s$ to get a smaller number of lines containing all 1s, contradicting the minimality of m.
By Hall's marriage theorem, we can find an SDR for $A_1, \ldots, A_r$. This implies that we can find r 1's in the first r rows and in the columns $\geq s + 1$ such that no 2 of these lie in the same line.
A similar argument gives s 1's in the first s columns and in the rows $\geq r + 1$ such that no 2 of these lie in the same line. By putting these $r + s$ entries together, we get $M \geq r + s = m$.

Birkhoff's theorem

Theorem 3 (Birkhoff). Let $A = (a_{i,j})$ be an $n \times n$ matrix, with all $a_{ij} \in \mathbf{Z}_{\geq 0}$. If A satisfies

i) For every j, we have $\sum_i a_{ij} = \ell$, and
ii) For every i, we have $\sum_j a_{ij} = \ell$,

then A is a sum of ℓ permutation matrices.

Proof. We argue by induction of $\ell \geq 0$. If $\ell = 0$, then $A = 0$ and the assertion is clear.

For the induction step, we put for $1 \leq i \leq n$

$$A_i = \{j \mid a_{ij} > 0\} \subseteq \{1, \ldots, n\}.$$

Note that $A_1, \ldots, A_n$ satisfy condition (H): if we consider $A_{i_1} \cup \ldots \cup A_{i_k}$, then

$$\sum_{p=1}^{k} \sum_{j \in A_{i_p}} a_{i_p j} = \sum_{p=1}^{k} \sum_{j=1}^{n} a_{i_p j} = k\ell.$$

Birkhoff's theorem, cont'd

Since the sum of all entries in a column is ℓ, it follows that there are at least k columns with nonzero entries in one of the rows $i_1, \ldots, i_k$. The sets $A_1, \ldots, A_n$ thus satisfy condition (H).

We can thus choose an SDR for $A_1, \ldots, A_n$: these are distinct $j_1, \ldots, j_n$ such that $a_{ij_i} > 0$ for all i.

We thus have a permutation matrix $P = (b_{i,j})$ with $b_{ij_i} = 1$ for all i and all other entries 0. In this case $A' = A - P$ satisfies the hypothesis in the theorem, with ℓ replaced by $\ell - 1$. Applying induction for A', we conclude the proof.

A lower bound for the number of SDRs

We end with the proof of a lower bound for the number of SDRs for a set of subsets that satisfy the (H) condition. In particular, this will give a second proof for Hall's marriage theorem.

Notation. Given integers $m_0 \leq m_1 \leq \ldots \leq m_{n-1}$, we put

$$F_n(m_0, \ldots, m_{n-1}) := \prod_{i=0}^{n-1} \max\{1, m_i - i\}.$$

Given arbitrary integers $a_0, \ldots, a_{n-1}$, we put

$$f_n(a_0, \ldots, a_{n-1}) := F_n(m_0, \ldots, m_{n-1}),$$

where $(m_0, \ldots, m_{n-1})$ is a nondecreasing rearrangement of $(a_0, \ldots, a_{n-1})$.

A lower bound for the number of SDRs, cont'd

Lemma. The function $f_n \colon \mathbf{Z}^n \to \mathbf{Z}_{>0}$ is nondecreasing with respect to each variable.

Proof. It is enough to show that for every i, we have

$$f_n(a_0, \ldots, a_i, \ldots, a_{n-1}) \leq f_n(a_0, \ldots, a_i + 1, \ldots, a_{n-1}).$$

Suppose that $(m_0, \ldots, m_{n-1})$ is a nondecreasing rearrangement of $(a_0, \ldots, a_{n-1})$ and let k be such that $m_k = a_i$ and either $k = n - 1$ or $m_k < m_{k+1}$. In this case, it follows that a nondecreasing rearrangement of $(a_0, \ldots, a_i + 1, \ldots, a_{n-1})$ is given by $(m_0, \ldots, m_k + 1, \ldots, m_{n-1})$. Since it is clear that we have

$$F_n(m_1, \ldots, m_k, \ldots, m_{n-1}) \leq F_n(m_1, \ldots, m_k + 1, \ldots, m_{n-1}),$$

we are done.

A lower bound for the number of SDRs, cont'd

Theorem 4. Given the subsets $A_0, \ldots, A_{n-1}$ of the finite set S, with $\#A_i = m_i$, if the sets satisfy property (H), then the number $N(A_0, \ldots, A_{n-1})$ of SDRs for $A_0, \ldots, A_{n-1}$ satisfies.

$$N(A_0, \ldots, A_{n-1}) \geq f_n(m_0, \ldots, m_{n-1}).$$

In particular, there is an SDR for $A_0, \ldots, A_{n-1}$.

Proof. We argue by induction on $n \geq 1$. If $n = 1$, then condition (H) says that $m_0 \geq 1$ and it is clear that $N(A_0) = m_0 = F_1(m_0)$.

For the induction step, we may permute the subsets and thus assume that $m_0 \leq m_1 \leq \ldots \leq m_{n-1}$. We distinguish two separate cases.

Case 1. The sets $A_0, \ldots, A_{n-1}$ contain no critical block, that is, the union of any k of these subsets has $> k$ elements.

A lower bound for the number of SDRs, cont'd

We choose $a \in A_0$ and put $A_i(a) = A_i \setminus \{a\}$ for $1 \leq i \leq n-1$. Note that $A_1(a), \ldots, A_{n-1}(a)$ again satisfy property (H): by the assumption in this case, the union of any k of $A_1, \ldots, A_{n-1}$ have $> k$ elements, hence the union of any k of $A_1(a), \ldots, A_{n-1}(a)$ have $\geq k$ elements.
It is clear that we have

$$N(A_0, \ldots, A_{n-1}) \geq \sum_{a \in A_0} N\big(\#A_1(a), \ldots, \#A_{n-1}(a)\big).$$

Since $\#A_i(a) \geq m_i - 1$ for all i and since f_{n-1} is nondecreasing in each variable, we conclude using the inductive hypothesis that

$$N(A_0, \ldots, A_{n-1}) \geq \sum_{a \in A_0} f_{n-1}\big(\#A_1(a), \ldots, \#A_{n-1}(a)\big)$$

$$\geq \sum_{a \in A_0} F_{n-1}(m_1 - 1, \ldots, m_{n-1} - 1) = m_0 \cdot F_{n-1}(m_1 - 1, \ldots, m_n - 1)$$

$$= F_n(m_0, \ldots, m_{n-1}).$$

A lower bound for the number of SDRs, cont'd

Case 2. We assume that we have a critical block $A_{\nu_0}, \ldots, A_{\nu_{k-1}}$, with $\nu_0 < \nu_1 < \ldots < \nu_{k-1}$, for some k with $0 < k < n$.

Remove all elements of $A_{\nu_0} \cup \ldots \cup A_{\nu_{k-1}}$ from the other A_i to get the sets $A'_{\mu_0}, \ldots, A'_{\mu_{\ell-1}}$, with $\mu_0 < \ldots < \mu_{\ell-1}$, where $k + \ell = n$ and

$$\{\nu_0, \ldots, \nu_{k-1}, \mu_0, \ldots, \mu_{\ell-1}\} = \{0, 1, \ldots, n-1\}.$$

• It is clear that the sets $A_{\nu_0}, \ldots, A_{\nu_{k-1}}$ satisfy property (H).
• The sets $A'_{\mu_0}, \ldots, A'_{\mu_{\ell-1}}$ also satisfy property (H): if B is the union of j of these sets, the union of B and of $A_{\nu_0} \cup \ldots \cup A_{\nu_{k-1}}$ has $\geq j + k$ elements. Since $\#\left(A_{\nu_0} \cup \ldots \cup A_{\nu_{k-1}}\right) = k$, it follows that $\#B \geq j$.

Moreover, the set of SDRs for $A_0, \ldots, A_{n-1}$ is in bijection with the product between the set of SDRs for $A_{\nu_0}, \ldots, A_{\nu_{k-1}}$ and that of SDRs for $A'_{\mu_0}, \ldots, A'_{\mu_{\ell-1}}$.

A lower bound for the number of SDRs, cont'd

We thus have

$$N(A_0, \ldots, A_{n-1}) = N(A_{\nu_0}, \ldots, A_{\nu_{k-1}}) \cdot N(A'_{\mu_0}, \ldots, A'_{\mu_{\ell-1}}).$$

Using the induction hypothesis for k and ℓ, the facts that $\#A'_{\mu_i} \geq m_{\mu_i} - k$ for $0 \leq i \leq \ell - 1$, and $m_{\nu_j} \geq m_j$ for $0 \leq j \leq k - 1$, together with the fact that both f_k and f_ℓ are nondecreasing in each variable, we conclude that

$$N(A_0, \ldots, A_{n-1}) \geq f_k(m_{\nu_0}, \ldots, m_{\nu_{k-1}}) \cdot f_\ell(\#A'_{\mu_0}, \ldots, \#A'_{\mu_{\ell-1}})$$

$$\geq F_k(m_0, \ldots, m_{k-1}) \cdot F_\ell(m_{\mu_0} - k, \ldots, m_{\mu_{\ell-1}} - k).$$

If we show that this last product is equal to $F_n(m_0, \ldots, m_{n-1})$, then the proof of the induction step is complete.

A lower bound for the number of SDRs, cont'd

By definition of F_k, F_ℓ, and F_n, we have

$$F_k(m_0, \ldots, m_{k-1}) = \prod_{i=0}^{k-1} \max\{m_i - i, 1\},$$

$$F_n(m_0, \ldots, m_{n-1}) = \prod_{i=0}^{n-1} \max\{m_i - i, 1\}, \quad \text{and}$$

$$F_\ell(m_{\mu_0} - k, \ldots, m_{\mu_{\ell-1}} - k) = \prod_{i=0}^{\ell-1} \max\{m_{\mu_i} - i - k, 1\}.$$

Hence in order to complete the proof, it is enough to show that

$$\max\{m_{i+k} - i - k, 1\} = \max\{m_{\mu_i} - i - k, 1\} \quad \text{for} \quad 0 \le i \le \ell - 1.$$

A lower bound for the number of SDRs, cont'd

We want to show that

$$\max\{m_{i+k} - i - k, 1\} = \max\{m_{\mu_i} - i - k, 1\} \quad \text{for} \quad 0 \leq i \leq \ell - 1.$$

Since $\mu_i \leq i + k$ for all i, the inequality "$\geq$" is clear. Moreover, note that if $\mu_i < i + k$, then $i + k \leq \mu_{k-1}$.

Observe now that

$$m_{\nu_{k-1}} = \#A_{\nu_{k-1}} \leq \#(A_{\nu_1} \cup \ldots \cup A_{\nu_{k-1}}) = k.$$

It follows that if $\mu_i < i + k$, then $m_{i+k} \leq m_{\nu_{k-1}} \leq k - 1$, hence in this case

$$\max\{m_{i+k} - i - k, 1\} = \max\{m_{\mu_i} - i - k, 1\} = 1.$$

This completes the proof of the theorem.