Lecture 7: Introduction to Ramsey theory, part II
The Ramsey numbers $N(p, q; 2)$

September 29, 2020

A first upper-bound

In this lecture we discuss the problem of estimating the Ramsey numbers $N(p, q; 2)$. In fact, determining/estimating these numbers is very much an open problem and we will only touch upon this topic

Recall first the meaning of the Ramsey number $R(p, q; r)$: this is the smallest N such that for every set S with $n \geq N$ elements, if all r-subsets are colored red or blue, then either there is a p-subset of S, all of whose r-subsets are red, or there is a q-subset of S, all of whose r-subsets are blue.

Recall that in the proof of Ramsey's theorem, we showed the following:

$$N(p, q; r) \leq N\big(N(p-1, q; r), N(p, q-1; r); r-1\big) + 1.$$

Recall that $N(p, q; 1) = p + q - 1$, hence for $r = 2$, we get

A first upper-bound, cont'd

$$N(p,q;2) \leq N(p-1,q;2) + N(p,q-1;2). \tag{1}$$

Corollary. For every $p, q \geq 2$, we have

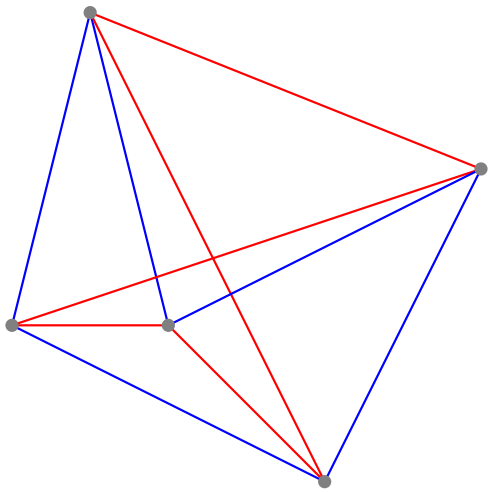
$$N(p,q;2) \leq \binom{p+q-2}{p-1}.$$

Proof. If $p = 2$, then we have seen that $N(p,q;2) = q$, hence the above inequality is an equality. Similarly, we are OK if $q = 2$.
The general case follows by induction on $p+q$ using (1):

$$N(p,q;2) \leq N(p-1,q;2) + N(p,q-1;2)$$
$$\leq \binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}.$$

The value of $N(3,3;2)$

We have seen last time that $N(3,3;2) \leq 6$. In fact, this is an equality: the following coloring of the edges of K_5



does not have any monochromatic triangle.

Upper bound for $N(4,3;2)$

We next consider $N(4,3;2)$. The inequality (1) gives

$$N(4,3;2) \leq N(3,3;2) + N(4,2;2) = 6 + 4 = 10.$$

In fact we can do better. More generally, we have the following:
Lemma. The inequality in (1) is strict if both $N(p-1,q;2)$ and $N(p,q-1;2)$ are even.

Proof. Suppose that we have equality in (1). In other words, if $N_1 = N(p-1,q;2)$ and $N_2 = N(p,q-1;2)$, we have a red/blue coloring of the edges of $K_{N_1+N_2-1}$ such that no p-subset of vertices has all edges joining them red and no q-subset of vertices has all edges joining them blue.

Upper bound for $N(4,3;2)$, cont'd

Using the argument in the proof of Ramsey's theorem, we see that for every vertex a, if we look at the edges with one end a, precisely $N_1 - 1$ of these are red and $N_2 - 1$ of these are blue.

In this case, we see that the total number of red edges is obtained by summing over the vertices a and dividing by 2:

$$\#(\text{red edges}) = \frac{1}{2}(N_1 - 1)(N_1 + N_2 - 1).$$

However, this is not an integer (since both N_1 and N_2 are even), a contradiction.

We thus conclude that

$$N(4,3;2) \leq N(3,3;2) + N(4,2;2) - 1 = 6 + 4 - 1 = 9.$$

Precise value of $N(4,3;2)$

We next show that in fact $N(4,3;2) = 9$. In order to do this, we will give a red/blue coloring of the edges of K_8 such that

- no triangle is fully blue and
- no K_4-subgraph is fully red.

In order to do this, we label the vertices of K_8 by the elements of $\mathbf{Z}_8 = \big\{[0],[1],\ldots,[7]\big\}$. We color the edge $\{i,j\}$ blue precisely if $i-j = \pm[3]$ or $i-j = [4]$.

If i,j,k give a blue triangle, it follows that $i-j, j-k \in \big\{[3],[4],[5]\big\}$. In this case, we have $i-k = (i-j)+(j-k) \in \big\{[6],[7],[0],[1],[2]\big\}$, hence $\{i,k\}$ is red, a contradiction.

Precise value of $N(4,3;2)$, cont'd

Suppose now that a, b, c, d give a fully red K_4 subgraph. First, we have $a - b \in \{[1], [2], [6], [7]\}$ (note that $a - b \neq [0]$ since a and b are distinct). The same argument applies to the other differences, e.g. b-c, b-d, etc.

If $a - b = [2]$, then $b - c$ can't be equal to [1], [2], or [6] (otherwise, by considering $a - c$, we get a contradiction).
Similarly, $b - d$ can't be equal to [1], [2], or [6]. This implies $b - c = b - d$, hence $c = d$, a contradiction.

The same argument applies to the other differences. Hence $a - d$, $b - d$, and $c - d$ are distinct elements of $\{[1], [6], [7]\}$. Say we have $a - d = [1]$, $b - d = [6]$, and $c - d = [7]$. in this case $b - a = [6] - [1] = [5]$, a contradiction.

Hence the given coloring satisfies the desired property.

Other values of $N(p, q; 2)$ for small p and q

Applying again the inequality (1), we obtain

$$N(4, 4; 2) \leq N(3, 4; 2) + N(4, 3; 2) = 9 + 9 = 18.$$

One can show that this is an equality.

We similarly have

$$N(5, 3; 2) \leq N(4, 3; 2) + N(5, 2; 2) = 9 + 5 = 14.$$

One can show that this is an equality.

Other values of $N(p, q; 2)$ for small p and q, cont'd

The following values are harder to prove:

$$N(6, 3; 2) = 18 \ (< 6 + 14 - 1 = 19).$$

$$N(7, 3; 2) = 23 \ (< 18 + 7 = 25).$$

$$N(8, 3; 2) = 28 \ (< 23 + 8 = 31).$$

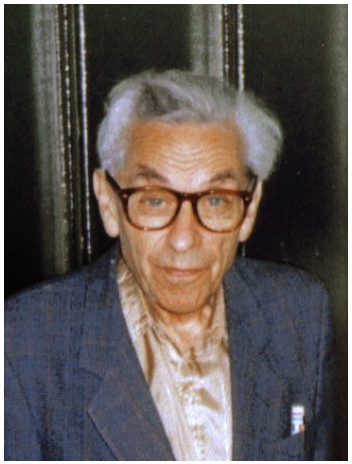
$$N(9, 3; 2) = 36 \ (< 28 + 9 = 37).$$

$$N(5, 4; 2) = 25 \ (< 18 + 14 - 1 = 31).$$

No other values of $N(p, q; 2)$ are known!

A quote from Erdös

A famous quote from the famous Hungarian mathematician Paul Erdös:



"Imagine an alien force, vastly more powerful than us, demanding the value of N(5, 5; 2) or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for N(6,6; 2). Then we should attempt to destroy the aliens."

Remark. It is known that $43 \leq N(5,5;2) \leq 48$ and $102 \leq N(6,6;2) \leq 165$.

An exponential lower bound

We will next be concerned with the asymptotic behavior of $N(p, p; 2)$. Our first result gives an exponential lower bound.

Theorem 1. For every $p \geq 2$, we have $N(p, p; 2) \geq 2^{\lfloor p/2 \rfloor}$, with strict inequality for $p \geq 3$.

Proof (Erdös). Consider the graph K_n. We have $2^{\binom{n}{2}}$ ways a coloring its edges with red and blue (to give such a coloring, we need to pick a subset of the set of all unordered pairs in $\{1, \ldots, n\}$).

Given a fixed subgraph of K_n isomorphic to K_p, the number of colorings of K_n for which this K_p is monochromatic is

$$2 \cdot 2^{\binom{n}{2} - \binom{p}{2}} = 2^{\binom{n}{2} - \binom{p}{2} + 1}$$

(note that there are $\binom{n}{2} - \binom{p}{2}$ edges that don't have both ends in the given subgraph).

An exponential lower bound

We conclude that the number of colorings for which some K_p is monochromatic is at most

$$\binom{n}{p} \cdot 2^{\binom{n}{2}-\binom{p}{2}+1}$$

(since we have $\binom{n}{p}$ subgraphs of K_n isomorphic to K_p).

For $p = 2$ we have equality in the theorem. To conclude the proof, it is thus enough to show that for $p \geq 3$, if $n = 2^{\lfloor p/2 \rfloor}$, then

$$2^{\binom{n}{2}} > \binom{n}{p} \cdot 2^{\binom{n}{2}-\binom{p}{2}+1}, \quad \text{or equivalently}$$

$$\binom{n}{p} < 2^{\binom{p}{2}-1}.$$

An exponential lower bound, cont'd

We have

$$\binom{n}{p} = \frac{n(n-1)\cdots(n-p+1)}{p!} < \frac{n^p}{p!} \quad \text{for} \quad p \geq 2.$$

Hence it is enough to show that

$$\frac{n^p}{p!} \leq 2^{\frac{p(p-1)}{2}-1} \quad \text{for} \quad p \geq 3.$$

Since $n \leq 2^{p/2}$, we have $n^p \leq 2^{p^2/2}$, and thus it is enough to show that

$$2^{p^2/2} \leq p! \cdot 2^{\frac{p(p-1)}{2}-1}, \quad \text{or equivalently}$$

$$2^{\frac{p}{2}+1} \leq p!.$$

An exponential lower bound, cont'd

It is easy to check that

$$2^{\frac{p}{2}+1} \leq p! \quad \text{for} \quad p \geq 3$$

by induction on p. Indeed, for $p = 3$, this is equivalent to $2^{3/2} \leq 3$, which is OK. If we know the assertion for p, then

$$2^{\frac{p+1}{2}+1} = 2^{1/2} \cdot 2^{\frac{p}{2}+1} \leq 2^{1/2} \cdot p! \leq (p+1)!$$

This completes the proof of the theorem.

Remark. A bit more care with the bounds shows that in fact we have

$$N(p, p; 2) > 2^{p/2} \quad \text{for all} \quad p \geq 3.$$

Interesting feature of the proof: we didn't actually construct any coloring.

An exponential upper bound

Remark. We also have the exponential upper-bound $N(p,p;2) \leq 4^p$, hence

$$\sqrt{2} \leq N(p,p;2)^{1/p} \leq 4 \quad \text{for} \quad p \geq 2.$$

It is not known whether $\lim_{p\to\infty} N(p,p;2)^{1/p}$ exists.

In order to prove this upper bound, recall that we have shown that $N(p,q;2) \leq \binom{p+q-2}{p-1}$ for $p,q \geq 2$. In particular, we have $N(p,p;2) \leq \binom{2p-2}{p-1}$ for $p \geq 2$.

We prove by induction on $p \geq 2$ that $\binom{2p-2}{p-1} \leq 4^p$. The case $p = 2$ is clear. Suppose now that we know the assertion for p. We then have

$$\binom{2p}{p} = \binom{2p-2}{p-1} \cdot \frac{2p(2p-1)}{p^2} \leq 4^p \cdot \frac{4p-2}{p} \leq 4^{p+1}.$$

Introduction to probabilities

Next goal: prove a stronger version of the lower bound theorem, with $2^{p/2}$ replaced by $cp2^{p/2}$ for some $c > 0$. This will allow us to introduce the Probabilistic Method, a powerful technique in combinatorics (a baby version was implicitly used in the proof of Theorem 1).

We will work in the following setup:

- We have a probability space X. This is simply a finite set of possibilities.
- We have events $A_1, \ldots, A_n$ in this space (simply subsets of the probability space).

We consider the probability that A_i happens

$$\Pr[A_i] := \frac{\#\{\text{possibilities when } A_i \text{ holds}\}}{\#\{\text{possibilities in the probability space}\}} = \frac{\#A_i}{\#X}.$$

We write $\overline{A_i}$ for the complement of A_i, so that

$$\Pr[\overline{A_i}] = 1 - \Pr[A_i].$$

Introduction to the probabilities, cont'd

For us, A_i will stand for an event that we do not want to happen. More precisely, we will have events $A_1, \ldots, A_n$ and we will want to guarantee that there is a positive probability that none of $A_1, \ldots, A_n$ occur.
The easiest scenario is the following:

$$\text{If} \quad \sum_{i=1}^{n} \Pr[A_i] < 1, \quad \text{then} \quad \bigcap_{i=1}^{n} \overline{A_i} \neq \emptyset.$$

However, we can do better if we keep track of dependency relations.
Recall: two events A and B are independent if $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.
If $\Pr[B] \neq 0$, then we put $\Pr[A|B] := \frac{\Pr[A \cap B]}{\Pr[B]}$, and A and B are independent if and only if $\Pr[A] = \Pr[A|B]$.

Note. $\Pr[A|B]$ is the probability that A holds if we require B to hold; hence A and B are independent if the probability that A holds does not change if we require B to hold.

Introduction to the probabilities, cont'd

Lemma 1. If A and B are independent, then A and $\overline{B}$ are independent, too (and thus also $\overline{A}$ and $\overline{B}$ are independent).

Proof. We have

$$\Pr[A \cap \overline{B}] = \Pr[A] - \Pr[A \cap B] = \Pr[A] - \Pr[A] \cdot \Pr[B]$$

$$= \Pr[A] \cdot \big(1 - \Pr[B]\big) = \Pr[A] \cdot \Pr[\overline{B}].$$

More generally, we say that the events $A_1, \ldots, A_n$ are mutually independent if for every $J \subseteq \{1, \ldots, n\}$, we have

$$\Pr[\cap_{j \in J} A_j] = \prod_{j \in J} \Pr[A_j].$$

Equivalently, for every $J \subseteq \{1, \ldots, n\}$ and every $i \in \{1, \ldots, n\} \setminus J$, the events $\cap_{j \in J} A_j$ and A_i are independent.

Introduction to the probabilities, cont'd

Why this is mutual independence relevant for us: recall that we have events $A_1, \ldots, A_n$ and want to guarantee that $\Pr[\cap_i \overline{A_i}] \neq 0$. If the events $\overline{A_1}, \ldots, \overline{A_n}$ are mutually independent, then

$$\Pr[\cap_i \overline{A_i}] = \prod_i \Pr[\overline{A_i}].$$

Hence in this case it is enough to check that $\Pr[A_i] < 1$ for all i.

Remark/Exercise. The events $A_1, \ldots, A_n$ are mutually independent if and only if $A_1, \ldots, A_{n-1}, \overline{A_n}$ are mutually independent if and only if $\overline{A_1}, \ldots, \overline{A_n}$ are mutually independent.

Dependency graphs

Given events $A_1, \ldots, A_n$, a dependency graph for these events is a graph G on the set $\{1, \ldots, n\}$ with the property that for every i and every $j_1, \ldots, j_r$, with all j_k different from i and non-adjacent to i, the events $\overline{A_i}$ and $\overline{A_{j_1}} \cap \ldots \cap \overline{A_{j_r}}$ are independent.

Note: this does not uniquely characterize the graph. For example, the complete graph on $\{1, \ldots, n\}$ is always a dependency graph for $A_1, \ldots, A_n$. What is interesting is to have a dependency graph with small $E(G)$.

Example. If the events $\overline{A_1}, \ldots, \overline{A_n}$ are mutually independent, then every graph on $\{1, \ldots, n\}$ is a dependency graph for these events.

The main ingredient will be a result saying that if we have a dependency graph with a bound for the degree of every vertex and each A_i has small probability (depending on the previous bound), then $\bigcap_i \overline{A_i} \neq \emptyset$.

Dependency graphs, cont'd

Proposition. Suppose that G is a dependency graph for the events $A_1, \ldots, A_n$ and that the following two conditions hold:

1) Every vertex of G has the degree $\leq d$ and
2) We have $\Pr[A_i] \leq p < 1$, where $4pd < 1$.

Then $\bigcap_{i=1}^n \overline{A_i} \neq \emptyset$.

Remark. Note that if $d = 0$, then $\overline{A_1}, \ldots, \overline{A_n}$ are mutually independent. The assertion in the proposition then follows from the fact that if $\overline{A_i} \neq \emptyset$ for all i, then $\bigcap_i \overline{A_i} \neq \emptyset$. The interesting case is thus when $d \geq 1$, which we assume from now on.

Before giving the proof, recall the following notation: if $\Pr[B] \neq 0$, then $\Pr[A|B] := \frac{\#(A \cap B)}{\#B}$. More generally, if $B_1, \ldots, B_r$ are events such that $B_1 \cap \ldots \cap B_r \neq \emptyset$, we put

$$\Pr[A|B_1, \ldots, B_r] = \frac{\#(A \cap B_1 \cap \ldots \cap B_r)}{\#(B_1 \cap \ldots \cap B_r)}.$$

Proof of the proposition

We prove by induction on $m \geq 2$ that for every subset $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, we have

$$\overline{A_{i_2}} \cap \ldots \cap \overline{A_{i_m}} \neq \emptyset \quad \text{and} \quad \Pr[A_{i_1} | \overline{A_{i_2}}, \ldots, \overline{A_{i_m}}] \leq \frac{1}{2d}.$$

This gives the assertion in the proposition: the above formula implies $\Pr[\overline{A_{i_1}} | \overline{A_{i_2}}, \ldots, \overline{A_{i_m}}] \neq 0$. Using the assertion for $m \leq n$, we thus have

$$\Pr[\overline{A_1} \cap \ldots \cap \overline{A_n}] = \Pr[\overline{A_1}] \cdot \prod_{i=2}^{n} \Pr[\overline{A_i} | \overline{A_1}, \ldots, \overline{A_{i-1}}]$$

$$\geq \left(1 - \frac{1}{4d}\right) \cdot \left(1 - \frac{1}{2d}\right)^{n-1} > 0.$$

In particular, we have $\overline{A_1} \cap \ldots \cap \overline{A_n} \neq \emptyset$.

Proof of the proposition, cont'd

Let's prove first the case $m = 2$: we need to show that if $i_1, i_2 \in \{1, \ldots, n\}$ are distinct, then $\overline{A_{i_2}} \neq \emptyset$ (this is clear, since $\Pr[\overline{A_{i_2}}] \geq 1 - p > 1 - \frac{1}{4d}$, by hypothesis) and $\Pr[A_{i_1} | \overline{A_{i_2}}] < \frac{1}{2d}$.

Since $\Pr[A_{i_1}], \Pr[A_{i_2}] \leq p$ and $p \leq \frac{1}{4d}$, we have

$$\Pr[A_{i_1} | \overline{A_{i_2}}] = \frac{\Pr[A_{i_1} \cap \overline{A_{i_2}}]}{\Pr[\overline{A_{i_2}}]} \leq \frac{\Pr[A_{i_1}]}{\Pr[\overline{A_{i_2}}]} \leq \frac{p}{1-p}$$

$$\leq \frac{\frac{1}{4d}}{1 - \frac{1}{4d}} = \frac{1}{4d - 1} < \frac{1}{2d}.$$

Hence the case $m = 2$ is OK.

Proof of the proposition, cont'd

Proof of the induction step. Suppose that $m \geq 3$ and that we know the assertion for $m-1$. After relabeling the events, we see that we may assume that $i_j = j$ for $1 \leq j \leq m$. The inductive hypothesis implies, in particular, that $\overline{A_2} \cap \ldots \cap \overline{A_m} \neq \emptyset$.
Furthermore, we may assume that 1 is adjacent to $2, \ldots, q$ in G and it is not adjacent to $q+1, \ldots, m$ (hence $q \leq d+1$ by hypothesis). We have

$$\Pr[A_1 | \overline{A_2}, \ldots, \overline{A_m}] = \frac{\Pr[A_1 \cap \overline{A_2} \cap \ldots \cap \overline{A_q} | \overline{A_{q+1}}, \ldots, \overline{A_m}]}{\Pr[\overline{A_2} \cap \ldots \cap \overline{A_q} | \overline{A_{q+1}}, \ldots, \overline{A_m}]}$$

Since A_1 and $\overline{A_{q+1}} \cap \ldots \cap \overline{A_m}$ are mutually independent (by hypothesis and Lemma 1), we have

$$\Pr[A_1 | \overline{A_{q+1}}, \ldots, \overline{A_m}] = \Pr[A_1] \leq \frac{1}{4d}, \quad \text{hence}$$

$$\Pr[A_1 \cap \overline{A_2} \cap \ldots \cap \overline{A_q} | \overline{A_{q+1}}, \ldots, \overline{A_m}] \leq \frac{1}{4d}.$$

Proof of the proposition, cont'd

On the other hand, for every events $B_1, \ldots, B_r$, we have

$$\Pr[\overline{B_1} \cap \ldots \cap \overline{B_r}] \geq 1 - \sum_{i=1}^{r} \Pr[B_i].$$

This gives

$$\Pr[\overline{A_2} \cap \ldots \cap \overline{A_q} | \overline{A_{q+1}}, \ldots, \overline{A_m}] \geq 1 - \sum_{i=2}^{q} \Pr[A_i | \overline{A_{q+1}}, \ldots, \overline{A_m}].$$

The induction hypothesis thus gives

$$\Pr[\overline{A_2} \cap \ldots \cap \overline{A_q} | \overline{A_{q+1}}, \ldots, \overline{A_m}] \geq 1 - \frac{q-1}{2d} \geq \frac{1}{2}.$$

We thus conclude that

$$\Pr[A_1 | \overline{A_2}, \ldots, \overline{A_m}] \leq \frac{1/4d}{1/2} = \frac{1}{2d},$$

completing the proof of the induction step and that of the proposition.

A stronger lower bound for Ramsey numbers

We use the result in the proposition to prove the following
Theorem 2. There is $c > 0$ such that $N(p, p; 2) \geq c \cdot p \cdot 2^{p/2}$ for $p \geq 2$.
Remark. Since c is not specified, this is really a result about $N(p, p; 2)$ for $p \gg 0$.

Proof of the theorem. Consider the complete graph K_n on $\{1, \ldots, n\}$, with $n \geq p$, and the probability space consisting of all possible colorings of the edges of this graph with red and blue.

Given $S \subseteq \{1, \ldots, n\}$ with $\#S = p$, let A_S be the set of such colorings such that the subgraph spanned by S is monochromatic.

We want to arrange, for suitable n, that we have such a coloring such that no K_p-subgraph is monochromatic, that is, $\bigcap_S \overline{A_S} \neq \emptyset$.

A stronger lower bound for Ramsey numbers, cont'd

We define a simple graph G on the set of p-subsets of $\{1, \ldots, n\}$ such that S and T are adjacent precisely when $\#(S \cap T) \geq 2$.

For a fixed p-subset S and a 2-subset S_0 of S, the number of p-subsets $T \neq S$ of $\{1, \ldots, n\}$ that contain S_0 is $\binom{n-2}{p-2} - 1$. It follows that $\deg(S) \leq \binom{p}{2} \cdot \binom{n-2}{p-2}$.
Note also that for every S, we have

$$\Pr[A_S] = \frac{2 \cdot 2^{\binom{n}{2} - \binom{p}{2}}}{2^{\binom{n}{2}}} = 2^{1 - \binom{p}{2}}.$$

Finally, G is a dependency graph for our set of events. In order to see this, we need to show that if $T, S_1, \ldots, S_r$ are distinct p-subsets of $\{1, \ldots, n\}$, with $\#(T \cap S_i) \leq 1$ for all i, then $\overline{A_T}$ and $\overline{A_{S_1}} \cap \ldots \cap \overline{A_{S_r}}$ are independent.

A stronger lower bound for Ramsey numbers, cont'd

Indeed, since there are no edges common to T and any of the S_i, the condition that T is not monochromatic is independent of the condition that neither of the S_i is monochromatic. Therefore G is a dependency graph.

We can thus apply the proposition to conclude that $\bigcap_S \overline{A_S} \neq \emptyset$ (and thus $N(p,p;2) > n$) as long as

$$4 \cdot 2^{1-\binom{p}{2}} \cdot \binom{p}{2} \cdot \binom{n-2}{p-2} < 1. \tag{2}$$

Since $\binom{n-2}{p-2} = \frac{(n-2)\cdots(n-p+1)}{(p-2)!} \leq \frac{n^{p-2}}{(p-2)!}$ for $p \geq 3$, in order to have (2), it is enough to have

$$\binom{p}{2} \cdot \frac{n^{p-2}}{(p-2)!} < 2^{\binom{p}{2}-3}. \tag{3}$$

A stronger lower bound for Ramsey numbers, cont'd

After taking $(p-2)$-roots, we see that (3) is equivalent to

$$n < \left((p-2)!\right)^{1/(p-2)} \cdot 2^{\frac{1}{p-2}\left(\frac{p(p-1)}{2}-3\right)} \cdot \binom{p}{2}^{-\frac{1}{p-2}}. \tag{4}$$

Recall that Stirling's formula says that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1, \quad \text{hence} \quad \lim_{n\to\infty} \frac{(n!)^{1/n}}{n/e} = 1.$$

This implies that

$$\lim_{p\to\infty} \frac{\left((p-2)!\right)^{1/(p-2)} \cdot 2^{\frac{1}{p-2}\left(\frac{p(p-1)}{2}-3\right)} \cdot \binom{p}{2}^{-\frac{1}{p-2}}}{\frac{p}{e} \cdot 2^{\frac{p+1}{2}}} = 1.$$

A stronger lower bound for Ramsey numbers, cont'd

We conclude that if $c < \frac{\sqrt{2}}{e}$, then for $p \gg 0$, we can choose n such that the bound (4) holds and $n \geq c \cdot p \cdot 2^{p/2}$. In other words, we have

$$N(p, p; 2) \geq c \cdot p \cdot 2^{p/2} \quad \text{for} \quad p \gg 0.$$