Lecture 5: Colorings of graphs

September 15, 2020

Proper colorings

In this lecture we will discuss various questions and results having to do with colorings of graphs. Fix a finite graph G.

Definition. A proper coloring of G is a function from $V(G)$ to a set C of "colors" such that the ends of every edge in G have different colors. If $\#C = k$, we say that G is k-colored.

Remark. Note that if G has a loop, then it has no proper colorings.
Convention. Because of the above remark, in this lecture we assume that all graphs have no loops. Also, since multiple edges do not impose any extra restriction, after possibly removing some edges, we may and will assume that G has no multiple edges. Therefore G is simple.

Definition. The chromatic number $\chi(G)$ of G is the minimal number k such that G can be k-colored.

First examples of chromatic numbers

Remark. Note that the chromatic number is well-defined. In fact, we have $\chi(G) \leq \#V(G)$.

Examples.
1) We have $\chi(G) = 1$ if and only if $E(G) = \emptyset$.
2) We have $\chi(G) \leq 2$ if and only if G is bipartite (recall that this means that there is a decomposition $V(G) = A \sqcup B$ such that every edge in $E(G)$ has one end in A and one end in B).
3) $\chi(K_n) = n$.
4) If $n \geq 2$, then $\chi(P_n) = 2$ if n is even and $\chi(P_n) = 3$ if n is odd.
5) If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Remark. It is an important result that for every simple graph G, the number of m-colorings of G is given by a polynomial function of m. We will prove this later in the course.

Further examples

What is the chromatic number of a tree on $n \geq 2$ vertices?

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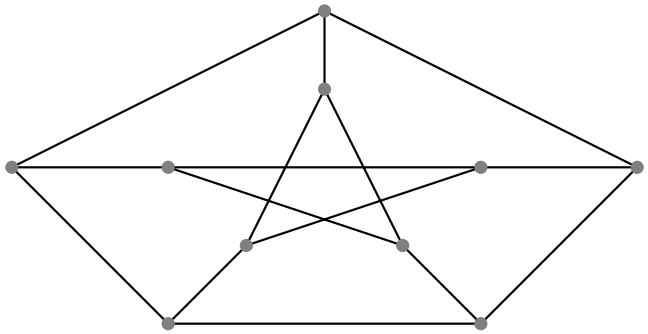
Answer: 2. Exercise: prove this!

Further examples, cont'd

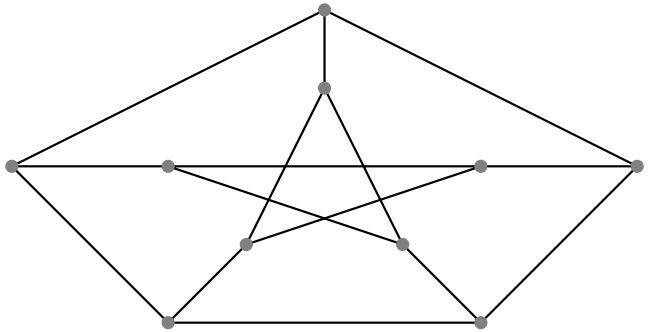
Consider the Petersen graph:



What is its chromatic number?

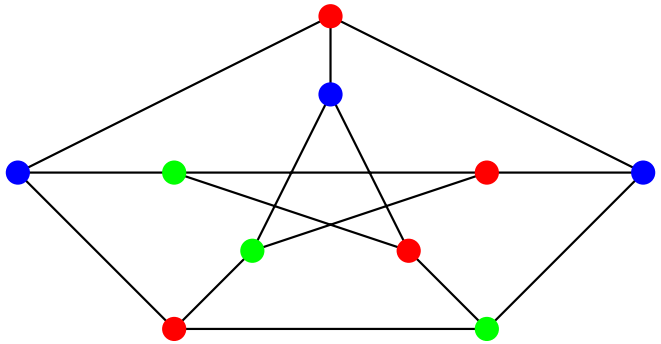
Further examples, cont'd

Consider the Petersen graph:



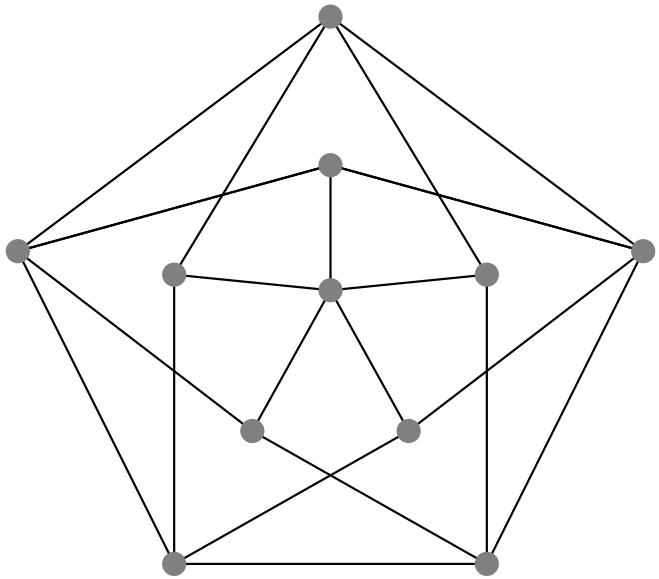
What is its chromatic number?

Answer: 3.



Further examples, cont'd

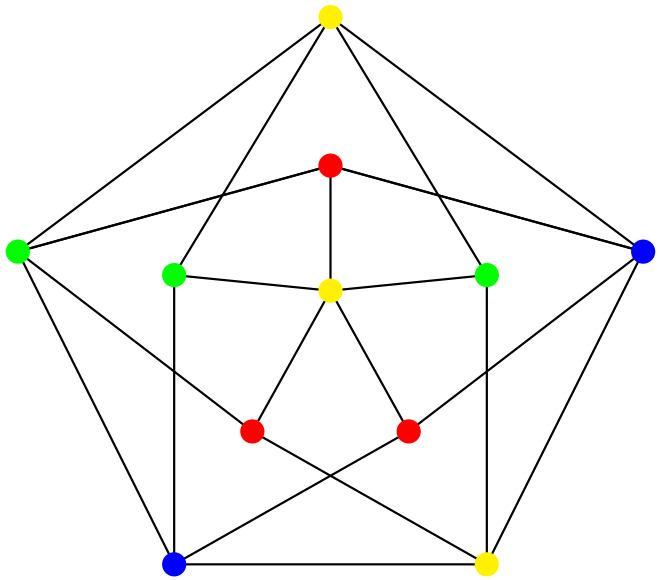
Consider the Grötzsch graph:



What is its chromatic number?

Further examples, cont'd

Answer: 4



Brooks' bound for the chromatic number

Theorem (Brooks). If $d \geq 3$ and G is a finite graph with no loops such that

1) K_{d+1} is not isomorphic with a subgraph of G, and
2) All vertices of G have degree $\leq d$,

then $\chi(G) \leq d$.

Remark 1. If condition 2) holds, then condition 1) fails if and only if G has a connected component that is isomorphic to K_{d+1}.

Remark 2. The assertion in the theorem can fail for $d = 2$: consider, for example, $G = P_5$.

As we will see, the proof of the theorem is quite intricate. However, there is a slightly weaker assertion that is easier to prove.

A bound in terms of the largest degree

For a finite graph G, let $\Delta(G) = \max_{x \in V(G)} \deg(x)$.
Remark. With this notation, for every G, we have

$$\chi(G) \leq \Delta(G) + 1.$$

This follows easily from Brooks' theorem (the case $\Delta(G) \leq 1$ being trivial), but it has a direct proof. Namely, let's order the vertices of G as $v_1, \ldots, v_n$. We color the vertices recursively with colors from $\{1, \ldots, N\}$, where $N = \Delta(G) + 1$.
At Step 1, we color v_1 with color 1. At Step i, we color v_i with the smallest j that has not been used to color the vertices v_m, with $m < i$, that are adjacent to v_i (this is well-defined since $N > \deg(v_i)$).

It is clear that after Step n, we have a proper coloring of G with $\leq N$ colors.

A bound in terms of the largest degree, cont'd

The bound in terms of $\Delta(G)$ can be achieved:
- If n is odd, then $\Delta(P_n) = 2$ and $\chi(G) = 3$.
- For every n, $\Delta(K_n) = n - 1$ and $\chi(K_n) = n$.

A consequence of Brooks' theorem: these are, essentially, the only cases when we have equality. More precisely:
Corollary. If G is a connected graph such that G is not isomorphic to some P_n, with n odd, or to some K_n, then $\chi(G) \leq \Delta(G)$.

Proof. Let $d = \Delta(G)$. Note that if $d = 0$, then $G \simeq K_1$ and if $d = 1$, then $G \simeq K_2$. Also: if $d = 2$, then $\chi(G) \leq 2$, unless $G \simeq P_n$, with n odd. Hence we may assume $d \geq 3$.
By Brooks' theorem and since G is connected, if $\chi(G) = d + 1$, then $G \simeq K_{d+1}$. This proves the corollary.

The proof of Brooks' theorem

Proof of Brooks' theorem. Suppose that the theorem fails and consider a counterexample G with a minimal number of vertices.
For a vertex x of G, consider the set $\Gamma(x)$ of vertices adjacent to x

$$\Gamma(x) = \{x_1, \ldots, x_\ell\}.$$

Note that $\ell \leq d$ by condition 2).

Let H be the graph obtained from G by removing the vertex x and removing all edges joining x with one of the x_i. It is clear that H still satisfies conditions 1) and 2). Since $\#V(H) < \#V(G)$, by the minimality condition in the choice of G, we see that H has a proper d-coloring, with colors $1, \ldots, d$. We may assume that all these colors are used in the coloring of $\Gamma(x)$: otherwise, we could use the left-over color to color x, leading to a d-coloring of G, a contradiction.

Since $\ell \leq d$, it follows that we have $\ell = d$. After relabeling the vertices, we may assume that x_i is colored with color i.

The proof of Brooks' theorem, cont'd

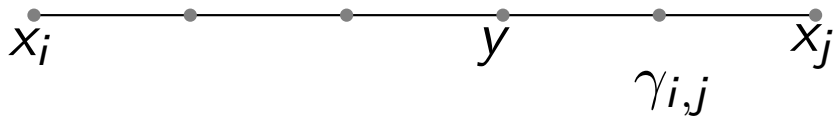
For $i \neq j$, let $H_{i,j}$ be the subgraph of H consisting of the vertices colored in either i or j (together with the edges in $E(G)$ between these vertices). If x_i and x_j lie in different connected components of $H_{i,j}$, then we can recolor the vertices in the connected component of x_i by switching the i and j colors; this will give a d-coloring of H such that x_i and x_j have the same color. As we have seen, this leads to a contradiction.

Therefore x_i and x_j lie in the same connected component of $H_{i,j}$ that we denote by $C_{i,j}$.

Since $\deg(x_i) \leq d$ by condition 2), it follows that x_i has at most $d-1$ neighbors in H. On the other hand, for every $j \neq i$, x_i has $\deg_{C_{i,j}}(x_i)$ neighbors of color j. Note that $\deg_{C_{i,j}}(x_i) \geq 1$ since $C_{i,j}$ is connected and contains at least one more vertex besides x_i (namely x_j). We thus conclude that for every $i \neq j$, x_i has precisely one neighbor in $C_{i,j}$ (and the same holds true, of course, for x_j).

The proof of Brooks' theorem, cont'd

Since $C_{i,j}$ is connected, we can find a path $\gamma_{i,j}$ from x_i to x_j. We may and will assume that $\gamma_{i,j}$ passes through each vertex only once.



Claim 1. Every vertex on this path different from x_i and x_j has degree 2 in $C_{i,j}$.

Arguing by contradiction, suppose that we have such a point y of degree ≥ 3 in $C_{i,j}$ and we chose the one that is closest to x_i.

Since y has 3 neighbors in $C_{i,j}$ (all of which are colored in the same color), it follows that the neighbors of y in H are covered in $\leq (d-2)$ colors. We can then recolor y with a color different from i and j to get a proper d-coloring for H. Of course, after our recoloring, y does not lie in $H_{i,j}$ anymore.

The proof of Brooks' theorem, cont'd

Since x_i has degree 1 in $H_{i,j}$ and the vertices on $\gamma_{i,j}$ strictly between x_i and y all have degree 2 in $H_{i,j}$, it follows that every walk from x_i to x_j in $H_{i,j}$, which passes through every vertex only once, has to agree with $\gamma_{i,j}$ between x_i and y. This implies that after recoloring, x_i and x_j lie in different connected components of $H_{i,j}$, a contradiction. This proves Claim 1.

Since $C_{i,j}$ is connected, we deduce from Claim 1 that all vertices in $C_{i,j}$ lie on $\gamma_{i,j}$. Moreover, $E(C_{i,j})$ is the set of edges in $\gamma_{i,j}$.

Claim 2. For every distinct i, j, and k, we have $C_{i,j} \cap C_{i,k} = \{x_i\}$.

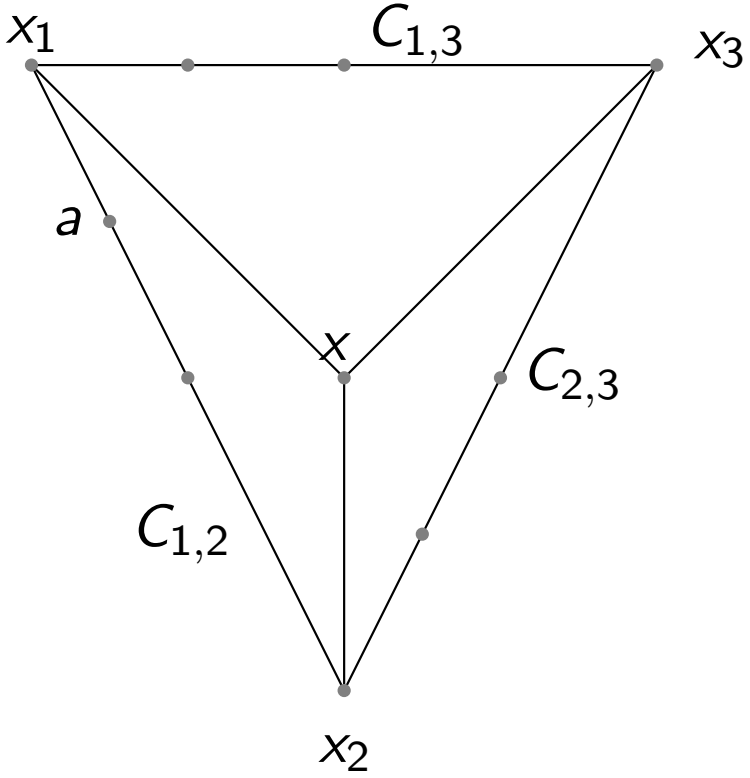
Indeed, suppose that $z \neq x_i$ lies in $C_{i,j} \cap C_{i,k}$. In this case z has color i; in particular, $z \neq x_j$ and $z \neq x_k$.

The proof of Brooks' theorem, cont'd

Therefore z has two neighbors in $C_{i,j}$ and 2 neighbors in $C_{i,k}$. The vertices in each of these two pairs are colored in the same color, hence the neighbors of z in H are colored in $\leq d-2$ colors. We may thus change the color of z to a color not in $\{i,j,k\}$ to get a proper coloring of H. Using what we know about the vertices and the edges of $C_{i,j}$, we conclude that with the new coloring, x_i and x_j do not lie in the same connected component of $H_{i,j}$, a contradiction. This proves Claim 2.

Finally, since G contains no subgraph isomorphic to K_{d+1}, it follows that two of $x_1, \ldots, x_d$ (say x_1 and x_2) are not adjacent. We then have the following picture:

The proof of Brooks' theorem, cont'd



Let a be the neighbor of x_1 in $C_{1,2}$ (different from x_2 by assumption); the color of a is 2. We switch colors 1 and 3 in $C_{1,3}$ and we get a proper d-coloring of H. Let $C'_{1,2}$ (resp. $C'_{2,3}$) be the new connected components containing x_2 and x_1 (resp. x_2 and x_3). Since a and x_1 are adjacent, it follows that $a \in C'_{1,2}$. On the other hand, the only vertex in $C_{1,2}$ that changed its color is x_1; using the path between a and x_2 in $C_{1,2}$, we see that $a \in C'_{2,3}$.

Since $a \in C'_{1,2} \cap C'_{2,3}$, this contradicts Claim 2. This completes the proof of the theorem.

The chromatic polynomial of a polygon

Let's take a break and discuss in groups the following problem:

We will prove later in the course that for every finite graph G, there is a polynomial Q_G (the *chromatic polynomial* of G) such that for every $m \geq 1$, the number of proper colorings of G with m colors is equal to $Q_G(m)$.

Show that if Q_n is the chromatic polynomial of the polygon P_n (where $n \geq 1$), then

$$Q_n(x) = (x-1)^n + (-1)^n(x-1).$$

Kneser's example

We now discuss of interesting example of a coloring problem, due to Kneser. Let's first introduce the relevant graphs.

For a positive integer n, we write $[n] = \{1, \ldots, n\}$. For $1 \leq k \leq n$, let

$$\binom{[n]}{k} := \{A \subseteq [n] \mid \#A = k\}.$$

This is a set with $\binom{n}{k}$ elements.

For every n, k as above, let $KG_{n,k}$ be the graph with vertex set $\binom{[n]}{k}$, in which A and B are joined by an edge if and only if $A \cap B = \emptyset$.

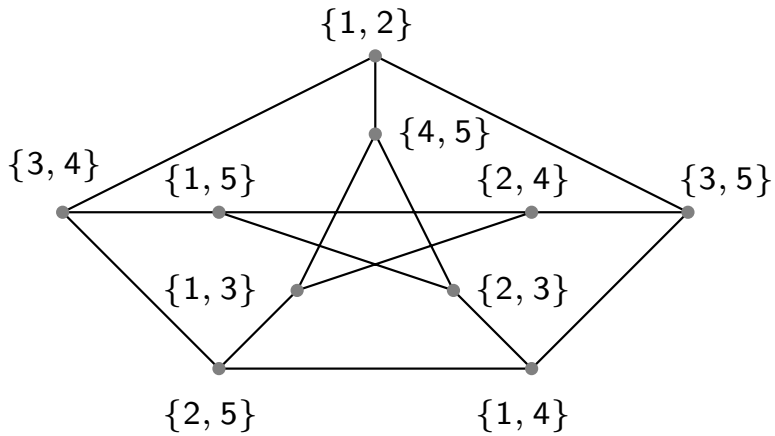
Kneser's example, cont'd

Examples of graphs $KG_{n,k}$:

1) If $k > \frac{n}{2}$, then $KG_{n,k}$ has no edges (not interesting). From now on, we always assume $2k \leq n$.

2) If $2k = n$, then $KG_{n,k}$ has the property that for every vertex x, there is a unique other vertex adjacent to it. In particular, we have $\chi(KG_{2k,k}) = 2$.

3) $KG_{5,2}$ is the Petersen graph:



Kneser's bound for $\chi(KG_{n,k})$

Proposition (Kneser). For every $n \geq 1$ and every k, with $1 \leq k \leq \frac{n}{2}$, we have

$$\chi(KG_{n,k}) \leq n - 2k + 2.$$

Proof. We define $c\colon \binom{[n]}{k} \to [n-2k+2]$ by

$$c(A) = \min\left\{ \min\{x \mid x \in A\}, n - 2k + 2\right\}.$$

Suppose that $A, B \in \binom{[n]}{k}$ are such that $A \cap B = \emptyset$, but $c(A) = c(B)$. If this common value is $\leq n - 2k + 1$, then there is $x \in A \cap B$, a contradiction.

Therefore the common value must be $\geq n - 2k + 2$. In this case, both A and B are contained in $\{n - 2k + 2, \ldots, n\}$, a set with $2k - 1$ elements. Since both A and B have k elements, we again get $A \cap B \neq \emptyset$, a contradiction. Hence c is a proper coloring.

Kneser's conjecture for $\chi(KG_{n,k})$

Conjecture (Kneser). For every $n \geq 1$ and every k, with $0 \leq k \leq \frac{n}{2}$, we have

$$\chi(KG_{n,k}) = n - 2k + 2.$$

This is now a theorem, with several known proofs. We give a proof due to Greene (after Baranyi also gave a similar one).

The proof will make use of the following result in topology.
Theorem (Borsuk-Ulam). If $f : S^n \to \mathbf{R}^n$ is a continuous function, then there is $x \in S^n$ such that $f(x) = f(-x)$.

Here $S^n \subseteq \mathbf{R}^n$ is the n-dimensional sphere

$$S^n = \left\{(a_0, \ldots, a_n) \in \mathbf{R}^{n+1} \mid a_0^2 + \ldots + a_n^2 = 1\right\}.$$

A consequence of Borsuk-Ulam

We will make use of the following consequence of Borsuk-Ulam:
Corollary. If $A_0, \ldots, A_n \subseteq S^n$ are open or closed subsets such that $S^n = A_0 \cup \ldots \cup A_n$, then there is $x \in S^n$ and $0 \leq i \leq n$ such that $x, -x \in A_i$.
Proof. We only give the proof when all A_i are closed. Consider the continuous function

$$f : S^n \to \mathbf{R}^n, \ f(x) = \big(d(x, A_1), \ldots, d(x, A_n)\big)$$

It follows from the Borsuk-Ulam theorem that there is $x \in S^n$ such that $f(x) = f(-x)$.
Note that since the A_i are closed, we have $d(y, A_i) = 0$ iff $y \in A_i$.
If there is i, with $1 \leq i \leq n$, such that $x \in A_i$ or $-x \in A_i$, then then by above the other condition also holds. If this is not the case, since $X = A_0 \cup \ldots \cup A_n$, we have $x, -x \in A_0$. This completes the proof.

Proof of Kneser's conjecture

Greene's proof of Kneser's conjecture:
Suppose n and k with $2k \leq n$ are such that $\chi(KG_{n,k}) \leq n - 2k + 1 =: d$ and we have a proper coloring $c\colon \binom{[n]}{k} \to [d]$.

Consider the sphere $S^d \subseteq \mathbf{R}^{d+1}$ and choose n points on S^d such that any $d+1$ of these are linearly independent (note that $n \geq d+1$). We will identify $[n]$ with the set X of these points on S^d.

For $1 \leq i \leq d$, let $U_i \subseteq S^d$ consist of those $x \in S^d$ for which there is a k-set $S \subseteq X$ such that $c(S) = i$ and S is contained in the half-space $\{y \mid \langle x, y\rangle > 0\}$. Note that U_i is open in S^d.

We also put $A = S^d \setminus (U_1 \cup \ldots \cup U_d)$, which is closed in S^d. Applying the consequence of Borsuk-Ulam to $A, U_1, \ldots, U_n$, we obtain $x \in S^d$ such that

1) Either there is i, with $1 \leq i \leq d$, such that $x, -x \in U_i$,
2) Or $x, -x \in A$.

Proof of Kneser's conjecture, cont'd

Case 1. We have $x, -x \in U_i$. In this case, we have k-subsets $S_1, S_2 \subseteq X$, with $c(S_1) = c(S_2)$ and

$$S_1 \subseteq \{y \mid \langle x, y\rangle > 0\} \quad \text{while} \quad S_2 \subseteq \{y \mid \langle x, y\rangle < 0\}.$$

This implies $S_1 \cap S_2 = \emptyset$, contradicting the fact that c gives a proper coloring.

Case 2. We have $x, -x \in A$. In this case, for every k-subset $S \subseteq X$, there are $w, w' \in S$ such that $\langle x, w\rangle \leq 0$ and $\langle x, w'\rangle \geq 0$.

This implies that

$$\#\big(X \cap \{y \mid \langle x, y\rangle > 0\}\big) \leq k-1 \quad \text{and} \quad \#\big(X \cap \{y \mid \langle x, y\rangle > 0\}\big) \leq k-1.$$

This implies that

$$\#\big(X \cap \{y \mid \langle x, y\rangle = 0\}\big) \geq n - 2k + 2 = d + 1,$$

contradicting the fact that any $d+1$ elements of X are linearly independent. This completes the proof of the theorem.

Further connections to algebraic topology

In fact, the first proof of Kneser's theorem was due to Lovász. His proof was very interesting since he made a more general connection to algebraic topology.

More precisely, Lovász associated to any finite simple graph G a simplicial complex $\mathcal{N}_G$. Recall that a simplicial complex Δ on a finite set S consists of a collection of subsets of S with the property that if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$.

Given a finite simple graph G, the neighborhood complex $\mathcal{N}_G$ of G is the simplicial complex on the set $V(G)$ such that a subset $A \subseteq V(G)$ is in $\mathcal{N}_G$ if there is $b \in V(G)$ that is a neighbor of all elements of A.

Why is this useful: attached to a simplicial complex Δ there is a topological space $|\Delta|$. Various topological invariants (such as homology or homotopy groups) provide interesting invariants for the simplicial complex. In the case of a graph G, one tries to use such invariants for $|\mathcal{N}_G|$ to get information about G.

Lovász's approach to Kneser's conjecture

Recall that a nonempty, path connected topological space X is k-connected if $\pi_i(X) = 0$ for $1 \leq i \leq k$ (convention: (-1)-connected means "nonempty" and 0-connected means "path connected").

Theorem (Lovász). If G is a finite simple graph such that $|\mathcal{N}_G|$ is k-connected, then

$$\chi(G) \geq k + 3.$$

Example. Consider the case when $G = K_n$. In this case $\mathcal{N}_G$ is the simplicial complex on $\{1, \ldots, n\}$ such that $A \subseteq \{1, \ldots, n\}$ lies in $\mathcal{N}_G$ iff $A \neq \{1, \ldots, n\}$.
Therefore $|\mathcal{N}_G|$ is homeomorphic to the sphere S^{n-2}. It is known that S^{n-2} is k-connected iff $k \leq n-3$. Since $\chi(K_n) = (n-3)+3$, we see that we have equality in the bound in the theorem.

Lovász's approach to Kneser's conjecture, cont'd

Lovász also showed: if $2k \leq n$, then $|\mathcal{N}_{KG_{n,k}}|$ is homotopically equivalent to a bouquet of S^{n-2k}.

In particular, this implies that $|\mathcal{N}_{KG_{n,k}}|$ is $(n-2k-1)$-connected. The theorem then implies

$$\chi(KG_{n,k}) \geq n - 2k + 2.$$

For details and further connections between graph theory and combinatorics, see
Mark de Longueville, A course in topological combinatorics.