Math 565. Introduction to Combinatorics
Lecture 1: Generalities about graphs

September 1, 2020

Outline of the course

Main goal of the course: give an introduction to some important concepts in combinatorics. The course will consist roughly of two parts:
1. Introduction to graph theory (the first half of the semester).
- Basic concepts about graphs
- Trees
- Hall's marriage problem
- Introduction to Ramsey theory
- Planar graphs
- Spectral graph theory
- Extremal graph theory

2. Lattices, combinatorial geometries, and matroids (the rest of the semester). We will discuss the properties of these of objects and the connections between them. We will devote some time also to the combinatorial study of arrangements of hyperplanes.

Outline of the course, cont'd

Prerequisites: familiarity with linear algebra, especially with vector spaces over an arbitrary field (almost always, this will be $\mathbf{R}$ or $\mathbf{C}$) and eigenvalues of matrices. We will occasionally encounter other objects from abstract algebra (groups, rings, etc).

Main reference: J. H. van Lint and R. M. Wilson, A Course in Combinatorics, Cambridge Univ. Press. For some topics not covered in the book, I will suggest other sources, when the time comes.

There will be one midterm in the middle of the semester and one final at the end. I will also give a number of homework assignments (probably 8 or 9). The grade will also include a class participation component. For more details about the grade, see the syllabus.

Definition of a graph

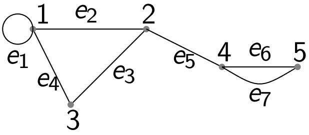
Definition. A graph G consists of the following data:
• A set $V = V(G)$ of vertices.
• A set $E = E(G)$ of edges.
• A map from E to the set of unordered pairs of vertices, which assigns to $e \in E$ its endpoints (or ends, for short).
We will say that an edge e is incident with its ends, or joins its ends.

Graphs have many applications in various areas of mathematics, but also in other fields, such as:
• Computer science (where they represent, for example, networks of communication)
• Linguistics (since various structures can be studied by graph-theoretic notions)
• Chemistry (where graphs can be used to model molecules).

Definition of a graph, cont'd

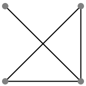
Graphs are useful when we are interested in some basic objects and in whether these are related (but not in the nature of the relation).

We represent a graph as follows:



This is a graph with $V = \{1, 2, 3, 4, 5\}$ and $E = \{e_1, \dots, e_7\}$ such that e_6 and e_7 are incident to 4 and 5 etc.

or



Note: the lines joining two vertices are allowed to intersect

Definition of a graph, cont'd

Note: we allow the ends of an edge to be equal. Such an edge is called a loop.

We will often deal with *simple* graphs: these are graphs G such that

1) G has no loops
2) There is at most one edge that is incident to any given pair of vertices.

Notation. If G is a graph that has at most one edge incident to any given pair of vertices, we sometimes use the notation $\{x, y\}$ for the edge joining x and y (and write $\{x, y\} \in E(G)$ to mean that there is such an edge in the graph).

A vertex in a graph is isolated if it is incident to no edges.

We will almost exclusively consider finite graphs: these are graphs such that both the set of vertices and the set of edges are finite.

A bit of history

• First appearance in Euler's work (1735) on the Königsberg problem (which we will discuss in the next lecture), that later led to the concept of Eulerian graph.

• Work of Kirkman (1856) investigating the polyhedra for which one can find a cycle passing through all the vertices just once. Also related work by Hamilton in the case of the icosahedron, motivated by his work in noncommutative algebra. This led to the notion of Hamiltonian graph.

• The concept of a tree appeared implicitly in the work of Kirchhoff (1824–1887) on electrical networks. Later, trees appeared in the work of Cayley (1821–1895) and Sylvester (1806–1897) in connection with the enumeration of certain chemical molecules. In particular, the term graph was introduced by Sylvester in a paper published in 1878 in Nature.

A bit of history, cont'd

One of the most famous (and influential) problems in graph theory is the Four Color problem: any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors.

The problem was posed by Guthrie in 1852 and had a lot of wrong proofs, including some by famous mathematicians like Cayley, Kempe, etc. It led to a lot of developments in graph theory.

A solution that relied heavily on computer calculations was given in 1976 by Appel and Haken, building on ideas of Heesch. Due to the complexity of the proof, which required checking the properties of 1,936 configurations by computer, there was some debate whether this was indeed a mathematical proof.

Degree

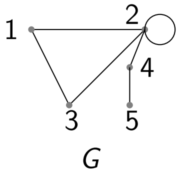
Definition. If G is a graph and x is a vertex of G, then the degree or valency $\deg(x)$ is the number of edges incident to x.

Note: a loop at x contributes 2 to $\deg(x)$.

Definition. If G is a graph and x is a vertex of G, then the set of vertices adjacent to x (also called neighbors of x) is the set

$$\Gamma(x) = \{y \in V(G) \mid \text{there is } e \in E(G) \text{ with ends } x \text{ and } y\}.$$

Example. Consider the graph:



We have $\deg(2) = 5$ and $\Gamma(2) = \{1, 2, 3, 4\}$

An exercise

Let's take a break from all the definitions and let's discuss the following

Exercise. Show that if G is a finite simple graph with at least two vertices, then there are two distinct vertices in G with the same degree.

The sum of degrees

Proposition. For every finite graph G, we have

$$\sum_{x \in V(G)} \deg(x) = 2 \cdot \# E(G).$$

Proof. Let's consider all edges $e \in E(G)$. Since every edge has two ends, if we count all ends, of all edges, we get $2 \cdot \# E(G)$. On the other hand, every $x \in V(G)$ is counted precisely $\deg(x)$ times. This gives the equality in the proposition.

Corollary. If G is a finite graph, then G has an even number of vertices with odd degree.

Proof. It follows from the proposition that $\sum_{x \in V(G)} \deg(x)$ is even, hence

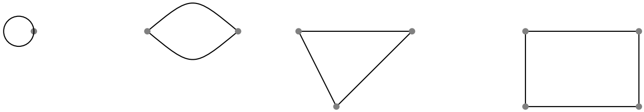
$$\sum_{x \in V(G), \deg(x) \text{ odd}} \deg(x)$$

is even. Hence there is an even number of vertices with odd degree.

First examples

1) The complete graph K_n on n vertices is the simple graph with n vertices and $\binom{n}{2}$ edges.

2) The polygon with n vertices P_n is the graph with vertices $x_1, x_2, \ldots, x_n$ and edges joining x_1 and x_2, x_2 and x_3, ..., x_{n-1} and x_n, and x_n and x_1. Some cases of small n:



P_1 $\qquad$ P_2 $\qquad$ P_3 $\qquad$ P_4

3) A simple graph G is bipartite if $V(G)$ can be written as a disjoint union $A \sqcup B$ such that every edge $e \in E(G)$ has one end in A and the other in B. Also say: $G[A, B]$ is a bipartite graph.

A proposition about bipartite graphs

Proposition. If $G[A, B]$ is a bipartite graph without isolated vertices such that $\deg(x) \geq \deg(y)$ for every $x \in A$ and $y \in B$ that are joined by an edge, then $\#A \leq \#B$, with equality if and only if $\deg(x) = \deg(y)$ whenever x and y are joined by an edge.

Proof. We rely again on changing the summation order:

$$\#A = \sum_{x \in A} \sum_{\substack{y \in B \\ \{x,y\} \in E(G)}} \frac{1}{\deg(x)} = \sum_{\substack{x \in A \\ y \in B}} \sum_{\{x,y\} \in E(G)} \frac{1}{\deg(x)}$$

$$\leq \sum_{\substack{x \in A \\ y \in B}} \sum_{\{x,y\} \in E(G)} \frac{1}{\deg(y)} = \sum_{y \in B} \sum_{\substack{x \in A \\ \{x,y\} \in E(G)}} \frac{1}{\deg(y)} = \#B.$$

Moreover, we see that $\#A = \#B$ if and only if we have $\deg(x) = \deg(y)$ for every $\{x, y\} \in E(G)$.

Isomorphisms of graphs

Recall that in many fields of mathematics, we want to identify certain objects that are isomorphic. We now introduce the corresponding notion for graphs.

Definition. Given two graphs G_1 and G_2, an isomorphism from G_1 to G_2 is given by two bijective maps $\varphi\colon V(G_1) \to V(G_2)$ and $\psi\colon E(G_1) \to E(G_2)$ such that e joins a and b in G_1 if and only if $\psi(e)$ joins $\varphi(a)$ and $\varphi(b)$ in G_2.
We say that G_1 is isomorphic to G_2 if there is an isomorphism as above from G_1 to G_2.

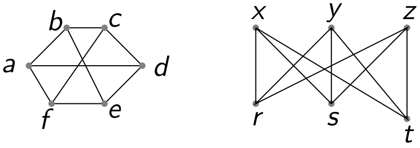
Remark. It is straightforward to see that if φ and ψ are as in the definition of an isomorphism, then φ^{-1} and ψ^{-1} satisfy the same condition. Moreover, if (φ_1, ψ_1) give an isomorphism from G_1 to G_2 and (φ_2, ψ_2) give an isomorphism from G_2 to G_3, then $(\varphi_2 \circ \varphi_1, \psi_2 \circ \psi_1)$ give an isomorphism from G_1 to G_3. Using these two properties, it follows that the relation of isomorphism is an equivalence relation.

Isomorphisms of graphs, cont'd

Remark. If G_1 and G_2 are simple graphs, then an isomorphism from G_1 to G_2 is uniquely determined by the corresponding bijection $\varphi\colon V(G_1) \to V(G_2)$. This is required to satisfy the condition that $x, y \in V(G_1)$ are joined by an edge in G_1 if and only if $\varphi(x), \varphi(x_2) \in V(G_2)$ are joined by an edge in G_2.

Definition. Automorphism of a graph G: an isomorphism from G to itself.

Example. The following simple graphs are isomorphic



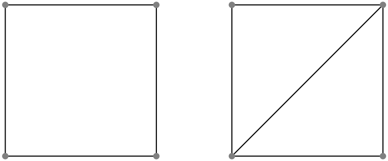
with the isomorphism given by $\varphi\colon \{a, b, c, d, e, f\} \to \{x, y, z, r, s, t\}$, $\varphi(a) = x$, $\varphi(c) = y$, $\varphi(e) = z$, $\varphi(b) = r$, $\varphi(d) = y$, $\varphi(f) = t$.

Preservation of properties

The idea is that a isomorphism between the graphs G_1 and G_2 given by $\varphi\colon V(G_1) \to V(G_2)$ and $\psi\colon E(G_1) \to E(G_2)$ allows us to identify properties and invariants of G_1 and G_2.

Examples.
1) $\deg(x) = \deg\big(\varphi(x)\big)$ for every $x \in V(G_1)$.
2) G_1 is simple if and only if G_2 is simple.

By considering automorphisms, we can exploit the symmetry of a graph. Two vertices of a graph are similar if there is an automorphism of the graph that maps one point to the other one.



For example, in the graph on the left all vertices are similar, while in the graph on the right, this is not the case.

Subgraphs

As in other fields of mathematics, in graph theory we also have a notion of "subobject".

Definition. Given a graph G, a subgraph H of G is a graph H such that

i) $V(H) \subseteq V(G)$.
ii) $E(H) \subseteq E(G)$.
iii) If $e \in E(H)$, then its ends in H are the same as its ends in G.

Example. Given any simple graph H on a set of n vertices, we can consider H as the subgraph of a graph isomorphic to K_n.

Definition. A subgraph H of G is a spanning subgraph if $V(H) = V(G)$.

Subgraphs, cont'd

Given a graph G and a subset S of $V(G)$, then the subgraph spanned by S is the subgraph of G whose

- set of vertices is S and
- set of edges consists of those edges in G that join vertices in S.

Another exercise

The girth of a graph G is the smallest n such that G contains a subgraph isomorphic to P_n (and it is infinite if no such n exists).

Exercise. Show that if G is a graph of girth ≥ 5 and d is such that every vertex of G has degree $\geq d$, then

$$\#V(G) \geq d^2 + 1.$$

Hint: Consider the set

$$F = \{(y, z) \in V(G) \times V(G) \mid y \neq z \text{ and } y, z \text{ have a common neighbor}\}.$$

Since we didn't have time to discuss this in class, I will put it on the first homework assignment.

Walks

Definition. A walk in a graph G is given by a sequence

$$x_0, e_1, x_1, e_2, \ldots, e_n, x_n$$

such that

1) Each x_i is a vertex of G and
2) Each e_i is an edge of G that joins x_i and x_{i-1}.

The length of a walk as above is n. We sometimes say that the above is an $x_0 - x_n$ walk.

Remark. If the graph G has no multiple edges between any two vertices, then a walk is determined by the corresponding sequence of vertices.

Walks, cont'd

Further definitions: consider a walk $x_0, e_1, x_1, e_2, \ldots, e_n, x_n$.

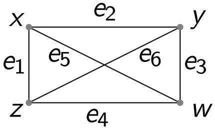
1) This is a path if the edges $e_1, \ldots, e_n$ are all distinct.

2) This is closed if $x_0 = x_n$.

3) This is a simple path if all vertices $x_0, x_1, \ldots, x_n$ are distinct (note that this is automatically a path).

4) This is a simple closed path if $x_0 = x_n$, but all $x_0, x_1, \ldots, x_{n-1}$ are distinct. By convention, we also require $n \geq 3$ (in which case, this is automatically a path).

Example. In the complete graph K_4



we have a walk of length 4 x, y, z, x, w and a simple closed path of length 4 x, y, z, w, x

Connectedness

Remarks. 1) If $x_0, e_1, x_1, \ldots, e_n, x_n$ is a walk, then $x_n, e_n, x_{n-1}, \ldots, e_1, x_0$ is a walk too.

2) If we have two walks

$$x_0, e_1, \ldots, e_n, x_n \quad \text{and} \quad x_n, e_{n+1}, x_{n+1}, \ldots, e_{n+m}, x_{n+m}$$

of lengths n and m, respectively, then by concatenation we get the walk

$$x_0, e_1, \ldots, e_n, x_n, e_{n+1}, x_{n+1}, \ldots, e_{n+m}, x_{n+m}$$

of length $m + n$.

It follows that if for $x, y \in V(G)$ we put $x \sim y$ if and only if there is a walk $x - y$, then this is an equivalence relation (note: $x \sim x$ since we allow length 0 walks).

Connectedness, cont'd

Definition. The subgraphs of G spanned by the equivalence classes of $V(G)$ with respect to this equivalence relation are called the connected components of G. The graph G is connected if $V(G)$ has only one connected component.

Convention. The empty graph (for which $V(G) = \emptyset$ and $E(G) = \emptyset$) is not connected.

Remark. If the G_i, for $i \in I$, are the connected components of G, then $E(G)$ is the disjoint union of the $E(G_i)$:

$$E(G) = \bigsqcup_{i \in I} E(G_i).$$

This follows from the fact that the ends of any edge in G lie in the same connected component of G.