Serre’s normality criterion

This write-up supplements the characterization of normal domains that we gave in class, now that we also discussed the notion of depth. The general characterization of normal rings is the content of a criterion due to Serre. We first introduce Serre’s conditions \((R_i)\) and \((S_i)\) and then prove the normality criterion.

1. Serre’s conditions

**Definition 1.1.** Given a Noetherian ring \(R\), we say that \(R\) satisfies Serre’s condition \((R_i)\) if for every prime ideal \(p\) in \(R\), with \(\text{codim}(R_p) \leq i\), the local ring \(R_p\) is regular.

**Example 1.2.** If \(X\) is an affine variety and \(A = \mathcal{O}(X)\), then \(A\) satisfies property \((R_i)\) if and only if \(\text{codim}_X(X_{\text{sing}}) \geq i + 1\).

**Definition 1.3.** We say that a Noetherian ring \(R\) satisfies Serre’s condition \((S_i)\) if for every prime ideal \(p\) in \(R\), we have \(\text{depth}(R_p) \geq \min\{\dim(R_p), i\}\).

**Example 1.4.** A Noetherian ring \(R\) satisfies \((S_1)\) if and only if every associated prime of \(R\) is minimal. It satisfies both \((R_0)\) and \((S_1)\) if and only if for every associated prime \(p\) of \(R\), we have \(pR_p = 0\). It is clear that this holds if \(R\) is reduced. The converse also holds: if \(0 = q_1 \cap \ldots \cap q_r\) is a minimal primary decomposition, then conditions \((R_0)\) and \((S_1)\) imply that if \(p_i = \text{rad}(q_i)\), then each \(p_i\) is a minimal prime ideal and \(q_i R_{p_i} \subseteq p_i R_{p_i} = 0\); since \(q_i\) is \(p_i\)-primary, it follows that \(q_i = p_i\) for all \(i\), hence \(R\) is reduced.

2. The normality criterion

As in the geometric setting, we say that an arbitrary Noetherian ring \(R\) is normal if \(R_p\) is an integrally closed domain for every prime ideal \(p\) in \(R\) (or, equivalently, for every maximal ideal \(p\) in \(R\)).

**Remark 2.1.** We note that a normal ring is isomorphic to a product of normal domains. Indeed, if \(R\) is normal and \(p_1, \ldots, p_r\) are the minimal prime ideals of \(R\), then \(p_i + p_j = R\) for every \(i \neq j\) (this is due to the fact that \(R_p\) is a domain for every maximal ideal \(p\) in \(R\)). Moreover, since al localizations of \(R\) are reduced, it follows that \(R\) is reduced, hence \(p_1 \cap \ldots \cap p_r = 0\). We thus conclude from the Chinese Remainder theorem that the canonical morphism

\[ R \to R/p_1 \times \ldots \times R/p_r \]

is an isomorphism. Furthermore, for every prime ideal \(q\) containing \(p_i\), the localization \(R_q\) is a normal domain, hence \((R/p_i)_q = R_q\) is normal. We thus deduce that each \(R/p_i\) is a normal domain.

**Theorem 2.2** (Serre). A Noetherian ring \(R\) is normal if and only if it satisfies conditions \((R_1)\) and \((S_2)\).
Proof. After localizing, we may assume that \((R, \mathfrak{m})\) is a local ring. It is straightforward to see that if \(R\) is a domain, then having \((R_1) + (S_2)\) is just a reformulation of conditions i) + ii) in Proposition E.5.1 in the notes. In particular, the “only if” assertion in the theorem is clear. For the “if” part, the subtlety is that we don’t know \textit{a priori} that \(R\) is a domain.

Suppose now that \(R\) satisfies conditions \((R_1)\) and \((S_2)\). In particular, it satisfies \((R_0) + (S_1)\), and thus \(R\) is reduced by Example 1.4. Let \(p_1, \ldots, p_r\) be the minimal prime ideals of \(R\), and let \(S = R \setminus \bigcup_{i=1}^r p_i\) be the set of non-zero-divisors in \(R\). Consider the inclusion map \(\phi: R \hookrightarrow K = S^{-1}R\). The Chinese Remainder theorem gives an isomorphism \(K \cong \prod_{i=1}^r K_i\), where \(K_i = \text{Frac}(R/p_i) = R_{p_i}\). If we can show that \(r = 1\), then \(R\) is a domain, in which case we are done. We follow the proof of Proposition E.5.1 in the notes to show that \(R\) is integrally closed in \(K\). If we know this, and \(e_i \in K\) is the idempotent corresponding to \(1 \in K_i\), then \(e_i^2 = e_i\) implies that \(e_i\) lies in \(R\). Since \(R\) is local, the only idempotents it has are 0 and 1, and these are mapped by \(\phi\) to 0 and 1, respectively, in \(K\). We thus see that \(r = 1\).

Suppose that \(\frac{b}{a} \in K\) is a non-zero element that is integral over \(R\) (note that \(a\) is a non-zero-divisor). Consider a minimal primary decomposition

\[(a) = q_1 \cap \ldots \cap q_s.\]

If \(\bar{q}_i = \text{rad}(q_i)\), then \(\bar{q}_i \in \text{Ass}(R/(a))\) by Remark E.3.13 in the notes. Condition \((S_2)\) implies that \(\text{codim}(\bar{q}_j) = 1\), and condition \((R_1)\) implies that \(R_{\bar{q}_j}\) is a DVR. Let \(j\) be fixed and consider \(i\) such that \(p_i \subseteq \bar{q}_j\). Since \(\frac{b}{a}\) is integral over \(R\), its image in \(K_i\) is integral over \(R\), and since \(R_{\bar{q}_j} \subseteq K_i\) is a DVR, hence integrally closed, we conclude that there is \(s \in R \setminus \bar{q}_j\) such that \(sb \in (a)\). Since \(q_j\) is a primary ideal, it follows that \(b \in q_j\). Since this holds for every \(j\), we conclude that \(b \in (a)\) and thus \(\frac{b}{a} \in R\). This completes the proof of the theorem. 

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