Problem Set 2

Problem 1. Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideals on the smooth variety X, then we have

$$\frac{1}{\mathrm{lct}(\mathfrak{a}\mathfrak{b})} \leq \frac{1}{\mathrm{lct}(\mathfrak{a})} + \frac{1}{\mathrm{lct}(\mathfrak{b})}$$

and for every $P \in X$, a similar inequality involving the log canonical thresholds at P.

Problem 2. Show that for every nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}_1, \ldots, b_s$ on the smooth variety X, and every $\lambda, \mu_1, \ldots, \mu_s \in \mathbb{R}_{\geq 0}$, we have

$$\mathcal{J}\big((\mathfrak{a}_1 + \ldots + \mathfrak{a}_r)^\lambda \mathfrak{b}_1^{\mu_1} \ldots \mathfrak{b}_s^{\mu_s}\big) = \sum_{\alpha_1+\ldots+\alpha_r=\lambda} \mathcal{J}(\mathfrak{a}_1^{\alpha_1} \ldots \mathfrak{a}_r^{\alpha_r} \mathfrak{b}_1^{\mu_1} \ldots \mathfrak{b}_s^{\mu_s}).$$

Problem 3. Let P be a point on the smooth n-dimensional variety X defined by the ideal $\mathfrak{m}_P$. Show that if $\mathfrak{a}$ is a nonzero ideal on X with $\mathrm{ord}_P(\mathfrak{a}) = d \geq 1$, then for every $r > d$, we have

$$\mathrm{lct}_P(\mathfrak{a} + \mathfrak{m}_P^r) \leq \frac{n + \mathrm{lct}_P(\mathfrak{a}) \cdot (r - d)}{r}.$$

Problem 4. Let $\mathfrak{a}$ be a nonzero ideal on the smooth variety X. Show that for every $\lambda \in \mathbb{R}_{\geq 0}$, we have

$$\frac{1}{\mathrm{lct}\left(\mathcal{J}(\mathfrak{a}^\lambda)\right)} \geq \frac{\lambda}{\mathrm{lct}(\mathfrak{a})} - 1,$$

with the convention that the quotient is 0 if the log canonical threshold at the denominator is infinite.

Problem 5. Let $\mathfrak{a}$ be a nonzero ideal on the normal variety X such that all associated subvarieties of $\mathfrak{a}$ have codimension $\leq r$ in X.

i) Show that around the generic point of each associated subvariety of $\mathfrak{a}$, there is a reduction of $\mathfrak{a}$ generated by r elements.
ii) Deduce that $\mathcal{J}(\mathfrak{a}^r) \subseteq \mathfrak{a}$.

Problem 6. Let $X = M_{m,n}(\mathbf{C})$, with $m \leq n$, and let $\mathfrak{a} \subseteq \mathbf{C}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ be the ideal generated by the maximal minors of the $m \times n$ matrix of indeterminates $(x_{i,j})$. Show that $\mathrm{lct}(\mathfrak{a}) = n - m + 1$.

Problem 7. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $g \in \mathbb{C}[y_1, \ldots, y_m]$ be nonzero, with $f(0) = 0 = g(0)$. Show that if $h(x, y) = f(x) + g(y)$, then we have the following formula due to Thom and Sebastiani:

$$\mathrm{lct}_0(h) = \min\{\mathrm{lct}_0(f) + \mathrm{lct}_0(g), 1\}.$$