Problem Set 1

Problem 1. Let $g\colon Z \to Y$ and $f\colon Y \to X$ be two proper birational morphisms between normal varieties.

i) Show that a prime divisor E on Z is $(f \circ g)$-exceptional if and only if either E is g-exceptional or it is the strict transform of an f-exceptional prime divisor on Y.
ii) Show that $\mathrm{Exc}(f \circ g) = \mathrm{Exc}(g) \cup g^{-1}\big(\mathrm{Exc}(f)\big)$.

Problem 2. Let X be a smooth variety and let Z be a smooth subvariety of X. Let $f\colon Y \to X$ be the blow-up along Z, with exceptional divisor E. Show that if D is an effective divisor on X and $\mathrm{ord}_Z(D) = q$, then $f^*(D) = \widetilde{D} + qE$, where if $D = \sum_i a_i D_i$, we put $\widetilde{D} = \sum_i a_i \widetilde{D_i}$.

Problem 3. Let X be a smooth variety, E a prime divisor on X, and $D = \sum_{i=1}^r a_i D_i$ a divisor on X such that $E \neq D_i$ for all i.

i) Show that if $D + E$ has simple normal crossings, then E is smooth and $D|_E$ has simple normal crossings. Moreover, the divisors $D_i|_E$ are smooth (possibly disconnected), without common components.
ii) If E is smooth, the $D_i|_E$ have no common components, and $D|_E$ has simple normal crossings, then there is an open neighborhood U of E_1 such that $(D + E)|_U$ has simple normal crossings.

Problem 4. Let X be a smooth variety over a ground field k, with $\mathrm{char}(k) = 0$, let Z be a smooth subvariety of X of codimension r, and let $\mathfrak{a}$ be an ideal in $\mathcal{O}_X$.

i) Suppose that $x_1, \ldots, x_n$ are algebraic coordinates on an open subset U of X such that $Z \cap U$ is nonempty and defined in U by $(x_1, \ldots, x_r)$. Let $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \in \mathrm{Der}_k\big(\mathcal{O}_X(U)\big)$ be the dual basis of $dx_1, \ldots, dx_n$, and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we consider $\frac{\partial^{|\alpha|}}{\partial x^\alpha}\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$. Show that for $f \in \mathcal{O}_X(U)$ we have $\mathrm{ord}_Z(f) \geq m$ if and only if $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}$ vanishes on X for all $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0)$, with $|\alpha| = \alpha_1 + \ldots + \alpha_r \leq m - 1$.
ii) Show that $\mathrm{ord}_Z(\mathfrak{a}) \geq m$ if and only if $\mathrm{ord}_x(\mathfrak{a}) \geq m$ for all $x \in Z$.
iii) Show that for every m, the set $\{x \in X \mid \mathrm{ord}_x(\mathfrak{a}) \geq m\}$ is closed in X.

Problem 5. Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideals on X, then for every $\lambda \in \mathbb{R}_{\geq 0}$ we have

$$\mathfrak{a} \cdot \mathcal{J}(\mathfrak{b}^\lambda) \subseteq \mathcal{J}(\mathfrak{a} \cdot \mathfrak{b}^\lambda).$$

Problem 6. Let X be a smooth variety. Show that if $\mathfrak{a}$ is a nonzero ideal on X and $\mathrm{lct}(\mathfrak{a}) = c$, then for every positive integer m, the locus

$$\{x \in X \mid \mathrm{ord}_x(\mathfrak{a}) \geq m\}$$

has codimension $\geq \lceil cm \rceil$.