CHAPTER 7. THE GRASSMANN VARIETY AND OTHER EXAMPLES

1. The Grassmann Variety

Let $V = k^n$ and let $r$ be an integer with $0 \leq r \leq n$. In this section we describe the structure of algebraic variety on the set $G(r, n)$ parametrizing the $r$-dimensional linear subspaces of $V$. These are the Grassmann varieties. Given an $r$-dimensional linear subspace $W$ of $V$, we denote by $[W]$ the corresponding point of $G(r, n)$.

This is trivial for $r = 0$ or $r = n$: in this case $G(r, n)$ is just a point. The first non-trivial case that we have already encountered is for $r = 1$: in this case $G(r, n) = P^{n-1}$. A similar description holds for $r = n - 1$: hyperplanes in $k^n$ are in bijection with lines in $(k^n)^* \simeq k^n$, hence these are again parametrized by $P^{n-1}$ (cf. Exercise 2.18 in Chapter 4).

We now proceed with the description in the general case. Given an $r$-dimensional linear subspace $W$ of $k^n$, choose a basis $u_1, \ldots, u_r$ of $W$. By writing $u_i = (a_{i,1}, \ldots, a_{i,n})$ for $1 \leq i \leq r$, we obtain a matrix $A = (a_{i,j}) \in M_{r,n}(k)$. Note that we have an action of $GL_r(k)$ on $M_{r,n}(k)$ given by left multiplication. Choosing a different basis of $W$ corresponds to multiplying the matrix on the left by an element of $GL_r(k)$. Moreover a matrix in $M_{r,n}(k)$ corresponds to some $r$-dimensional linear subspace in $k^n$ if and only if it has maximal rank $r$. We can thus identify $G(r, n)$ with the quotient set $U/GL_r(k)$, where $U$ is the open subset of $M_{r,n}(k)$ consisting of matrices of rank $r$.

For every subset $I \subseteq \{1, \ldots, n\}$ with $r$ elements, let $U_I$ be the open subset of $U$ given by the non-vanishing of the $r$-minor on the columns indexed by the elements of $I$. Note that this subset is preserved by the $GL_r(k)$-action and let $V_I$ be the corresponding subset of $G(r, n)$. We now construct a bijection

$$\varphi_I: V_I \to M_{r,n-r}(k) = A^{r(n-r)}.$$  

In order to simplify the notation, say $I = \{1, \ldots, r\}$. Given any matrix $A \in U_I$, let us write it as $A = (A', A'')$ for matrices $A' \in M_{r,r}(k)$ and $A'' \in M_{r,n-r}(k)$. Note that by assumption $\det(A') \neq 0$. In this case there is a unique matrix $B \in GL_r(k)$ such that $B \cdot A = (I_r, C)$, for some matrix $C \in M_{r,n-r}(k)$ (namely $B = (A')^{-1}$, in which case $C = (A')^{-1} \cdot A''$).

Therefore every matrix class in $V_I$ is the class of a unique matrix of the form $(I_r, C)$, with $C \in M_{r,n-r}(k)$. This gives the desired bijection between $V_{\{1, \ldots, r\}} \to A^{r(n-r)}$, and a similar argument works for every $V_I$.

We put on each $V_I$ the topology and the sheaf of functions that make the above bijection an isomorphism in $\mathcal{T}_{op_k}$. We need to show that these glue to give on $G(r, n)$ a structure of a prevariety: we need to show that for every subsets $I$ and $J$ as above, the subset $\varphi_I(V_I \cap V_J)$ is an open subset of $A^{r(n-r)}$ and the map

$$\varphi_J \circ \varphi_I^{-1}: \varphi_I(V_I \cap V_J) \to \varphi_J(V_I \cap V_J)$$

(1)
is a morphism of algebraic varieties (in which case, by symmetry, it is an isomorphism).
In order to simplify the notation, suppose that \( I = \{1, \ldots, r\} \). It is then easy to see that if \( \#(I \cap J) = \ell \), then \( \varphi_I(V_I \cap V_J) \subseteq \mathbb{A}^{r(n-r)} \) is the principal affine open subset defined by
the non-vanishing of the \((r-\ell)\)-minor on the rows indexed by those \( i \in I \setminus J \) and on the columns indexed by those \( j \in J \setminus I \). Moreover, the map (1) is given by associating to a matrix \( C \) the \( r \times n \) matrix \( M = (I_r, C) \), multiplying it on the left with the inverse of the \( r \times r\)-submatrix of \( M \) on the columns in \( J \) to get \( M' \), and then keeping the \( r \times (n-r) \) submatrix of \( M' \) on the columns in \( \{1, \ldots, n\} \setminus J \). It is clear that this is a morphism.

We thus conclude that \( G(r, n) \) is an object in \( \text{Top}_k \). In fact, it is a prevariety, since it is covered by open subsets isomorphic to affine varieties. In fact, since each \( V_I \) is isomorphic to an affine space, it is smooth and irreducible, and since we have seen that any two \( V_I \) intersect, we conclude that \( G(r, n) \) is irreducible by Exercise 3.17 in Chapter 1. Furthermore, since each \( V_I \) has dimension \( r(n-r) \), we conclude that \( \dim(G(r, n)) = r(n-r) \). We collect these facts in the following proposition.

**Proposition 1.1.** The Grassmann variety \( G(r, n) \) is a smooth, irreducible prevariety of dimension \( r(n-r) \), that has a cover by open subsets isomorphic to \( \mathbb{A}^{r(n-r)} \).

**Example 1.2.** If \( r = 1 \), the algebraic variety \( G(1, n) \) is just \( \mathbb{P}^{n-1} \), described via the charts \( U_i = (x_i \neq 0) \cong \mathbb{A}^{n-1} \).

**Example 1.3.** If \( r = n-1 \), the algebraic variety \( G(n-1, n) \) has an open cover
\[
G(n-1, n) = U_1 \cup \ldots \cup U_n.
\]
For every \( i \), we have an isomorphism \( \mathbb{A}^{n-1} \cong U_i \) such that \((\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)\) is mapped to the hyperplane generated by \( \{e_j + \lambda_j e_i \mid j \neq i\} \). This is the hyperplane defined by the equation
\[
eq 0.
\]
We thus see that the variety structure on \( G(n-1, n) \) is the same as on \( (\mathbb{P}^{n-1})^* \), which is isomorphic to \( \mathbb{P}^{n-1} \) (cf. Exercise 2.18 in Chapter 4).

Our next goal is to show that, in fact, \( G(r, n) \) is a projective variety. Note that if \( W \) is an \( r \)-dimensional linear subspace of \( V = k^n \), then \( \wedge^r W \) is a 1-dimensional linear subspace of \( \wedge^r V \cong k^d \), where \( d = \binom{n}{r} \). If \( e_1, \ldots, e_n \) is the standard basis of \( k^n \), then we have a basis of \( \wedge^r V \) given by the \( e_I = e_{i_1} \wedge \ldots \wedge e_{i_r} \), where \( I = \{i_1, \ldots, i_r\} \) is a subset of \( \{1, \ldots, n\} \) with \( r \)-elements (and where, in order to write \( e_I \), we order the elements \( i_1 < \ldots < i_r \)). We correspondingly denote the coordinates on the projective space of lines in \( \wedge^r V \) by \( x_I \).

**Proposition 1.4.** The map \( f: G(r, n) \to \mathbb{P}^{d-1} \) that maps \([W]\) to \([\wedge^r W]\) is a closed immersion. In particular, \( G(r, n) \) is a projective variety.

The embedding in the above proposition is the *Plücker embedding* of the Grassmann variety.

**Proof of Proposition 1.4.** If \( W \subseteq V \) is an \( r \)-dimensional linear subspace described by the matrix \( A \), then \( f([W]) \in \mathbb{P}^{d-1} \) is given in the above homogeneous coordinates by the \( r \)-minors of \( A \). In particular, we see that the inverse image of the affine chart \( W_I = (x_I \neq 0) \) is the affine open subset \( V_I \subseteq G(r, n) \).
In order to complete the proof, it is enough to show that for every $I$, the induced map $V_I \to W_I$ is a morphism and the corresponding ring homomorphism

\[ \mathcal{O}(W_I) \to \mathcal{O}(V_I) \]

is surjective. The argument is the same for all $I$, but in order to simplify the notation, we assume $I = \{1, \ldots, r\}$. Note that the map $V_I \to W_I$ gets identified to $M_{r,n-r}(k) \to A^{r\cdot n-1}$, then maps a matrix $B$ to all $r$-minors of $(I, B)$, with the exception of the one on the first $r$ columns. In particular, we see that this map is a morphism. By choosing $r-1$ columns of the first $r$ ones and an additional column of the last $(n-r)$ ones, we obtain every entry of $B$ as an $r$-minor as above. This implies that the homomorphism (2) is surjective. \[ \square \]

**Remark 1.5.** The algebraic group $GL_n(k)$ acts on $k^n$ and thus acts on $G(r, n)$ by $g \cdot [W] = [g \cdot W]$. Note that if $W$ is described by the matrix $A \in M_{r,n}(k)$, then $g \cdot W$ is described by $A \cdot g^t$. It is straightforward to see that this is an algebraic action. Since any two linear subspaces can by mapped one to the other by a linear automorphism of $k^n$, we see that the $GL_n(k)$-action on $G(r, n)$ is transitive.

**Remark 1.6.** If $W$ is an $r$-dimensional linear subspace of $V = k^n$, then we have an induced surjection $V^* \to W^*$, whose kernel is an $(n-r)$-dimensional linear subspace of $(k^n)^* \simeq k^n$. In this way we get a bijection $G(r, n) \to G(n-r, n)$ and it is not hard to check that this is, in fact, an isomorphism of algebraic varieties.

**Remark 1.7.** Given an arbitrary $n$-dimensional vector space $V$ over $k$, let $G(r, V)$ be the set of $r$-dimensional linear subspaces of $V$. By choosing an isomorphism $V \simeq k^n$, we obtain a bijection $G(r, V) \simeq G(r, n)$ and we put on $G(r, V)$ the structure of an algebraic variety that makes this an isomorphism. Note that this is independent of the choice of isomorphism $V \simeq k^n$: for a different isomorphism, we have to compose the map $G(r, V) \to G(r, n)$ with the action on $G(r, n)$ of a suitable element in $GL_n(k)$.

**Remark 1.8.** It is sometimes convenient to identify $G(r, n)$ with the set of $(r-1)$-dimensional linear subspaces in $P^{n-1}$.

**Notation 1.9.** Given a finite-dimensional $k$-vector space $V$, we denote by $P(V)$ the projective space parametrizing hyperplanes in $V$. Therefore the homogeneous coordinate ring of $P(V)$ is given by the symmetric algebra $Sym^*(V)$. With this notation, the projective space parametrizing the lines in $V$ is given by $P(V^*)$.

We end this section by discussing the incidence correspondence for the Grassmann variety and by giving some applications. More applications will be given in the next sections.

Consider the set of $r$-dimensional linear subspaces in $P^n$, parametrized by $G = G(r+1, n+1)$. The incidence correspondence is the subset

\[ Z = \{(q, [V]) \in P^n \times G \mid q \in V\} . \]

Note that this is a closed subset of $P^n \times G$. Indeed, if we represent $[W]$ by the matrix $A = (a_{i,j})_{0 \leq i \leq r+1, 0 \leq j \leq n}$, then $([b_0, \ldots, b_n], [W])$ lies in $Z$ if and only if the rank of the
matrix

\[
B = \begin{pmatrix}
b_0 & b_1 & \ldots & b_n \\
a_{0,0} & a_{0,1} & \ldots & a_{0,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{r,0} & a_{r,1} & \ldots & a_{r,n}
\end{pmatrix}
\]

is \( \leq r + 1 \). This is the case if and only if all \((r + 2)\)-minors of \( B \) vanish. By expanding along the first row, we can write each such minor as \( \sum_{j \in I} b_j \delta_j \), where \( I \subseteq \{0, \ldots, n\} \) is the subset with \( r + 2 \) elements determining the minor and each \( \delta_j \) is a suitable minor of \( A \). Consider the closed immersion

\[
P^n \times G \overset{i}{\hookrightarrow} P^n \times P^N \overset{j}{\hookrightarrow} P^M,
\]

where \( i \) is given by \( i(u, v) = (u, \varphi(v)) \), with \( \varphi \) being the Plücker embedding, and \( j \) is the Segre embedding. It follows from the above discussion that via this embedding, \( Z \) is the inverse image of a suitable linear subspace of \( P^M \), and therefore it is closed in \( P^n \times G \). Since both \( P^n \) and \( G \) are projective varieties, we conclude that \( Z \) is a projective variety.

The projections onto the two components induce the morphisms \( \pi_1: Z \to P^n \) and \( \pi_2: Z \to G \). It follows from the definition that for every \([W] \in G\), we have \( \pi_2^{-1}([W]) \simeq W \).

**Exercise 1.10.** Show that the morphism \( \pi_2: Z \to G \) is locally trivial, with fiber\(^1\) \( P^r \).

Since all fibers of \( \pi_2 \) are irreducible, of dimension \( r \), we conclude from Proposition 5.1 in Chapter 5 that \( Z \) is irreducible, of dimension

\[
\dim(Z) = r + \dim(G) = r + (r + 1)(n - r).
\]

(we use here the fact that \( G \) is irreducible and \( Z \) is a projective variety).

Given a point \( q \in P^n \), the fiber \( \pi^{-1}(q) \subseteq G \) consists of all \( r \)-dimensional linear subspaces of \( P^n \) containing \( q \) (equivalently, these are the \((r + 1)\)-dimensional linear subspaces of \( k^{n+1} \) containing a given line). These are in bijection with the Grassmann variety \( G(r, n) \).

**Exercise 1.11.** Show that the morphism \( \pi_1: Z \to P^n \) is locally trivial, with fiber \( G(r, n) \).

We use the incidence correspondence to prove the following

**Proposition 1.12.** Let \( X \subseteq P^n \) be a closed subvariety of dimension \( d \) and let \( G = G(r + 1, n + 1) \). If we put

\[
M_r(X) = \{ [W] \in G \mid W \cap X \neq \emptyset \},
\]

then the following hold:

i) The set \( M_r(X) \) is a closed subset of \( G \), which is irreducible if \( X \) is irreducible.

ii) We have \( \dim(M_r(X)) = \dim(G) - (n - r - d) \) for \( 0 \leq r \leq n - d \).

\(^1\)Given a variety \( F \), we say that a morphism \( f: X \to Y \) is locally trivial, with fiber \( F \), if there is an open cover \( Y = U_1 \cup \ldots \cup U_r \) such that for every \( i \), we have an isomorphism \( f^{-1}(U_i) \simeq U_i \times F \) of varieties over \( U_i \).
Exercise 1.13. Consider the Grassmann variety $G = G(r + 1, n + 1)$ parametrizing the $r$-dimensional linear subspaces in $\mathbb{P}^n$. Show that if $Z$ is a closed subset of $G$, then the set 

$$\tilde{Z} := \bigcup_{[V] \in Z} V \subseteq \mathbb{P}^n$$

is a closed subset of $\mathbb{P}^n$, with $\dim(\tilde{Z}) \leq \dim(Z) + r$.

Exercise 1.14. Show that if $X$ and $Y$ are disjoint closed subvarieties of $\mathbb{P}^n$, then the join $J(X,Y) \subseteq \mathbb{P}^n$, defined as the union of all lines in $\mathbb{P}^n$ joining a point in $X$ and a point in $Y$, is a closed subset of $\mathbb{P}^n$, with

$$\dim(J(X,Y)) \leq \dim(X) + \dim(Y) + 1.$$
the projective space $\mathbb{P}^{N_d}$ consisting of classes of homogeneous polynomials $F \in S_d$ such that the ideal $(F)$ is radical. We will denote by $[H]$ the point of $\mathcal{H}_d$ corresponding to the hypersurface $H \subseteq \mathbb{P}^n$.

**Lemma 2.1.** The subset $\mathcal{H}_d \subseteq \mathbb{P}^{N_d}$ is a non-empty open subset.

**Proof.** Note that given $F \in S_d$, the ideal $(F)$ is not reduced if and only if there is a positive integer $e$ and a homogeneous polynomial $G \in S_e$ such that $G^2$ divides $F$. For every $e$ such that $0 < 2e \leq d$, consider the map

$$\alpha_e : \mathbb{P}^{N_e} \times \mathbb{P}^{N_d-2e} \to \mathbb{P}^{N_d}$$

that maps $([G], [H])$ to $[G^2 H]$. It is straightforward to see that this is a morphism. Since the source is a projective variety, it follows that the image of $\alpha_e$ is closed. Since $\mathcal{H}_d$ is equal to $\mathbb{P}^{N_d} \setminus \bigcup_{1 \leq e \leq \lfloor d/2 \rfloor} \text{Im}(\alpha_e)$, we see that this set is open in $\mathbb{P}^{N_d}$. In order to see that it is non-empty, it is enough to consider $f \in S_d$ which is the product of $d$ distinct linear forms. $\square$

**Exercise 2.2.** Show that if $X \subseteq \mathbb{P}^n$ is a smooth closed subvariety, then for a general hypersurface $H$ of degree $d$ in $\mathbb{P}^n$, the intersection $X \cap H$ is smooth.

**Exercise 2.3.** Show that the set of those $[F] \in \mathcal{H}_d$ defining a smooth hypersurface is a non-empty open subset of $\mathcal{H}_d$.

We next construct the *universal hypersurface* over $\mathcal{H}_d$. In fact, for many purposes, it is more convenient to work with the whole space $\mathbb{P}^{N_d}$ instead of restricting to $\mathcal{H}_d$ (this is due to the fact that $\mathbb{P}^{N_d}$ is complete, while $\mathcal{H}_d$ is not). Define

$$\mathcal{Z}_d := \{ (p, [F]) \in \mathbb{P}^n \times \mathbb{P}^{N_d} \mid F(p) = 0 \}.$$ 

It is easy to see that via the composition of closed embeddings

$$\mathbb{P}^n \times \mathbb{P}^{N_d} \xrightarrow{\nu_x \times 1} \mathbb{P}^{N_d} \times \mathbb{P}^{N_d} \xrightarrow{\beta} \mathbb{P}^M,$$

where $\nu_x$ is the $d^{th}$ Veronese embedding and $\beta$ is the Segre embedding, $\mathcal{Z}_d$ is the inverse image of a hyperplane, hence it is a closed subset of $\mathbb{P}^n \times \mathbb{P}^{N_d}$.

Note that the projections onto the two components induce two morphisms

$$\varphi : \mathcal{Z}_d \to \mathbb{P}^n \quad \text{and} \quad \psi : \mathcal{Z}_d \to \mathbb{P}^{N_d}.$$ 

Since $\mathbb{P}^n$ and $\mathbb{P}^{N_d}$ are projective varieties, we deduce that both $\varphi$ and $\psi$ are proper morphisms. It follows from definition that for every $[H] \in \mathcal{H}_d$, we have $\psi^{-1}([H]) = H$.

On the other hand, for every $p \in \mathbb{P}^n$, the fiber $\varphi^{-1}(p)$ consists of the classes of those $F \in S_d$ such that $F(p) = 0$. This is a hyperplane in $\mathbb{P}^{N_d}$. We deduce from Proposition 5.1 in Chapter 5 that $\mathcal{Z}_d$ is irreducible, of dimension $N_d + n - 1$.

We now turn to linear subspaces on projective hypersurfaces. Given $r < n$, let $G = G(r + 1, n + 1)$ be the Grassmann variety parametrizing the $r$-dimensional linear
subspaces in \( P^n \). Consider the incidence correspondence \( I \subseteq P^N \times G \) consisting of pairs \(([F],[\Lambda])\) such that \( F \) vanishes on \( \Lambda \).

We first show that \( I \) is closed in \( P^N \times G \). Suppose that we are over the open subset \( V = V_{1,...,r} \simeq A^{(r+1)(n-r)} \) of \( G \), where a subspace \( \Lambda \) is described by the linear span of the rows of the matrix

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & a_{0,r+1} & \cdots & a_{0,n} \\
0 & 1 & \cdots & 0 & a_{1,r+1} & \cdots & a_{1,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & a_{r,r+1} & \cdots & a_{r,n}
\end{pmatrix}
\]

The hypersurface corresponding to \( c = (c_\alpha) \), which is defined by \( f_c = \sum_\alpha c_\alpha x^\alpha \) contains the subspace corresponding to the above matrix if and only if

\[
f_c \left( x_0, \ldots, x_r, \sum_{0 \leq i \leq r} a_{i,r+1} x_i, \ldots, \sum_{0 \leq i \leq r} a_{i,n} x_i \right) = 0 \quad \text{in} \quad k[x_0, \ldots, x_r].
\]

We can write

\[
f_c \left( x_0, \ldots, x_r, \sum_{0 \leq i \leq r} a_{i,r+1} x_i, \ldots, \sum_{0 \leq i \leq r} a_{i,n} x_i \right) = \sum_\beta F_\beta(a, c) x^\beta,
\]

where the sum is running over those \( \beta = (\beta_0, \ldots, \beta_r) \) with \( \sum_i \beta_i = d \). Note that each \( F_\beta \) is a polynomial in the \( a_{i,j} \) and \( c_\alpha \) variables, homogeneous of degree 1 in the \( c_\alpha \)'s. With this notation, \( I \cap (P^N \times V) \) is the zero-locus in \( P^N \times V \) of the ideal generated by all \( F_\beta \); in particular, it is a closed subset. The equations over the other charts in \( G \) are similar.

In particular, we see that \( I \) is a projective variety. Let \( \pi_1: I \to P^N \) and \( \pi_2: I \to G \) be the morphisms induced by the projections onto the two factors.

**Definition 2.4.** For every hypersurface \( H \) of degree \( d \) in \( P^n \), the Fano variety of \( r \)-planes in \( H \), denoted \( F_r(H) \), is the fiber \( \pi_1^{-1}([H]) \) of \( \pi_1 \), parametrizing the \( r \)-dimensional linear subspaces contained in \( H \).

**Proposition 2.5.** The projective variety \( I \) is irreducible, of dimension

\[
(r + 1)(n - r) + \binom{n + d}{d} - \binom{r + d}{d} - 1.
\]

**Proof.** Consider the morphism \( \pi_2: I \to G \). By Proposition 5.1 in Chapter 5, it is enough to show that every fiber \( \pi^{-1}([\Lambda]) \) is isomorphic to a linear subspace of \( P^N \), of codimension \( \binom{r+d}{d} \). In order to see this, we may assume that \( \Lambda \) is defined by \( x_{r+1} = \ldots = x_n = 0 \). It is clear that a polynomial \( f \) vanishes on \( \Lambda \) if and only if all coefficients of the monomials in \( x_0, \ldots, x_r \) in \( f \) vanish; this gives a linear subspace of codimension \( \binom{r+d}{d} \). \( \square \)

**Example 2.6.** Consider lines on cubic surfaces: that is, we specialize to the case when \( n = 3 = d \) and \( r = 1 \). Note that in this case \( I \) is an irreducible variety of dimension 19, the same as the dimension of the projective space parametrizing homogeneous polynomials of degree 3 in \( S = k[x_0, x_1, x_2, x_3] \). We claim that the morphism \( \pi_1: I \to P^{19} \) is surjective; in
other words, every hypersurface in $\mathbb{P}^3$ which is the zero-locus of a degree 3 homogeneous polynomial contains at least one line. In order to see this, it is enough to exhibit such a hypersurface that only contains finitely many lines (this follows from Theorem 4.1 in Chapter 2). At least for $\text{char}(k) \neq 3$, such an example is given by the Fermat cubic surface below.

**Example 2.7.** Suppose that $\text{char}(k) \neq 3$ and let $X$ be the Fermat surface in $\mathbb{P}^3$ defined by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$ 

Of course, if $\text{char}(k) = 3$, then the zero locus of this polynomial is the hyperplane $x_0 + x_1 + x_2 + x_3 = 0$, which contains infinitely many lines.

Up to reordering the variables, every line $L \subseteq X$ can be given by equations of the form

$$x_0 = \alpha x_2 + \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2 + \delta x_3,$$
for some $\alpha, \beta, \gamma, \delta \in k$. This line lies on $X$ if and only if

$$(\alpha x_2 + \beta x_3)^3 + (\gamma x_2 + \delta x_3)^3 + x_2^3 + x_3^3 = 0 \quad \text{in} \quad k[x_2, x_3].$$

This is equivalent to the following system of equations:

$$\alpha^3 + \gamma^3 = -1, \quad \alpha^2 \beta + \gamma^2 \delta = 0, \quad \alpha \beta^2 + \gamma \delta^2 = 0, \quad \text{and} \quad \beta^3 + \delta^3 = -1.$$

If $\alpha, \beta, \gamma, \delta$ are all nonzero, then it follows from the third equation that

$$\gamma = -\alpha \beta^2 \delta^{-2},$$
and plugging in the second equation, we get

$$\alpha^2 \beta + \alpha^2 \beta^4 \delta^{-4} = 0,$$
which implies $\beta^3 = -\delta^3$, contradicting the fourth equation.

Suppose now, for example, that $\alpha = 0$. We deduce from the second equation that $\gamma \delta = 0$. Moreover, $\gamma^3 = -1$ by the first equation, hence $\delta = 0$ and $\beta^3 = -1$ by the fourth equation. We thus get in this way the 9 lines with the equations

$$x_0 = \beta x_3 \quad \text{and} \quad x_1 = \gamma x_2,$$
where $\beta, \gamma \in k$ are such that $\beta^3 = -1 = \gamma^3$. After permuting the variables, we obtain 2 more sets of lines on $X$, hence in total we have 27 lines.