CHAPTER 1. AFFINE ALGEBRAIC VARIETIES

During this first part of the course, we will establish a correspondence between various geometric notions and algebraic ones. Some references for this part of the course are [Har77, Chapter I], [Mum88, Chapter I], and [Sha13, Chapter I].

1. Algebraic subsets and ideals

Let k be a fixed algebraically closed field. We do not make any assumption on the characteristic. Important examples are $\mathbf{C}$, $\overline{\mathbf{Q}}$, and $\overline{\mathbf{F}_p}$, for a prime integer p.

For a positive integer n we denote by $\mathbf{A}^n$ the *n-dimensional affine space*. For now, this is just a set, namely k^n. We assume that n is fixed and denote the polynomial ring $k[x_1, \ldots, x_n]$ by R. Note that if $f \in R$ and $u = (u_1, \ldots, u_n)$, we may evaluate f at u to get $f(u) \in k$. This gives a surjective ring homomorphism

$$k[x_1, \ldots, x_n] \to k, \quad f \to f(u),$$

whose kernel is the (maximal) ideal $(x_1 - u_1, \ldots, x_n - u_n)$.

Our goal in this section is to establish a correspondence between certain subsets of $\mathbf{A}^n$ (those *defined by polynomial equations*) and ideals in R (more precisely, radical ideals). A large part of this correspondence is tautological. The non-trivial input will be provided by Hilbert's Nullstellensatz, which we will be prove in the next section.

Definition 1.1. Given a subset $S \subseteq R$, the *zero-locus of S* (also called *the subset of $\mathbf{A}^n$ defined by S*) is the set

$$V(S) := \{u \in \mathbf{A}^n \mid f(u) = 0 \text{ for all } f \in S\}.$$

An *algebraic subset* of $\mathbf{A}^n$ is a subset of the form $V(S)$ for some subset S of R.

Example 1.2. Any linear subspace of k^n is an algebraic subset; in fact, it can be written as $V(S)$, where S is a finite set of linear polynomials (that is, polynomials of the form $\sum_{i=1}^n a_i x_i$). More generally, any translation of a linear subspace (that is, an *affine subspace*) of k^n is an algebraic subset.

Example 1.3. A union of two lines in $\mathbf{A}^2$ is an algebraic subset (see Proposition 1.6). For example, the union of the two coordinate axes can be written as $V(x_1 x_2)$.

Example 1.4. Another example of an algebraic subset of $\mathbf{A}^2$ is the hyperbola

$$\{u = (u_1, u_2) \in \mathbf{A}^2 \mid u_1 u_2 = 1\}.$$

Remark 1.5. Recall that if S is a subset of R and I is the ideal of R generated by S, then we can write

$$I = \{g_1 f_1 + \ldots + g_m f_m \mid m \geq 0, f_1, \ldots, f_m \in S, g_1, \ldots, g_m \in R\}.$$

It is then easy to see that $V(S) = V(I)$. In particular, every algebraic subset of $\mathbf{A}^n$ can be written as $V(I)$ for some ideal I in R.

We collect in the following proposition the basic properties of taking the zero locus.

Proposition 1.6. *The following hold:*

1) $V(R) = \emptyset$*; in particular, the empty set is an algebraic subset.*
2) $V(0) = \mathbf{A}^n$*: in particular,* $\mathbf{A}^n$ *is an algebraic subset.*
3) *If* I *and* J *are ideals in* R *with* $I \subseteq J$*, then* $V(J) \subseteq V(I)$*.*
4) *If* $(I_\alpha)_\alpha$ *is a family of ideals in* R*, we have*
$$\bigcap_\alpha V(I_\alpha) = V\left(\bigcup_\alpha I_\alpha\right) = V\left(\sum_\alpha I_\alpha\right).$$
5) *If* I *and* J *are ideals in* R*, then*
$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

Proof. The assertions in 1)–4) are trivial to check. Note also that the inclusions
$$V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(I \cdot J)$$
follow directly from 3). In order to show that $V(I \cdot J) \subseteq V(I) \cup V(J)$, we argue by contradiction: suppose that $u \in V(I \cdot J) \smallsetminus \big(V(I) \cup V(J)\big)$. We can thus find $f \in I$ such that $f(u) \neq 0$ and $g \in J$ such that $g(u) \neq 0$. In this case $fg \in I \cdot J$ and $(fg)(u) = f(u)g(u) \neq 0$, a contradiction with the fact that k is a domain. □

An important consequence of the assertions in the above proposition is that the algebraic subsets of $\mathbf{A}^n$ form the closed subsets for a topology of $\mathbf{A}^n$. This is the *Zariski topology* on $\mathbf{A}^n$.

The Zariski topology provides a convenient framework for dealing with algebraic subsets of $\mathbf{A}^n$. However, we will see that it has a lot less subsets than one is used to from the case of the usual Euclidean space (over $\mathbf{R}$ or over $\mathbf{C}$).

We now define a map in the other direction, from subsets of $\mathbf{A}^n$ to ideals in R. Given a subset W of $\mathbf{A}^n$, we put
$$I(W) := \{f \in R \mid f(u) = 0 \text{ for all } u \in W\}.$$
It is straightforward to see that this is an ideal in R. In fact, it is a radical[1] ideal: indeed, since k is a reduced ring, if $f(u)^q = 0$ for some positive integer q, then $f(u) = 0$. We collect in the next proposition some easy properties of this definition.

Proposition 1.7. *The following hold:*

1) $I(\emptyset) = R$.

[1] An ideal I in a ring R is *radical* if whenever $f^q \in I$ for some $f \in R$ and some positive integer q, we have $f \in I$. A related concept is that of a *reduced* ring: this is a ring such that whenever $f^q = 0$ for some $f \in R$ and some positive integer q, we have $f = 0$. Note that an ideal I is radical if and only if R/I is a reduced ring.

2) *If* $(W_\alpha)_\alpha$ *is a family of subsets of* $\mathbf{A}^n$*, then* $I\left(\bigcup_\alpha W_\alpha\right) = \bigcap_\alpha I(W_\alpha)$.
3) *If* $W_1 \subseteq W_2$*, then* $I(W_2) \subseteq I(W_1)$.

Proof. All assertions follow immediately from definition. □

We have thus set up two maps between subsets of $\mathbf{A}^n$ and ideals in R and we are interested in the two compositions. Understanding one of these compositions is tautological, as follows:

Proposition 1.8. *For every subset* Z *of* $\mathbf{A}^n$*, the set* $V\big(I(Z)\big)$ *is equal to the closure* $\overline{Z}$ *of* Z*, with respect to the Zariski topology. In particular, if* Z *is an algebraic subset of* $\mathbf{A}^n$*, then* $V\big(I(Z)\big) = Z$.

Proof. We clearly have

$$Z \subseteq V\big(I(Z)\big),$$

and since the right-hand side is closed by definition, we have

$$\overline{Z} \subseteq V\big(I(Z)\big).$$

In order to prove the reverse inclusion, recall that by definition of the closure of a subset, we have

$$\overline{Z} = \bigcap_W W,$$

where W runs over all algebraic subsets of $\mathbf{A}^n$ that contain Z. Every such W can be written as $W = V(J)$, for some ideal J in R. Note that we have $J \subseteq I(W)$, while the inclusion $Z \subseteq W$ gives $I(W) \subseteq I(Z)$. We thus have $J \subseteq I(Z)$, hence $V\big(I(Z)\big) \subseteq V(J) = W$. Since $V\big(I(Z)\big)$ is contained in every such W, we conclude that

$$V\big(I(Z)\big) \subseteq \overline{Z}.$$

□

The interesting statement here concerns the other composition. Recall that if J is an ideal in a ring R, then the set

$$\{f \in R \mid f^q \in J \text{ for some } q \geq 1\}$$

is a radical ideal; in fact, it is the smallest radical ideal containing J, denoted $\mathrm{rad}(J)$.

Theorem 1.9. *(Hilbert's Nullstellensatz) For every ideal* J *in* R*, we have*

$$I\big(V(J)\big) = \mathrm{rad}(J).$$

The inclusion $J \subseteq I\big(V(J)\big)$ is trivial and since the right-hand side is a radical ideal, we obtain the inclusion

$$\mathrm{rad}(J) \subseteq I\big(V(J)\big).$$

This reverse inclusion is the subtle one and this is where we use the hypothesis that k is algebraically closed (note that this did not play any role so far). We will prove this in the next section, after some preparations. Assuming this, we obtain the following conclusion.

Corollary 1.10. *The two maps $I(-)$ and $V(-)$ between the algebraic subsets of $\mathbf{A}^n$ and the radical ideals in $k[x_1, \ldots, x_n]$ are inverse, order-reversing bijections.*

Remark 1.11. It follows from Corollary 1.10 that via the above bijection, the minimal nonempty algebraic subsets correspond to the maximal ideals in R. It is clear that the minimal nonempty algebraic subsets are precisely the points in $\mathbf{A}^n$. On the other hand, given $a = (a_1, \ldots, a_n) \in \mathbf{A}^n$, the ideal $I(u)$ contains the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$, hence the two ideals are equal. We thus deduce that *every* maximal ideal in R is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. We will see in the next section that the general statement of Theorem 1.9 is proved by reduction to this special case.

Exercise 1.12. Show that the closed subsets of $\mathbf{A}^1$ are $\mathbf{A}^1$ and its finite subsets.

Exercise 1.13. Show that if W_1 and W_2 are algebraic subsets of $\mathbf{A}^n$, then

$$I(W_1 \cap W_2) = \mathrm{rad}\big(I(W_1) + I(W_2)\big).$$

Exercise 1.14. For m and $n \geq 1$, let us identify $\mathbf{A}^{mn}$ with the set of all matrices $B \in M_{m,n}(k)$. Show that the set

$$M^r_{m,n}(k) := \{B \in M_{m,n}(k) \mid \mathrm{rk}(B) \leq r\}$$

is a closed algebraic subset of $M_{m,n}(k)$.

Exercise 1.15. Show that the following subset of $\mathbf{A}^3$

$$W_1 = \{(t, t^2, t^3) \mid t \in k\}$$

is a closed algebraic subset, and describe $I(W_1)$. Can you do the same for

$$W_2 = \{(t^2, t^3, t^4) \mid t \in k\}?$$

How about

$$W_3 = \{(t^3, t^4, t^5) \mid t \in k\}?$$

Exercise 1.16. For an arbitrary commutative ring R, one can define the *maximal spectrum* $\mathrm{MaxSpec}(R)$ of R, as follows. As a set, this is the set of all maximal ideals in R. For every ideal J in R, we put

$$V(J) := \{\mathfrak{m} \in \mathrm{MaxSpec}(R) \mid J \subseteq \mathfrak{m}\}$$

and for every subset $S \subseteq \mathrm{MaxSpec}(R)$, we define

$$I(S) := \bigcap_{\mathfrak{m} \in S} \mathfrak{m}.$$

i) Show that $\mathrm{MaxSpec}(R)$ has a structure of topological space in which the closed subsets are the subsets of the form $V(I)$, for an ideal I in R.
ii) Show that for every subset S of $\mathrm{MaxSpec}(R)$, we have $V\big(I(S)\big) = \overline{S}$.
iii) Show that if R is an algebra of finite type over an algebraically closed field k, then for every ideal J in S, we have $I\big(V(J)\big) = \mathrm{rad}(J)$.
iv) Show that if $X \subseteq \mathbf{A}^n$ is a closed subset, then we have a homeomorphism $X \simeq \mathrm{MaxSpec}(R/J)$, where $R = k[x_1, \ldots, x_n]$ and $J = I(X)$.

2. Noether normalization and Hilbert's Nullstellensatz

The proof of Hilbert's Nullstellensatz is based on the following result, known as *Noether's normalization lemma*. As we will see, this has many other applications.

Before stating the result, we recall that a ring homomorphism $A \to B$ is *finite* if B is finitely generated as an A-module. It is straightforward to check that a composition of two finite homomorphisms is again finite. Moreover, if $A \to B$ is a finite homomorphism, then for every homomorphism $A \to C$, the induced homomorphism $C = A \otimes_A C \to B \otimes_A C$ is finite. A more interesting property that we will need is that if $A \hookrightarrow B$ is an injective finite homomorphism, with A and B domains, then A is a field if and only if B is a field (for a proof, see Review Sheet 1).

Remark 2.1. If $A \hookrightarrow B$ is an injective, finite homomorphism between two domains, and $K = \mathrm{Frac}(A)$ and $L = \mathrm{Frac}(B)$, then the induced injective homomorphism $K \hookrightarrow L$ is finite. Indeed, by tensoring the inclusion $A \hookrightarrow B$ with K, we obtain a finite, injective homomorphism $K \hookrightarrow K \otimes_A B$ between domains. Note that $K \otimes_A B$ is a ring of fractions of B, hence the canonical homomorphism $K \otimes_A B \to L$ is injective. Since K is a field, it follows that $K \otimes_A B$ is a field, and thus $K \otimes_A B = L$. In particular, we see that $[L : K] < \infty$.

Theorem 2.2. *Let k be a field and A a finitely generated k-algebra which is an integral domain, with fraction field K. If $\mathrm{trdeg}(K/k) = n$, then there is a k-subalgebra B of A, such that*

1) *B is isomorphic as a k-algebra to $k[x_1, \dots, x_n]$, and*
2) *The inclusion $B \hookrightarrow A$ is finite.*

Proof. We only give the proof when k is infinite. This will be enough for our purpose, since in all our applications the field k will always contain an algebraically closed (hence infinite) field. For a proof in the general case, see [Mum88].

The fact that k is infinite will be used via the following property: for every nonzero polynomial $f \in k[x_1, \dots, x_r]$, there is $\lambda \in k^r$ such that $f(\lambda) \neq 0$. When $r = 1$, this follows from the fact that a nonzero polynomial in one variable has at most as many roots as its degree. The general case then follows by an easy induction on r.

Let $y_1, \dots, y_m \in A$ be generators of A as a k-algebra. In particular, we have $K = k(y_1, \dots, y_m)$, hence $m \geq n$. We will show, by induction on m, that we can find a change of variable of the form

$$y_i = \sum_{j=1}^{n} b_{i,j} z_j, \quad \text{for} \quad 1 \leq i \leq m, \quad \text{with} \quad \det(b_{i,j}) \neq 0,$$

(so that we have $A = k[z_1, \dots, z_m]$) such that the inclusion $k[z_1, \dots, z_n] \hookrightarrow A$ is finite. Note that this is enough: if $B = k[z_1, \dots, z_n]$, then it follows from Remark 2.1 that the induced field extension $\mathrm{Frac}(B) \hookrightarrow K$ is finite. Therefore we have

$$n = \mathrm{trdeg}(K/k) = \mathrm{trdeg}\big(k(z_1, \dots, z_n)/k\big),$$

hence $z_1, \ldots, z_n$ are algebraically independent.

If $m = n$, there is nothing to prove. Suppose now that $m > n$, hence $y_1, \ldots, y_m$ are algebraically dependent over k. Therefore there is a nonzero polynomial $f \in k[x_1, \ldots, x_m]$ such that $f(y_1, \ldots, y_m) = 0$. Suppose now that we write

$$y_i = \sum_{j=1}^{m} b_{i,j} z_j, \quad \text{with} \quad b_{i,j} \in k,\ \det(b_{i,j}) \neq 0.$$

Let $d = \deg(f)$ and let us write

$$f = f_d + f_{d-1} + \ldots + f_0, \quad \text{with} \quad \deg(f_i) = i \quad \text{or} \quad f_i = 0.$$

By assumption, we have $f_d \neq 0$. If we write

$$f = \sum_{\alpha \in \mathbf{Z}^m_{\geq 0}} c_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m},$$

then we have

$$\begin{aligned} 0 = f(y_1, \ldots, y_m) &= \sum_{\alpha} c_\alpha (b_{1,1} z_1 + \ldots + b_{1,m} z_m)^{\alpha_1} \cdots (b_{m,1} z_1 + \ldots + b_{m,m} z_m)^{\alpha_m} \\ &= f_d(b_{1,m}, \ldots, b_{m,m}) z_m^d + \text{lower degree terms in } z_m. \end{aligned}$$

Since we assume that k is infinite, we may choose the $b_{i,j}$ such that

$$\det(b_{i,j}) \cdot f_d(b_{1,m}, \ldots, b_{m,m}) \neq 0.$$

In this case, we see that after this linear change of variable, the inclusion

$$k[y_1, \ldots, y_{m-1}] \hookrightarrow k[y_1, \ldots, y_m]$$

is finite, since the right-hand side is generated as a module over the left-hand side by $1, y_m, \ldots, y_m^{d-1}$. Note that by Remark 2.1, the induced extension

$$k(y_1, \ldots, y_{m-1}) \hookrightarrow k(y_1, \ldots, y_m)$$

is finite, hence $\operatorname{trdeg}\big(k(y_1, \ldots, y_{m-1})/k\big) = n$. By induction, we can do a linear change of variable in $y_1, \ldots, y_{m-1}$, after which the inclusion

$$k[y_1, \ldots, y_n] \hookrightarrow k[y_1, \ldots, y_{m-1}]$$

is finite, in which case the composition

$$k[y_1, \ldots, y_n] \hookrightarrow k[y_1, \ldots, y_{m-1}] \hookrightarrow k[y_1, \ldots, y_m]$$

is finite. This completes the proof of the theorem. □

We will use Theorem 2.2 to prove Hilbert's Nullstellensatz in several steps.

Corollary 2.3. *If k is a field, A is a finitely generated k-algebra, and $K = A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal in A, then K is a finite extension of k.*

Proof. Note that K is a field which is finitely generated as a k-algebra. It follows from the theorem that if $n = \operatorname{trdeg}(K/k)$, then there is a finite injective homomorphism

$$k[x_1, \dots, x_n] \hookrightarrow K.$$

Since K is a field, it follows that $k[x_1, \dots, x_n]$ is a field, hence $n = 0$. Therefore K/k is finite. □

Corollary 2.4. *(Hilbert's Nullstellensatz, weak version) If k is an algebraically closed field, then every maximal ideal $\mathfrak{m}$ in $R = k[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$, for some $a_1, \dots, a_n \in k$.*

Proof. It follows from Corollary 2.3 that if $K = R/\mathfrak{m}$, the field extension K/k is finite. Since k is algebraically closed, the canonical homomorphism $k \to K$ is an isomorphism. In particular, for every i there is $a_i \in R$ such that $x_i - a_i \in \mathfrak{m}$. Therefore we have $(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$ and since both ideals are maximal, they must be equal. □

We can now prove Hilbert's Nullstellensatz, in its strong form.

Proof of Theorem 1.9. It follows from Corollary 2.4 that given any ideal $\mathfrak{a}$ of R, different from R, the zero-locus $V(\mathfrak{a})$ of $\mathfrak{a}$ is nonempty. Indeed, since $\mathfrak{a} \neq R$, there is a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. By Corollary 2.4, we have

$$\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n) \quad \text{for some} \quad a_1, \dots, a_n \in k.$$

In particular, we see that $a = (a_1, \dots, a_n) \in V(\mathfrak{m}) \subseteq V(J)$. We will use this fact in the polynomial ring $R[y] = k[x_1, \dots, x_n, y]$; this is *Rabinovich's trick*.

It is clear that for every ideal J in R we have the inclusion

$$\operatorname{rad}(J) \subseteq I\big(V(J)\big).$$

In order to prove the reverse inclusion, suppose that $f \in I\big(V(J)\big)$. Consider now the ideal $\mathfrak{a}$ in $R[y]$ generated by J and by $1 - fy$. If $\mathfrak{a} \neq R[y]$, we have seen that there is $(a_1, \dots, a_n, b) \in V(\mathfrak{a})$. By definition of $\mathfrak{a}$, this means that $g(a_1, \dots, a_n) = 0$ for all $g \in J$ (that is, $(a_1, \dots, a_n) \in V(J)$) and $1 = f(a_1, \dots, a_n)g(b)$. In particular, we have $f(a_1, \dots, a_n) \neq 0$, contradicting the fact that $f \in I\big(V(J)\big)$.

We thus conclude that $\mathfrak{a} = R$. Therefore we can find $f_1, \dots, f_r \in J$ and $g_1, \dots, g_{r+1} \in R[y]$ such that

$$\sum_{i=1}^{r} f_i(x)g_i(x, y) + \big(1 - f(x)y\big)g_{r+1}(x, y) = 1. \tag{1}$$

We now consider the R-algebra homomorphism $R[y] \to R_f$ that maps y to $\frac{1}{f}$. The relation (1) gives

$$\sum_{i=1}^{r} f_i(x)g_i\big(x, 1/f(x)\big) = 1$$

and after clearing the denominators (recall that R is a domain), we see that there is a positive integer N such that $f^N \in (f_1, \dots, f_r)$, hence $f \in \operatorname{rad}(J)$. This completes the proof of the theorem. □

3. The topology on the affine space

In this section we begin making use of the fact that the ring $k[x_1, \ldots, x_n]$ is Noetherian. Recall that a (commutative) ring R is Noetherian if the following equivalent conditions hold:

i) Every ideal in R is finitely generated.
ii) There is no infinite strictly increasing sequence of ideals of R.
iii) Every nonempty family of ideals of R has a maximal element

For this and other basic facts about Noetherian rings and modules, see Review Sheet 2. A basic result in commutative algebra is Hilbert's basis theorem: if R is a Noetherian ring, then $R[x]$ is Noetherian (a proof is given in Review Sheet 2). In particular, since a field k is trivially Noetherian, a recursive application of the theorem implies that every polynomial algebra $k[x_1, \ldots, x_n]$ is Noetherian.

As in the previous sections, we fix an algebraically closed field k and a positive integer n. The fact that the ring $R = k[x_1, \ldots, x_n]$ is Noetherian has two immediate consequences. First, since every ideal is finitely generated, it follows that for every algebraic subset $W \subseteq \mathbf{A}^n$, there are finitely many polynomials $f_1, \ldots, f_r$ such that $W = V(f_1, \ldots, f_r)$. Second, we see via the correspondence in Corollary 1.10 that there is no infinite strictly decreasing sequence of closed subsets in $\mathbf{A}^n$.

Definition 3.1. A topological space X is *Noetherian* if there is no infinite strictly decreasing sequence of closed subsets in X.

We have thus seen that with the Zariski topology $\mathbf{A}^n$ is a Noetherian topological space. This implies that every subspace of $\mathbf{A}^n$ is Noetherian, by the following

Lemma 3.2. *If X is a Noetherian topological space and Y is a subspace of X, then Y is Noetherian.*

Proof. If we have a infinite strictly decreasing sequence of closed subsets of Y

$$F_1 \supsetneq F_2 \supsetneq \ldots,$$

consider the corresponding sequence of closures in X:

$$\overline{F_1} \supseteq \overline{F_2} \supseteq \ldots.$$

Since F_i is closed in Y, we have $\overline{F_i} \cap Y = F_i$ for all i, which implies that $\overline{F_i} \neq \overline{F_{i+1}}$ for every i. This contradicts the fact that X is Noetherian. $\square$

Remark 3.3. Note that every Noetherian topological space is quasi-compact: this follows from the fact that there is no infinite strictly increasing sequence of open subsets.

Example 3.4. The real line $\mathbf{R}$, with the usual Euclidean topology, is not Noetherian.

We now introduce an important notion.

Definition 3.5. A topological space X is *irreducible* if it is nonempty and whenever we write $X = X_1 \cup X_2$, with both X_1 and X_2 closed, we have $X_1 = X$ or $X_2 = X$. We say that X is *reducible* when it is not irreducible.

Remark 3.6. By passing to complements, we see that a topological space is irreducible if and only if it is nonempty and for every two nonempty open subsets U and V, the intersection $U \cap V$ is nonempty (equivalently, every nonempty open subset of X is dense in X).

Remarks 3.7.
1) If Y is a subset of X (with the subspace topology), the closed subsets of Y are those of the form $F \cap Y$, where F is a closed subset of X. It follows that Y is irreducible if and only if it is nonempty and whenever $Y \subseteq Y_1 \cup Y_2$, with Y_1 and Y_2 closed in X, we have $Y \subseteq Y_1$ or $Y \subseteq Y_2$.
2) If Y is an irreducible subset of X and if $Y \subseteq Y_1 \cup \ldots \cup Y_r$, with all Y_i closed in X, then there is i such that $Y \subseteq Y_i$. This follows easily by induction on r.
3) If Y and F are subsets of X, with F closed, then $Y \subseteq F$ if and only if $\overline{Y} \subseteq F$. It then follows from the description in 1) that Y is irreducible if and only if $\overline{Y}$ is irreducible.
4) If X is irreducible and U is a nonempty open subset of X, then it follows from Remark 3.6 that U is dense in X. Since X is irreducible, it follows from 3) that U is irreducible.

In the case of closed subsets of $\mathbf{A}^n$, the following proposition describes irreducibility in terms of the corresponding ideal.

Proposition 3.8. *If $W \subseteq \mathbf{A}^n$ is a closed subset, then W is irreducible if and only if $I(W)$ is a prime ideal in R.*

Proof. Note first that $W \neq \emptyset$ if and only if $I(W) \neq R$. Suppose first that W is irreducible and let $f, g \in R$ be such that $fg \in I(W)$. We can then write

$$W = \big(W \cap V(f)\big) \cup \big(W \cap V(g)$$

Since both subsets on the right-hand side are closed and W is irreducible, it follows that we have either $W = W \cap V(f)$ (in which case $f \in I(W)$) or $W = W \cap V(g)$ (in which case $g \in I(W)$). Therefore $I(W)$ is a prime ideal.

Conversely, suppose that $I(W)$ is prime and we write $W = W_1 \cup W_2$, with W_1 and W_2 closed. Arguing by contradiction, suppose that $W \neq W_i$ for $i = 1, 2$, in which case $I(W) \subsetneq I(W_i)$, hence we can find $f_i \in I(W_i) \smallsetminus I(W)$. On the other hand, we have $f_1 f_2 \in I(W_1) \cap I(W_2) = I(W)$, contradicting the fact that $I(W)$ is prime. $\square$

Example 3.9. Since R is a domain, it follows from the proposition that $\mathbf{A}^n$ is irreducible.

Example 3.10. If $L \subseteq \mathbf{A}^n$ is a linear subspace, then L is irreducible. Indeed, after a linear change of variables, we have $R = k[y_1, \ldots, y_n]$ such that $I(L) = (y_1, \ldots, y_r)$ for some $r \geq 1$, and this is clearly a prime ideal in R.

Example 3.11. The union of two lines in $\mathbf{A}^2$ is a reducible closed subset.

Proposition 3.12. *Let X be a Noetherian topological space. Given a closed, nonempty subset Y, there are finitely many irreducible closed subsets $Y_1, \ldots, Y_r$ such that*

$$Y = Y_1 \cup \ldots \cup Y_r.$$

We may clearly assume that the decomposition is minimal*, in the sense that $Y_i \not\subseteq Y_j$ for $i \neq j$. In this case $Y_1, \ldots, Y_r$ are unique up to reordering.*

The closed subsets $Y_1, \ldots, Y_r$ in the proposition are the *irreducible components* of Y and the decomposition in the proposition is the *irreducible decomposition* of Y.

Proof of Proposition 3.12. Suppose first that there are nonempty closed subsets Y of X that do not have such a decomposition. Since X is Noetherian, we may choose a minimal such Y. In particular, Y is not irreducible, hence we may write $Y = Y_1 \cup Y_2$, with Y_1 and Y_2 closed and strictly contained in Y. Note that Y_1 and Y_2 are nonempty, hence by the minimality of Y, we may write both Y_1 and Y_2 as finite unions of irreducible subsets. In this case, Y is also a finite union of irreducible subsets, a contradiction.

Suppose now that we have two minimal decompositions

$$Y = Y_1 \cup \ldots \cup Y_r = Y'_1 \cup \ldots \cup Y'_s,$$

with the Y_i and Y'_j irreducible. For every $i \leq r$, we get an induced decomposition

$$Y_i = \bigcup_{j=1}^{s} (Y_i \cap Y'_j),$$

with the $Y_i \cap Y'_j$ closed for all j. Since Y_i is irreducible, it follows that there is $j \leq s$ such that $Y_i = Y_i \cap Y'_j \subseteq Y'_j$. Arguing in the same way, we see that there is $\ell \leq r$ such that $Y'_j \subseteq Y_\ell$. In particular, we have $Y_i \subseteq Y_\ell$, hence by the minimality assumption, we have $i = \ell$, and therefore $Y_i = Y'_j$. By iterating this argument and by reversing the roles of the Y_α and the Y'_β, we see that $r = s$ and the Y_α and the Y'_β are the same up to relabeling. $\square$

Remark 3.13. It is clear that if X is a Noetherian topological space, W is a closed subset of X, and Z is a closed subset of W, then the irreducible decomposition of Z is the same whether considered in W or in X.

Recall that by a theorem due to Gauss, if R is a UFD, then the polynomial ring $R[x]$ is a UFD. A repeated application of this result gives that every polynomial ring $k[x_1, \ldots, x_n]$ is a UFD. In particular, a nonzero polynomial $f \in k[x_1, \ldots, x_n]$ is irreducible if and only if the ideal (f) is prime.

Example 3.14. Given a polynomial $f \in k[x_1, \ldots, x_n] \smallsetminus k$, the subset $V(f)$ is irreducible if and only if f is a power of an irreducible polynomial. In fact, if the irreducible decomposition of f is $f = cf_1^{m_1} \cdots f_r^{m_r}$, for some $c \in k^*$, then the irreducible components of $V(f)$ are $V(f_1), \ldots, V(f_r)$.

Exercise 3.15. Let Y be the algebraic subset of $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find the corresponding prime ideals.

Exercise 3.16. Show that if X and Y are topological spaces, with X irreducible, and $f\colon X \to Y$ is a continuous map, then $\overline{f(X)}$ is irreducible.

Exercise 3.17. Let X be a topological space, and consider a finite open cover

$$X = U_1 \cup \ldots \cup U_n,$$

where each U_i is nonempty. Show that X is irreducible if and only if the following hold:

i) Each U_i is irreducible.
ii) For every i and j, we have $U_i \cap U_j \neq \emptyset$.

Exercise 3.18. Let X be a Noetherian topological space and Y a subset X. Show that if $Y = Y_1 \cup \ldots \cup Y_r$ is the irreducible decomposition of Y, then $\overline{Y} = \overline{Y_1} \cup \ldots \cup \overline{Y_r}$ is the irreducible decomposition of $\overline{Y}$.

Exercise 3.19. Let X be a Noetherian topological space and Y a nonempty closed subset of X, with irreducible decomposition

$$Y = Y_1 \cup \ldots \cup Y_r.$$

Show that if U is an open subset of X, then the irreducible decomposition of $U \cap Y$ is given by

$$U \cap Y = \bigcup_{i, U \cap Y_i \neq \emptyset} (U \cap Y_i).$$

We end these general topological considerations by discussing the notion of locally closed subsets.

Definition 3.20. Let X be a topological space. A subset V of X is *locally closed* if for every $x \in V$, there is an open neighborhood U_x of x in X such that $U_x \cap V$ is closed in U_x.

Remark 3.21. One should contrast the above definition with the local characterization of closed subsets: V is closed in X if and only if for every $x \in X$, there is an open neighborhood U_x of x in X such that $U_x \cap V$ is closed in U_x.

Proposition 3.22. *If V is a subset of a topological space X, then the following are equivalent:*

i) *V is a locally closed subset.*
ii) *V is open in $\overline{V}$.*
iii) *We can write $V = U \cap F$, with U open and F closed.*

Proof. If V is locally closed, let us choose for every $x \in V$ an open neighborhood U_x of x as in the definition. In this case V is closed in U by Remark 3.21, hence $V = U \cap F$ for some F closed in X, proving i)⇒iii). In order to see iii)⇒ii), note that if if $V = U \cap F$, with U open and F closed, then $\overline{V} \subseteq F$, hence $V = U \cap \overline{V}$ is open in $\overline{V}$. Finally, the implication ii)⇒i) is clear: if $V = W \cap \overline{V}$ for some W open in X, then for every $x \in V$, if we take $U_x = W$, we have $U_x \cap V$ closed in U_x. □

Let $X \subseteq \mathbf{A}^n$ be a closed subset. We always consider on X the subspace topology. We now introduce a basis of open subsets on X.

Definition 3.23. A *principal affine open subset* of X is an open subset of the form

$$D_X(f) := X \smallsetminus V(f) = \{x \in X \mid f(x) \neq 0\},$$

for some $f \in k[x_1, \ldots, x_n]$.

Note that $D_X(f)$ is nonempty if and only if $f \notin I(X)$. It is clear that $D_X(f) \cap D_X(g) = D_X(fg)$. Every open subset of X can be written as $X \smallsetminus V(J)$ for some ideal J in R. Since J is finitely generated, we can write $J = (f_1, \ldots, f_r)$, in which case

$$X \smallsetminus V(J) = D_X(f_1) \cup \ldots \cup D_X(f_r).$$

Therefore every open subset of X is a finite union of principal affine open subsets of X. We thus see that the principal affine open subsets give a basis for the topology of X.

Exercise 3.24. Let X be a topological space and Y a locally closed subset of X. Show that a subset Z of Y is locally closed in X if and only if it is locally closed in Y.

4. Regular functions and morphisms

Definition 4.1. An *affine algebraic variety* (or affine variety, for short) is a a closed subset of some affine space $\mathbf{A}^n$. A *quasi-affine variety* is a locally closed subset of some affine space $\mathbf{A}^n$, or equivalently, an open subset of an affine algebraic variety. A quasi-affine variety is always endowed with the subspace topology.

The above is only a temporary definition: a (quasi)affine variety is not just a topological space, but it comes with more information that distinguishes which maps between such objects are allowed. We will later formalize this as a ringed space. We now proceed describing the "allowable" maps.

Definition 4.2. Let $Y \subseteq \mathbf{A}^n$ be a locally closed subset. A *regular function* on Y is a map $\varphi\colon Y \to k$ that can locally be given by a quotient of polynomial functions, that is, for every $y \in Y$, there is an open neighborhood U_y of y in Y, and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that

$$g(u) \neq 0 \quad \text{and} \quad \varphi(u) = \frac{f(u)}{g(u)} \quad \text{for all} \quad u \in U_y.$$

We write $\mathcal{O}(Y)$ for the set of regular functions on Y. If Y is an affine variety, then $\mathcal{O}(Y)$ is also called the *coordinate ring* of Y. By convention, we put $\mathcal{O}(Y) = 0$ if $Y = \emptyset$.

Remark 4.3. It is easy to see that $\mathcal{O}(Y)$ is a subalgebra of the k-algebra of functions $Y \to k$, with respect to point-wise operations. For example, suppose that φ_1 and φ_2 are regular functions, $y \in Y$ and U_1 and U_2 are open neighborhoods of y, and $f_1, f_2, g_1, g_2 \in k[x_1, \ldots, x_n]$ are such that for all $u \in U_y$ we have

$$g_i(u) \neq 0 \quad \text{and} \quad \varphi_i(u) = \frac{f_i(u)}{g_i(u)} \quad \text{for} \quad i = 1, 2.$$

If we take $U = U_1 \cap U_2$ and $f = f_1 g_2 + f_2 g_1$, $g = g_1 g_2$, then for all $u \in U$, we have

$$g(u) \neq 0 \quad \text{and} \quad (\varphi_1 + \varphi_2)(u) = \frac{f(u)}{g(u)}.$$

Remark 4.4. It follows from definition that if $\varphi\colon Y \to k$ is a regular function such that $\varphi(y) \neq 0$ for every $y \in Y$, then the function $\frac{1}{\varphi}$ is a regular function, too.

Example 4.5. If X is a locally closed subset of $\mathbf{A}^n$, then the projection π_i on the ith component, given by

$$\pi_i(a_1, \ldots, a_n) = a_i$$

induces a regular function $X \to k$. Indeed, if $f_i = x_i \in k[x_1, \ldots, x_n]$, then $\pi_i(a) = f_i(a)$ for all $a \in X$.

When Y is closed in $\mathbf{A}^n$, one can describe more precisely $\mathcal{O}(Y)$. It follows by definition that we have a k-algebra homomorphism

$$k[x_1, \ldots, x_n] \to \mathcal{O}(Y)$$

that maps a polynomial f to the function $\big(u \to f(u)\big)$. By definition, the kernel of this map is the ideal $I(Y)$. With this notation, we have the following

Proposition 4.6. *The induced k-algebra homomorphism*

$$k[x_1, \ldots, x_n]/I(Y) \to \mathcal{O}(Y)$$

is an isomorphism.

A similar description holds for principal affine open subsets of affine varieties. Suppose that Y is closed in $\mathbf{A}^n$ and $U = D_Y(h)$, for some $h \in k[x_1, \ldots, x_n]$. We have a k-algebra homomorphism

$$\Phi\colon k[x_1, \ldots, x_n]_h \to \mathcal{O}(U),$$

that maps $\frac{f}{h^m}$ to the map $\big(u \to f(u)/h(u)^m\big)$. With this notation, we have the following generalization of the previous proposition.

Proposition 4.7. *The above k-algebra homomorphism induces an isomorphism*

$$k[x_1, \ldots, x_n]_h/I(Y)_h \to \mathcal{O}\big(D_Y(h)\big).$$

Of course it is enough to prove this more general version.

Proof of Proposition 4.7. The kernel of Φ consists of those fractions $\frac{f}{h^m}$ such that $\frac{f(u)}{h(u)} = 0$ for every $u \in D_Y(h)$. It is clear that this condition is satisfied if $f \in I(Y)$. Conversely, if this condition holds, then $f(u)h(u) = 0$ for every $u \in Y$. Therefore $fh \in I(Y)$, hence $\frac{f}{h^m} = \frac{fh}{h^{m+1}} \in I(Y)_h$. This shows that Φ is injective.

We now show that Φ is surjective. Consider $\varphi \in \mathcal{O}\big(D_Y(h)\big)$. Using the hypothesis and the fact that $D_Y(h)$ is quasi-compact (being a Noetherian topological space), we can write

$$D_Y(h) = V_1 \cup \ldots \cup V_r$$

and we have $f_i, g_i \in k[x_1, \ldots, x_n]$ for $1 \leq i \leq r$ such that $g_i(u) \neq 0$ and $\varphi(u) = \frac{f_i(u)}{g_i(u)}$ for all $u \in V_i$ and all i. Since the principal affine open subsets form a basis for the topology on Y, we may assume that $V_i = D_Y(h_i)$ for all i, for some $h_i \in k[x_1, \ldots, x_n] \smallsetminus I(Y)$. Since $g_i(u) \neq 0$ for all $u \in Y \smallsetminus V(h_i)$, it follows from Theorem 1.9 that

$$h_i \in \operatorname{rad}\big(I(Y) + (g_i)\big).$$

After possibly replacing each h_i by a suitable power, and then by a suitable element with the same class mod $I(Y)$, we may and will assume that $h_i \in (g_i)$. Finally, after multiplying both f_i and g_i by a suitable polynomial, we may assume that $g_i = h_i$ for all i.

We know that on $D_Y(g_i) \cap D_Y(g_j) = D_Y(g_i g_j)$ we have

$$\frac{f_i(u)}{g_i(u)} = \frac{f_j(u)}{g_j(u)}.$$

Applying the injectivity statement for $D_Y(g_i g_j)$, we conclude that

$$\frac{f_i}{g_i} = \frac{f_j}{g_j} \quad \text{in} \quad k[x_1, \ldots, x_n]_{g_i g_j}/I(Y)_{g_i g_j}.$$

Therefore there is a positive integer N such that

$$(g_i g_j)^N (f_i g_j - f_j g_i) \in I(Y) \quad \text{for all} \quad i, j.$$

After replacing each f_i and g_i by $f_i g_i^N$ and g_i^{N+1}, respectively, we may assume that

$$f_i g_j - f_j g_i \in I(Y) \quad \text{for all} \quad i, j.$$

On the other hand, we have

$$D_Y(h) = \bigcup_{i=1}^{r} D_Y(g_i),$$

hence $Y \cap V(h) = Y \cap V(g_1, \ldots, g_r)$, and by Theorem 1.9, we have

$$\operatorname{rad}\big(I(Y) + (h)\big) = \operatorname{rad}\big(I(Y) + (g_1, \ldots, g_r)\big).$$

In particular, we can write

$$h^m - \sum_{i=1}^{r} a_i g_i \in I(Y) \quad \text{for some} \quad m \geq 1 \quad \text{and} \quad a_1, \ldots, a_r \in k[x_1, \ldots, x_n].$$

We claim that

$$\varphi = \Phi\left(\frac{a_1 f_1 + \ldots + a_r f_r}{h^m}\right).$$

Indeed, for $u \in D_Y(g_j)$, we have

$$\frac{f_j(u)}{g_j(u)} = \frac{a_1(u) f_1(u) + \ldots + a_r(u) f_r(u)}{h(u)^m}$$

since

$$h(u)^m f_j(u) = \sum_{i=1}^{r} a_i(u) g_i(u) f_j(u) = \left(\sum_{i=1}^{r} a_i(u) f_i(u)\right) g_j(u).$$

This completes the proof of the claim and thus that of the proposition. $\square$

Example 4.8. In general, it is not the case that a regular function admits a global description as the quotient of two polynomial functions. Consider, for example the closed subset W of $\mathbf{A}^4$ defined by $x_1x_2 = x_3x_4$. Inside W we have the plane L given by $x_2 = x_3 = 0$. We define the regular function $\varphi\colon W \smallsetminus L \to k$ given by

$$\varphi(u_1, u_2, u_3, u_4) = \begin{cases} \frac{u_1}{u_3}, & \text{if } u_3 \neq 0; \\ \frac{u_4}{u_2}, & \text{if } u_2 \neq 0. \end{cases}$$

It is an easy exercise to check that there are no polynomials $P, Q \in k[x_1, x_2, x_3, x_4]$ such that

$$Q(u) \neq 0 \quad \text{and} \quad \varphi(u) = \frac{P(u)}{Q(u)} \quad \text{for all} \quad u \in W \smallsetminus L.$$

We now turn to maps between quasi-affine varieties. If Y is a subset of $\mathbf{A}^m$ and $f\colon X \to Y$ is a map, then the composition $X \to Y \hookrightarrow \mathbf{A}^m$ is written as $(f_1, \ldots, f_m)$, with $f_i\colon X \to k$. We often abuse notation writing $f = (f_1, \ldots, f_m)$.

Definition 4.9. If $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ are locally closed subsets, a map $f = (f_1, \ldots, f_m)\colon X \to Y$ is a *morphism* if $f_i \in \mathcal{O}(X)$ for all i.

Remark 4.10. It follows from definition that $f\colon X \to Y$ is a morphism if and only if the composition

$$X \to Y \hookrightarrow \mathbf{A}^m$$

is a morphism

Remark 4.11. If $X \subseteq \mathbf{A}^n$ is a locally closed subset, then a morphism $X \to \mathbf{A}^1$ is the same as a regular function $X \to k$.

Example 4.12. If X is a locally closed of $\mathbf{A}^n$, then the inclusion map $\iota\colon X \to \mathbf{A}^n$ is a morphism (this follows from Example 4.5). This implies that the identity map $1_X\colon X \to X$ is a morphism.

Proposition 4.13. *If X and Y are quasi-affine varieties, then every morphism $f\colon X \to Y$ is continuous.*

Proof. Suppose that X and Y are locally closed in $\mathbf{A}^n$ and $\mathbf{A}^m$, respectively, and write $f = (f_1, \ldots, f_m)$. We will show that if $V \subseteq Y$ is a closed subset, then $f^{-1}(V)$ is a closed subset of X. By assumption, we can write

$$V = Y \cap V(I) \quad \text{for some ideal} \quad I \subseteq k[x_1, \ldots, x_n].$$

In order to check that $f^{-1}(V)$ is closed, it is enough to find for every $x \in X$ an open neighborhood U_x of x in X such that $U_x \cap f^{-1}(V)$ is closed in U_x (see Remark 3.21). Since each f_i is a regular function, after replacing X by a suitable open neighborhood of x, we may assume that there are $P_i, Q_i \in k[x_1, \ldots, x_n]$ such that

$$Q_i(u) \neq 0 \quad \text{and} \quad f_i(u) = \frac{P_i(u)}{Q_i(u)} \quad \text{for all} \quad u \in X.$$

For every $h \in I$, there are polynomials $A_h, B_h \in k[x_1, \dots, x_n]$ such that

$$B_h(u) \neq 0 \quad \text{and} \quad h\left(\frac{P_1(u)}{Q_1(u)}, \dots, \frac{P_m(u)}{Q_m(u)}\right) = \frac{A_h(u)}{B_h(u)} \quad \text{for all} \quad u \in X.$$

It is then clear that for $u \in X$ we have $u \in f^{-1}(V)$ if and only if $A_h(u) = 0$ for all $h \in I$. Therefore $f^{-1}(V)$ is closed. □

Proposition 4.14. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms between quasi-affine varieties, the composition $g \circ f$ is a morphism.*

Proof. Suppose that $X \subseteq \mathbf{A}^m$, $Y \subseteq \mathbf{A}^n$ and $Z \subseteq \mathbf{A}^q$ are locally closed subsets and let us write $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_q)$. We need to show that $g_i \circ f \in \mathcal{O}(X)$ for $1 \leq i \leq q$. Let us fix such i, a point $x \in X$, and let $y = f(x)$. Since $g_i \in \mathcal{O}(Y)$ is a morphism, there is an open neighborhood V_y of y and $P, Q \in k[x_1, \dots, x_n]$ such that

$$Q(u) \neq 0 \quad \text{and} \quad g_i(u) = \frac{P(u)}{Q(u)} \quad \text{for all} \quad u \in V_y.$$

Similarly, since f is a morphism, we can find an open neighborhood U_x of x and $A_j, B_j \in k[x_1, \dots, x_m]$ for $1 \leq j \leq n$ such that

$$B_j(u) \neq 0 \quad \text{and} \quad f_j(u) = \frac{A_j(u)}{B_j(u)} \quad \text{for all} \quad u \in U_x.$$

It follows from Proposition 4.13 that $U_x \cap f^{-1}(V_y)$ is open and we have

$$g_i \circ f(u) = \frac{P\left(\frac{A_1(u)}{B_1(u)}, \dots, \frac{A_n(u)}{B_n(u)}\right)}{Q\left(\frac{A_1(u)}{B_1(u)}, \dots, \frac{A_n(u)}{B_n(u)}\right)}.$$

After clearing the denominators, we see that indeed, $g_i \circ f$ is a regular function in the neighborhood of x. □

It follows from Proposition 4.14 (and Example 4.12) that we may consider the *category of quasi-affine varieties* over k, whose objects are locally closed subsets of affine spaces over k, and whose arrows are the morphisms as defined above. Moreover, since a regular function on X is the same as a morphism $X \to \mathbf{A}^1$, we see that if $f\colon X \to Y$ is a morphism of quasi-affine varieties, we get an induced map

$$f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X), \quad f^{\#}(\varphi) = \varphi \circ f.$$

This is clearly a morphism of k-algebras. By mapping every quasi-affine variety X to $\mathcal{O}(X)$ and every morphism $f\colon X \to Y$ to $f^{\#}$, we obtain a contravariant functor from the category of quasi-affine varieties over k to the category of k-algebras.

Definition 4.15. A morphism $f\colon X \to Y$ is an *isomorphism* if it is an isomorphism in the above category. It is clear that this is the case if and only if f is bijective and f^{-1} is a morphism.

The following result shows that for affine varieties, this functor induces an anti-equivalence of categories. Let $\mathcal{A}fVar_k$ be the full subcategory of the category of quasi-affine varieties whose objects consist of the closed subsets of affine spaces over k and let $\mathcal{C}_k$ denote the category whose objects are reduced, finitely generated k-algebras and whose arrows are the morphisms of k-algebras.

Theorem 4.16. *The contravariant functor*

$$\mathcal{A}fVar_k \to \mathcal{C}_k$$

that maps X to $\mathcal{O}(X)$ and $f\colon X \to Y$ to $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is an anti-equivalence of categories.

Proof. Note first that if X is an affine variety, then $\mathcal{O}(X)$ is indeed a reduced, finitely generated k-algebra. Indeed, if X is a closed subset of $\mathbf{A}^n$, then it follows from Proposition 4.6 that we have an isomorphism $\mathcal{O}(X) \simeq k[x_1, \ldots, x_n]/I(X)$, which gives the assertion.

In order to show that the functor is an anti-equivalence of categories, it is enough to check two things:

i) For every affine varieties X and Y, the map

$$\mathrm{Hom}_{\mathcal{A}fVar_k}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_k}\big(\mathcal{O}(Y), \mathcal{O}(X)\big), \quad f \to f^{\#}$$

is a bijection.

ii) For every reduced, finitely generated k-algebra A, there is an affine variety X with $\mathcal{O}(X) \simeq A$.

The assertion in ii) is clear: since A is finitely generated, we can find an isomorphism $A \simeq k[x_1, \ldots, x_m]/J$, for some positive integer m and some ideal J. Moreover, since A is reduced, J is a radical ideal. If $X = V(J) \subseteq \mathbf{A}^m$, then it follows from Theorem 1.9 that $J = I(X)$ and therefore $\mathcal{O}(X) \simeq A$ by Proposition 4.6.

In order to prove the assertion in i), suppose that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. By Proposition 4.6, we have canonical isomorphisms

$$\mathcal{O}(X) \simeq k[x_1, \ldots, x_m]/I(X) \quad \text{and} \quad \mathcal{O}(Y) \simeq k[y_1, \ldots, y_n]/I(Y).$$

If $f\colon X \to Y$ is a morphism and we write $f = (f_1, \ldots, f_n)$, then $f^{\#}(\overline{y_i}) = \overline{f}_i$. Since f is determined by the classes $\overline{f_1}, \ldots, \overline{f_n}$ mod $I(X)$, it is clear that the map in i) is injective.

Suppose now that $\alpha\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a morphism of k-algebras and let $f_i \in k[x_1, \ldots, x_m]$ be such that $\overline{f_i} = \alpha(\overline{y_i}) \in \mathcal{O}(X)$. It is then clear that $f = (f_1, \ldots, f_n)$ gives a morphism $X \to \mathbf{A}^n$. Its image lies inside Y since for every $g \in I(Y)$ we have $g(f_1, \ldots, f_n) \in I(X)$, hence $g\big(f(u)\big) = 0$ for all $u \in X$. Therefore f gives a morphism $X \to Y$ such that $f^{\#} = \alpha$. $\square$

Definition 4.17. We extend somewhat the notion of affine variety by saying that a quasi-affine variety is affine if it is isomorphic (in the category of quasi-affine varieties) to a closed subset of some affine space.

An important example that does not come directly as a closed subset of an affine space is provided by the following proposition.

Proposition 4.18. *Let X be a closed subset of $\mathbf{A}^n$ and $U = D_X(g)$, for some $g \in k[x_1, \ldots, x_n]$. If J is the ideal in $k[x_1, \ldots, x_n, y]$ generated by $I(X)$ and $1 - g(x)y$, then U is isomorphic to $V(J)$. In particular, U is an affine variety*[2]*.*

Proof. Define $\varphi\colon U \to V(J)$ by $\varphi(u) = \big(u, 1/g(u)\big)$. It is clear that $\varphi(u)$ lies indeed in $V(J)$ and that φ is a morphism. Moreover, we also have a morphism $\psi\colon V(J) \to U$ induced by the projection onto the first n components. It is straightforward to check that φ and ψ are inverse to each other. □

Notation 4.19. If X is a quasi-affine variety and $f \in \mathcal{O}(X)$, then we put

$$D_X(f) = \{u \in X \mid f(u) \neq 0\}.$$

If X is affine, say it is isomorphic to the closed subset Y of $\mathbf{A}^n$, then f corresponds to the restriction to Y of some $g \in k[x_1, \ldots, x_n]$. In this case, it is clear that $D_X(f)$ is isomorphic to $D_Y(g)$, hence it is an affine variety.

Remark 4.20. If X is a locally closed subset of $\mathbf{A}^n$, then X is open in $\overline{X}$. Since the principal affine open subsets of $\overline{X}$ give a basis of open subsets for the topology of X, it follows from Proposition 4.18 that the open subsets of X that are themselves affine varieties give a basis for the topology of X.

Exercise 4.21. Suppose that $f\colon X \to Y$ is a morphism of affine algebraic varieties, and consider the induced homomorphism $f^\sharp\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. Show that if $u \in \mathcal{O}(Y)$, then

i) We have $f^{-1}(D_Y(u)) = D_X(w)$, where $w = f^\sharp(u)$.
ii) The induced ring homomorphism

$$\mathcal{O}(D_Y(u)) \to \mathcal{O}(D_X(w))$$

can be identified with the homomorphism

$$\mathcal{O}(Y)_u \to \mathcal{O}(X)_w$$

induced by $f^\sharp$ by localization.

Exercise 4.22. Let X be an affine algebraic variety, and let $\mathcal{O}(X)$ be the ring of regular functions on X. For every ideal J of $\mathcal{O}(X)$, let

$$V(J) := \{p \in X \mid f(p) = 0 \text{ for all } f \in J\}.$$

For $S \subseteq X$, consider the following ideal of $\mathcal{O}(X)$

$$I_X(S) := \{f \in \mathcal{O}(X) \mid f(p) = 0 \text{ for all } p \in S\}.$$

Show that for every subset S of X and every ideal J in $\mathcal{O}(X)$, we have

$$V\big(I_X(S)\big) = \overline{S} \quad \text{and} \quad I_X\big(V(J)\big) = \operatorname{rad}(J).$$

In particular, the maps $V(-)$ and $I_X(-)$ define order-reversing inverse bijections between the closed subsets of X and the radical ideals in $\mathcal{O}(X)$. Via this correspondence, the

[2]This justifies calling these subsets principal affine open subsets.

points of X correspond to the maximal ideals in $\mathcal{O}(X)$. This generalizes the case $X = \mathbf{A}^n$ that was discussed in Section 1.

We have seen that a morphism $f\colon X \to Y$ between affine varieties is determined by the corresponding k-algebra homomorphism $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. For such a morphism, it follows from the above exercise that the closed subsets in X and Y are in bijection with the radical ideals in $\mathcal{O}(X)$ and, respectively, $\mathcal{O}(Y)$. In the next proposition we translate the operations of taking the image and inverse image as operations on ideals.

Proposition 4.23. *Let $f\colon X \to Y$ be a morphism of affine varieties and $\varphi = f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the corresponding k-algebra homomorphism. For a point x in X or Y, we denote by $\mathfrak{m}_x$ the corresponding maximal ideal.*

i) *If $x \in X$ and $y = f(x)$, then $\mathfrak{m}_y = \varphi^{-1}(\mathfrak{m}_x)$.*
ii) *More generally, if $\mathfrak{a}$ is an ideal in $\mathcal{O}(X)$ and $W = V(\mathfrak{a})$, then $I_Y\big(\overline{f(W)}\big) = \varphi^{-1}\big(I_X(W)\big)$.*
iii) *In particular, we have $I_Y\big(\overline{f(X)}\big) = \ker(\varphi)$. Therefore $\overline{f(X)} = Y$ if and only if φ is injective.*
iv) *If $\mathfrak{b}$ is an ideal in $\mathcal{O}(Y)$ and $Z = V(\mathfrak{b})$, then $f^{-1}(Z) = V\big(\mathfrak{b} \cdot \mathcal{O}(X)\big)$.*

Proof. The assertion in i) is a special case of that in ii), hence we begin by showing ii). We have

$$\begin{aligned} I_Y\big(\overline{f(W)}\big) = I_Y\big(f(W)\big) &= \{g \in \mathcal{O}(Y) \mid g\big(f(x)\big) = 0 \text{ for all } x \in W\} \\ &= \{g \in \mathcal{O}(Y) \mid \varphi(g) \in I_X(W)\} = \varphi^{-1}\big(I_X(W)\big). \end{aligned}$$

By taking $W = X$, we obtain the assertion in iii)

Finally, if $\mathfrak{b}$ and Z are as in iv), we see that

$$f^{-1}(Z) = \{x \in X \mid g\big(f(x)\big) = 0 \text{ for all } g \in \mathfrak{b}\} = V\big(\mathfrak{b} \cdot \mathcal{O}(X)\big).$$

□

Remark 4.24. If $f\colon X \to Y$ is a morphism of affine varieties, then $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective if and only if f factors as $X \xrightarrow{g} Z \xrightarrow{\iota} Y$, where Z is a closed subset of Y, ι is the inclusion map, and g is an isomorphism.

Exercise 4.25. Let $Y \subseteq \mathbf{A}^2$ be the *cuspidal curve* defined by the equation $x^2 - y^3 = 0$. Construct a bijective morphism $f\colon \mathbf{A}^1 \to Y$. Is it an isomorphism ?

Exercise 4.26. Suppose that $\mathrm{char}(k) = p > 0$, and consider the map $f\colon \mathbf{A}^n \to \mathbf{A}^n$ given by $f(a_1, \ldots, a_n) = (a_1^p, \ldots, a_n^p)$. Show that f is a morphism of affine algebraic varieties, and that it is a homeomorphism, but it is not an isomorphism.

Exercise 4.27. Use Exercise 3.16 to show that the affine variety

$$M^r_{m,n}(k) := \{B \in M_{m,n}(k) \mid \mathrm{rk}(B) \leq r\}$$

is irreducible.

Exercise 4.28. Let $n \geq 2$ be an integer.

i) Show that the set

$$B_n = \left\{ (a_0, a_1, \ldots, a_n) \in \mathbf{A}^{n+1} \mid \operatorname{rank} \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} \leq 1 \right\}$$

is a closed subset of $\mathbf{A}^{n+1}$.

ii) Show that

$$B_n = \{(s^n, s^{n-1}t, \ldots, t^n) \mid s, t \in k\}.$$

Deduce that B_n is irreducible.

Exercise 4.29. In order to get an example of a quasi-affine variety which is not affine, consider $U = \mathbf{A}^2 \smallsetminus \{0\}$. Show that the canonical homomorphism $\mathcal{O}(\mathbf{A}^2) \to \mathcal{O}(U)$ is an isomorphism and deduce that U is not affine.

Exercise 4.30. Show that $\mathbf{A}^1$ is *not* isomorphic to any proper open subset of itself.

Exercise 4.31. Show that if X is a quasi-affine variety such that $\mathcal{O}(X) = k$, then X consists of only one point.

5. Local rings and rational functions

Let X be a quasi-affine variety and W an irreducible closed subset of X.

Definition 5.1. The *local ring* of X at W is the k-algebra

$$\mathcal{O}_{X,W} := \varinjlim_{U \cap W \neq \emptyset} \mathcal{O}(U).$$

Here the direct limit is over the open subsets of X with $U \cap W \neq \emptyset$, ordered by reverse inclusion, and where for $U_1 \subseteq U_2$, the map $\mathcal{O}(U_2) \to \mathcal{O}(U_1)$ is given by restriction of functions.

Remark 5.2. Note that the poset indexing the above direct limit is filtering: given any two open subsets U_1 and U_2 that intersect W nontrivially, we have $U_1 \cap U_2 \cap W \neq \emptyset$ (we use here the fact that W is irreducible). Because of this, the elements of $\mathcal{O}_{X,W}$ can be described as pairs (U, φ), where U is open with $W \cap U \neq \emptyset$ and $\varphi \in \mathcal{O}(U)$, modulo the following equivalence relation:

$$(U_1, \varphi_1) \sim (U_2, \varphi_2)$$

if there is an open subset $U \subseteq U_1 \cap U_2$, with $U \cap W \neq \emptyset$, such that $\varphi_1|_U = \varphi_2|_U$. Operations are defined by restricting to the intersection: for example, we have

$$(U_1, \varphi_1) + (U_2, \varphi_2) = (U_1 \cap U_2, \varphi_1|_{U_1 \cap U_2} + \varphi_2|_{U_1 \cap U_2}).$$

In order to describe $\mathcal{O}_{X,W}$, we begin with the following lemma.

Lemma 5.3. *If W is an irreducible closed subset of X and V is an open subset of X with $V \cap W \neq \emptyset$, we have a canonical k-algebra isomorphism*

$$\mathcal{O}_{X,W} \simeq \mathcal{O}_{V, W \cap V}.$$

Proof. The assertion follows from the fact that the following subset

$$\{U \subseteq V \mid U \text{ open}, U \cap W \neq \emptyset\} \subseteq \{U \subseteq X \mid U \text{ open}, U \cap W \neq \emptyset\}$$

is final. Explicitly, we have the morphism

$$\mathcal{O}_{V,W\cap V} \to \mathcal{O}_{X,W}, \quad (U,\varphi) \to (U,\varphi),$$

with inverse

$$\mathcal{O}_{X,W} \to \mathcal{O}_{V,W\cap V}, \quad (U,\varphi) \to (U \cap V, \varphi|_{U\cap V}).$$

□

Given a quasi-affine variety X, the open subsets of X that are affine varieties give a basis for the topology of X (see Remark 4.20). By Lemma 5.3, we see that it is enough to compute $\mathcal{O}_{X,W}$ when X is an affine variety. This is the content of the next result.

Proposition 5.4. *Let X be an affine variety and W an irreducible closed subset of X. If $\mathfrak{p} \subseteq \mathcal{O}(X)$ is the prime ideal corresponding to W, then we have a canonical isomorphism*

$$\mathcal{O}_{X,W} \simeq \mathcal{O}(X)_{\mathfrak{p}}.$$

In particular, $\mathcal{O}_{X,W}$ is a local ring, with maximal ideal consisting of classes of pairs (U,φ), with $\varphi_{U\cap W} = 0$.

Proof. Since the principal affine open subsets of X form a basis for the topology of X, we obtain using Proposition 4.7 a canonical isomorphism

$$\mathcal{O}_{X,W} \simeq \varinjlim_{f} \mathcal{O}(X)_f,$$

where the direct limit on the right-hand side is over those $f \in \mathcal{O}(X)$ such that $D_X(f) \cap W \neq \emptyset$. This condition is equivalent to $f \notin \mathfrak{p}$ and it is straightforward to check that the maps $\mathcal{O}(X)_f \to \mathcal{O}(X)_{\mathfrak{p}}$ induce an isomorphism

$$\varinjlim_{f} \mathcal{O}(X)_f \simeq \mathcal{O}(X)_{\mathfrak{p}}.$$

The last assertion in the proposition follows easily from the fact that $\mathcal{O}(X)_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}\mathcal{O}(X)_{\mathfrak{p}}$ □

There are two particularly interesting cases of this definition. First, if we take $W = \{x\}$, for a point $x \in X$, we obtain the local ring $\mathcal{O}_{X,x}$ of X *at* x. Its elements are *germs of regular functions* at x. This is a local ring, whose maximal ideal consists of germs of functions vanishing at x. As we will see, this local ring is responsible for the properties of X in a neighborhood of x. If X is an affine variety and $\mathfrak{m}$ is the maximal ideal corresponding to x, then Proposition 5.4 gives an isomorphism

$$\mathcal{O}_{X,x} \simeq \mathcal{O}(X)_{\mathfrak{m}}.$$

Exercise 5.5. Let $f\colon X \to Y$ be a morphism of quasi-affine varieties, and let $Z \subseteq X$ be a closed irreducible subset. Recall that by Exercise 3.16, we know that $W := \overline{f(Z)}$ is irreducible. Show that we have an induced morphism of k-algebras

$$g\colon \mathcal{O}_{Y,W} \to \mathcal{O}_{X,Z}$$

and that g is a local homomorphism of local rings (that is, it maps the maximal ideal of $\mathcal{O}_{Y,W}$ inside the maximal ideal of $\mathcal{O}_{X,Z}$). If X and Y are affine varieties, and

$$\mathfrak{p} = I_X(Z) \quad \text{and} \quad \mathfrak{q} = I_Y(W) = (f^{\#})^{-1}(\mathfrak{p}),$$

then via the isomorphisms given by Proposition 5.4, g gets identified to the homomorphism

$$\mathcal{O}(Y)_{\mathfrak{q}} \to \mathcal{O}(X)_{\mathfrak{p}}$$

induced by $f^{\#}$ via localization.

Exercise 5.6. Let X and Y be quasi-affine varieties. By the previous exercise, if $f\colon X \to Y$ is a morphism, $p \in X$ is a point, and $f(p) = q$, then f induces a local ring homomorphism $\varphi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$.

i) Show that if $f'\colon X \to Y$ is another morphism with $f'(p) = q$, and induced homomorphism $\varphi'\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, then $\varphi = \varphi'$ if and only if there is an open neighborhood U of p such that $f|_U = g|_U$.
ii) Show that given any local morphism of local k-algebras $\psi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, there is an open neighborhood W of p, and a morphism $g\colon W \to Y$ with $g(p) = q$, and inducing ψ.
iii) Deduce that $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,q}$ are isomorphic as k-algebras if and only if there are open neighborhoods W of p and V of q, and an isomorphism $h\colon W \to V$, with $h(p) = q$.

Another important example of local ring of X occurs when X is an irreducible variety and we take $W = X$. The resulting local ring is, in fact, a field, the *field of rational functions* $k(X)$ of X. Indeed, if $U \subseteq X$ is an affine open subset, then it follows from Lemma 5.3 and Proposition 5.4 that $k(X)$ is isomorphic to the field of fractions of the domain $\mathcal{O}(X)$. The elements of $k(X)$ are *rational functions* on X, that is, pairs (U, φ), where U is a nonempty open subset of X and $\varphi\colon U \to k$ is a regular function, where we identify two such pairs if the two functions agree on some nonempty open subset of their domains (in fact, as we will see shortly, in this case they agree on the intersection of their domains). We now discuss in more detail rational functions and, more generally, rational maps.

Lemma 5.7. *If X and Y are quasi-affine varieties and f_1 and f_2 are two morphisms $X \to Y$, then the subset*

$$\{a \in X \mid f_1(a) = f_2(a)\} \subseteq X$$

is closed.

Proof. If Y is a locally closed subset in $\mathbf{A}^n$, then we write $f_i = (f_{i,1}, \ldots, f_{i,n})$ for $i = 1, 2$. With this notation, we have

$$\{a \in X \mid f_1(a) = f_2(a)\} = \bigcap_{j=1}^{n} \{a \in X \mid (f_{1,j} - f_{2,j})(a) = 0\},$$

hence this set is closed in X, since each function $f_{1,j} - f_{2,j}$ is regular, hence continuous. $\square$

Definition 5.8. Let X and Y be quasi-affine varieties. A *rational map* $f\colon X \dashrightarrow Y$ is given by a pair (U, φ), where U is a dense, open subset of X and $\varphi\colon U \to Y$ is a morphism, and where we identify (U_1, φ_1) with (U_2, φ_2) if there is an open dense subset $V \subseteq U_1 \cap U_2$ such that $\varphi_1|_V = \varphi_2|_V$. In fact, in this case we have $\varphi_1|_{U_1 \cap U_2} = \varphi_2|_{U_1 \cap U_2}$ by Lemma 5.7. We also note that since U_1 and U_2 are dense open subsets of X, then also $U_1 \cap U_2$ is a dense subset of X.

Remark 5.9. If $f\colon X \dashrightarrow Y$ is a rational map and (U_i, φ_i) are the representatives of f, then we can define a map $\varphi\colon U = \bigcup_i U_i \to Y$ by $\varphi(u) = \varphi_i(u)$ if $u \in U_i$. This is well-defined and it is a morphism, since its restriction to each of the U_i is a morphism. Moreover, (U, φ) is a representative of f. The open subset U, the largest one on which a representative of f is defined, is the *domain of definition* of f.

Definition 5.10. Given a quasi-affine variety X, the set of rational functions $X \dashrightarrow k$ is denoted by $k(X)$. Since the intersection of two dense open sets is again open and dense, we may define the sum and product of two rational functions. For example, given two rational functions with representatives (U_1, φ_1) and (U_2, φ_2), we define their sum by the representative

$$(U_1 \cap U_2, \varphi_1|_{U_1 \cap U_2} + \varphi_2|_{U_1 \cap U_2}),$$

and similarly for the product. It is straightforward to see that using also scalar multiplication, $k(X)$ is a k-algebra. Note that when X is irreducible, we recover our previous definition.

Exercise 5.11. Let X be a quasi-affine variety, and let $X_1, \ldots, X_r$ be its irreducible components. Show that there is a canonical isomorphism

$$k(X) \simeq k(X_1) \times \cdots \times k(X_r).$$

Exercise 5.12. Let W be the closed subset in $\mathbf{A}^2$, defined by $x^2 + y^2 = 1$. What is the domain of definition of the rational function on W given by $\frac{1-y}{x}$?

Our next goal is to define a category in which the arrows are given by rational function. For simplicity, we only consider irreducible varieties.

Definition 5.13. A morphism $f\colon X \to Y$ is *dominant* if $Y = \overline{f(X)}$. Equivalently, for every nonempty open subset $V \subseteq Y$, we have $f^{-1}(V) \neq \emptyset$. Note that if U is open and dense in X, then f is dominant if and only if the composition $U \hookrightarrow X \xrightarrow{f} Y$ is dominant. We can thus define the same notion for rational maps: if $f\colon X \dashrightarrow Y$ is a rational map with representative (U, φ), we say that f is *dominant* if $\varphi\colon U \to Y$ is dominant.

Suppose that X, Y, and Z are irreducible quasi-affine varieties and $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ are rational maps, with f dominant. In this case we may define the composition $g \circ f$, which is a rational map; moreover, if g is dominant, too, then $g \circ f$ is dominant. Indeed, choose a representative (U, φ) for f and a representative (V, ψ) for g. Since the morphism $\varphi\colon U \to Y$ is dominant, it follows that $W := \varphi^{-1}(V)$ is nonempty. We then take $g \circ f$ to be the rational function defined by the composition

$$W \xrightarrow{f|_W} V \longrightarrow Z.$$

It is straightforward to see that this independent of the representatives for f and g. Moreover, if g is dominant, then $g \circ f$ is dominant: if Z' is a nonempty open subset of Z, then $\psi^{-1}(Z')$ is nonempty and open since g is dominant and therefore $\varphi^{-1}\big(\psi^{-1}(Z')\big)$ is nonempty, since f is dominant.

It is clear that the identity map is dominant. Moreover, composition of dominant rational map is associative. We thus obtain a category in which the objects are the irreducible quasi-affine varieties over k and the set $\mathrm{Hom}_{\mathrm{rat}}(X, Y)$ of arrows from X to Y consists of the dominant rational maps $X \dashrightarrow Y$, with the composition defined above. We are then led to the following important concept.

Definition 5.14. A rational dominant map $f\colon X \dashrightarrow Y$ between irreducible quasi-affine varieties is *birational* if it is an isomorphism in the above category. More precisely, this is the case if there is a dominant rational map $g\colon Y \dashrightarrow X$ such that

$$g \circ f = 1_X \quad \text{and} \quad f \circ g = 1_Y.$$

A *birational morphism* is a morphism which is birational as a rational map. Two irreducible quasi-affine varieties X and Y are *birational* if there is a birational map $X \dashrightarrow Y$.

This notion plays a fundamental role in the classification of algebraic varieties. On one hand, birational varieties share interesting geometric properties. On the other hand, classifying algebraic varieties up to birational equivalence turns out to be a more reasonable endeavor than classifying varieties up to isomorphism.

Example 5.15. If U is an open subset of the irreducible quias-affine variety X, then the inclusion map $i\colon U \hookrightarrow X$ is a birational morphism. Its inverse is given by the rational map represented by the identity morphism of U.

Example 5.16. An interesting example, which we will come back to later, is given by the morphism

$$f\colon \mathbf{A}^n \to \mathbf{A}^n, \quad f(x_1, \dots, x_n) = (x_1, x_1x_2, \dots, x_1x_n).$$

Note that the linear subspace given $L = (x_1 = 0)$ is mapped to 0, but f induces an isomorphism

$$\mathbf{A}^n \smallsetminus L = f^{-1}(\mathbf{A}^n \smallsetminus L) \to \mathbf{A}^n \smallsetminus L,$$

with inverse given by $g(y_1, \dots, y_n) = (y_1, y_2/y_1, \dots, y_n/y_1)$.

Example 5.17. Let X be the closed subset of $\mathbf{A}^2$ (on which we denote the coordinates by x and y), defined by $x^2 - y^3 = 0$. Let $f\colon \mathbf{A}^1 \to X$ be the morphism given by $f(t) = (t^3, t^2)$. Note that f is birational: if $g\colon X \smallsetminus \{(0,0)\} \to \mathbf{A}^1$ is the morphism given by $g(u, v) = \frac{u}{v}$, then g gives a rational map $X \dashrightarrow \mathbf{A}^1$ that is an inverse of f. Note that since $f^{-1}(0,0) = \{0\}$, it follows that the morphism f is bijective, However, f is not an isomorphism: otherwise, by Theorem 4.16 the induced homomorphism

$$f^{\#}\colon \mathcal{O}(X) = k[x, y]/(x^2 - y^3) \to k[t], \quad f^{\#}(x) = t^3, f^{\#}(y) = t^2$$

would be an isomorphism. However, it is clear that t is not in the image.

If $f\colon X \dashrightarrow Y$ is a rational, dominant map, then by taking $Z = \mathbf{A}^1$, we see that by precomposing with f we obtain a map

$$f^{\#}\colon k(Y) \to k(X).$$

It is straightforward to see that this is a field homomorphism.

Theorem 5.18. *We have an anti-equivalence of categories between the category of irreducible quasi-affine varieties and dominant rational maps and the category of finite type field extensions of k and k-algebra homomorphisms, that maps a variety X to $k(X)$ and a rational dominant map $f\colon X \dashrightarrow Y$ to $f^{\#}\colon k(Y) \to k(X)$.*

Proof. It is clear that we have a contravariant functor as described in the theorem. Note that if X is an irreducible quasi-affine variety, then $k(X)$ is a finite type extension of k: indeed, if U is an affine open subset of X, then we have $k(X) \simeq k(U) \simeq \mathrm{Frac}\big(\mathcal{O}(U)\big)$.

In order to show that this functor is an anti-equivalence, it is enough to prove the following two statements:

i) Given any two quasi-affine varieties X and Y, the map
$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) \to \mathrm{Hom}_{k-\mathrm{alg}}\big(k(Y), k(X)\big)$$
is bijective.

ii) Given any finite type field extension K/k, there is an irreducible quasi-affine variety X such that $k(X) \simeq K$.

The assertion in ii) is easy to see: if $K = k(a_1, \ldots, a_n)$, let $A = k[a_1, \ldots, a_n]$. We can thus write $A \simeq k[x_1, \ldots, x_n]/P$ for some (prime) ideal P and if $X = V(P) \subseteq \mathbf{A}^n$, then X is irreducible and $k(X) \simeq K$.

In order to prove i), suppose that X and Y are locally closed in $\mathbf{A}^m$ and, respectively, $\mathbf{A}^n$. Since X and Y are open in $\overline{X}$ and $\overline{Y}$, respectively, by Proposition 3.22, and since inclusions of open subsets are birational, it follows that the inclusions $X \hookrightarrow \overline{X}$ and $Y \hookrightarrow \overline{Y}$ induce an isomorphism

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) \simeq \mathrm{Hom}_{\mathrm{rat}}(\overline{X}, \overline{Y}),$$

and also isomorphisms

$$k(X) \simeq k(\overline{X}) \quad \text{and} \quad k(Y) \simeq k(\overline{Y}).$$

We may thus replace X and Y by $\overline{X}$ and $\overline{Y}$, respectively, in order to assume that X and Y are closed subsets of the respective affine spaces.

It is clear that

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) = \bigcup_{g \in \mathcal{O}(X)} \mathrm{Hom}_{\mathrm{dom}}\big(D_X(g), Y\big),$$

where each set on the right-hand side consists of the dominant morphisms $D_X(g) \to Y$. Moreover, since $\mathcal{O}(Y)$ is a finitely generated k-algebra, we have

$$\mathrm{Hom}_{k-\mathrm{alg}}\big(k(Y), k(X)\big) = \bigcup_{g \in \mathcal{O}(X)} \mathrm{Hom}_{\mathrm{inj}}\big(\mathcal{O}(Y), \mathcal{O}(X)_g\big),$$

where each set on the right-hand side consists of the injective k-algebra homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)_g$. Since the map $f \to f^\#$ gives a bijection

$$\mathrm{Hom}_{\mathrm{dom}}\big(D_X(g), Y\big) \simeq \mathrm{Hom}_{\mathrm{inj}}\big(\mathcal{O}(Y), \mathcal{O}(X)_g\big)$$

by Theorem 4.16 and Proposition 4.23, this completes the proof. □

Corollary 5.19. *A dominant rational map $f\colon X \to Y$ between irreducible quasi-affine varieties X and Y is birational if and only if the induced homomorphism $f^\#\colon k(Y) \to k(X)$ is an isomorphism.*

Remark 5.20. A rational map $f\colon X \dashrightarrow Y$ between the irreducible quasiaffine varieties X and Y is birational if and only if there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces an isomorphism $U \simeq V$. The "if" assertion is clear, so we only need to prove the converse. Suppose that f is defined by the morphism $\varphi\colon X' \to Y$ and its inverse g is defined by the morphism $\psi\colon Y' \to X$, where $X' \subseteq X$ and $Y' \subseteq Y$ are open subsets. The equality $f \circ g = 1_Y$ as rational functions implies by Lemma 5.7 that the composition

$$\psi^{-1}(X') \xrightarrow{\psi} X' \xrightarrow{\varphi} Y$$

is the inclusion. In particular, we deduce that

$$\psi\big(\psi^{-1}(X')\big) \subseteq \varphi^{-1}\big(\psi^{-1}(X')\big) \subseteq \varphi^{-1}(Y').$$

Similarly, the equality of rational functions $g \circ f = 1_X$ shows that the composition

$$\varphi^{-1}(Y') \xrightarrow{\varphi} Y' \xrightarrow{\psi} X$$

is the inclusion; in particular, we obtain

$$\varphi\big(\varphi^{-1}(Y')\big) \subseteq \psi^{-1}(X').$$

It is now clear that φ and ψ induce inverse morphisms between $\varphi^{-1}(Y')$ and $\psi^{-1}(X')$.

Exercise 5.21. Let $X \subset \mathbf{A}^n$ be a hypersurface defined by an equation $f(x_1, \ldots, x_n) = 0$, where $f = f_{d-1} + f_d$, with f_{d-1} and f_d nonzero, homogeneous of degrees $d-1$ and d, respectively. Show that if X is irreducible, then X is birational to $\mathbf{A}^{n-1}$.

6. Products of (quasi-)affine varieties

We begin by showing that for positive integers m and n, the Zariski topology on $\mathbf{A}^m \times \mathbf{A}^n = \mathbf{A}^{m+n}$ is finer than the product topology.

Proposition 6.1. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets, then $X \times Y$ is a closed subset of $\mathbf{A}^{m+n}$.*

Proof. The assertion follows from the fact that if $X = V(I)$ and $Y = V(J)$, for ideals $I \subseteq k[x_1, \ldots, x_m]$ and $J \subseteq k[y_1, \ldots, y_n]$, then

$$X \times Y = V(I \cdot R + J \cdot R),$$

where $R = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. □

Corollary 6.2. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are open (respectively, locally closed) subsets, then $X \times Y$ is an open (respectively, locally closed) subset of $\mathbf{A}^{m+n} = \mathbf{A}^m \times \mathbf{A}^n$. In particular, the topology on $\mathbf{A}^m \times \mathbf{A}^n$ is finer than the product topology.*

Proof. The assertion for open subsets follows from Proposition 6.1 and the fact that

$$\mathbf{A}^{m+n} \smallsetminus X \times Y = \big(\mathbf{A}^m \times (\mathbf{A}^n \smallsetminus Y)\big) \cup \big((\mathbf{A}^m \smallsetminus X) \times \mathbf{A}^n\big).$$

The assertion for locally closed subsets follows immediately from the assertions for open and closed subsets. □

Corollary 6.3. *Given any quasi-affine varieties X and Y, the topology on $X \times Y$ is finer than the product topology.*

Proof. If X and Y are locally closed subsets of $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively, then $X \times Y$ is a locally closed subset of $\mathbf{A}^{m+n}$. Since the topology on $\mathbf{A}^{m+n}$ is finer than the product topology by the previous corollary, we are done. □

Example 6.4. The topology on $\mathbf{A}^m \times \mathbf{A}^n$ is strictly finer than the product topology. For example, the diagonal in $\mathbf{A}^1 \times \mathbf{A}^1$ is closed (defined by $x - y \in k[x, y]$), but it is not closed in the product topology.

Remark 6.5. If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are locally closed subsets, then $X \times Y \subseteq \mathbf{A}^{m+n}$ is a locally closed subset, and the two projections induce morphisms $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$. These make $X \times Y$ the product of X and Y in the category of quasi-affine varieties over k. Indeed, given two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, it is clear that there is a unique morphism $\varphi\colon Z \to X \times Y$ such that $p \circ \varphi = f$ and $q \circ \varphi = g$, namely $\varphi = (f, g)$.

This implies, in particular, that if $f\colon X \to X'$ and $g\colon Y \to Y'$ are isomorphisms, then the induced map $X \times Y \to X' \times Y'$ is an isomorphism.

Proposition 6.6. *If $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are locally closed subsets, then the two projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are open*[3].

Proof. We show that p is open, the argument for q being entirely similar. Note first that by Remark 6.5, we may replace X and Y by isomorphic quasi-affine varieties. Moreover, if we write $X = \bigcup_i X_i$ and $Y = \bigcup_j Y_j$, then for any open subset W of $X \times Y$, we have

$$p(W) = p\left(\bigcup_{i,j} W \cap (X_i \times Y_j)\right),$$

hence if order to show that p is open, it is enough to show that each projection $X_i \times Y_j \to X_i$ is open. By Remark 4.20, both X and Y can be covered by open subsets that are affine varieties. We may thus assume that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. Let $k[x_1, \ldots, x_m]$ and $k[y_1, \ldots, y_n]$ be the rings corresponding to $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively. Using again the fact that every open subset of $X \times Y$ is a union of principal affine open

[3]Recall that a continuous map $\varphi\colon Z_1 \to Z_2$ is *open* if for every open subset U of Z_1, its image $\varphi(U)$ is open in Z_2.

subsets, we see that it is enough to show that $p(W)$ is open in $\mathbf{A}^m$ for a nonempty subest $W = D_{X\times Y}(h)$, where $h \in k[x, y]$.

Let us write

$$h = \sum_{i=1}^{r} f_i(x)g_i(y). \tag{2}$$

We may and will assume that for the given set W, h and the expression (2) are chosen such that r is minimal. Note that in this case, the classes $\overline{g_1}, \ldots, \overline{g_r}$ in $\mathcal{O}(Y)$ are linearly independent over k. Indeed, if this is not the case and $\sum_{i=1}^{r} \lambda_i g_i = P(y) \in I(Y)$, such that $\lambda_j \neq 0$ for some j, then we may take $h' = h - \lambda_j^{-1} f_j(x)P(y)$; we then have $D_{X\times Y}(h') = D_{X\times Y}(h)$ and we can write

$$h' = \sum_{i, i\neq j} \big(f_i(x) - \lambda_i\lambda_j^{-1} f_j(x)\big)g_i(y),$$

contradicting the minimality of r.

Suppose now that $u \in p(W)$. This implies that $u \in X$ such that there is $v \in Y$, with $h(u, v) \neq 0$. In particular, there is j such that $f_j(u) \neq 0$. It is enough to show that in this case $D_X(f_j)$, which contains u, is contained in $p(W)$. Suppose, arguing by contradiction, that there is $u' \in D_X(f_j) \smallsetminus p(W)$. This implies that for every $v \in Y$, we have $\sum_{i=1}^{r} f_i(u')g_i(v) = 0$, hence $\sum_{i=1}^{r} f_i(u')g_i \in I(Y)$. Since $f_j(u') \neq 0$, this contradicts the fact that the classes $\overline{g_1}, \ldots, \overline{g_r}$ in $\mathcal{O}(Y)$ are linearly independent over k. □

Corollary 6.7. *If X and Y are irreducible quasi-affine varieties, then $X\times Y$ is irreducible.*

Proof. We need to show that if U and V are nonempty, open subsets of $X \times Y$, then $U \cap V$ is nonempty. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the two projections. By the proposition, the nonempty subsets $p(U)$ and $p(V)$ of X are open. Since X is irreducible, we can find $a \in p(U) \cap p(V)$. In this case, the subsets $\{b \in Y \mid (a, b) \in U\}$ and $\{b \in Y \mid (a, b) \in V\}$ of Y are nonempty. They are also open: this follows from the fact that the map $Y \to X \times Y$, $y \to (a, y)$ is a morphism, hence it is continuous. Since Y is irreducible, these two subsets must intersect, hence there is a point $(a, b) \in U \cap V$. □

Our next goal is to describe the ideal defining the product of two affine varieties. Suppose that $X \subseteq \mathbf{A}^m$ and $Y \subseteq \mathbf{A}^n$ are closed subsets. We have seen in the proof of Proposition 6.1 that if $I(X) \subseteq \mathcal{O}(\mathbf{A}^m)$ and $I(Y) \subseteq \mathcal{O}(\mathbf{A}^n)$ are the ideals defining X and Y, respectively, then $X \times Y$ is the algebraic subset of $\mathbf{A}^{m+n}$ defined by

$$J := I(X) \cdot \mathcal{O}(\mathbf{A}^{m+n}) + I(Y) \cdot \mathcal{O}(\mathbf{A}^{m+n}).$$

We claim that, in fact, J is the ideal defining $X \times Y$, that is, J is a radical ideal. Note that $\mathcal{O}(\mathbf{A}^{m+n})$ is canonically isomorphic to $\mathcal{O}(\mathbf{A}^m) \otimes_k \mathcal{O}(\mathbf{A}^n)$ and by the right-exactness of the tensor product, we have

$$\mathcal{O}(\mathbf{A}^{m+n})/J \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$

The assertion that J is a radical ideal (or equivalently, that $\mathcal{O}(\mathbf{A}^{m+n})/J$ is a reduced ring is the content of the following

Proposition 6.8. *If X and Y are affine varieties, then the ring $\mathcal{O}(X)\otimes_k\mathcal{O}(Y)$ is reduced.*

Before giving the proof of the proposition, we need some algebraic preparations concerning separable extensions.

Lemma 6.9. *If k is any field and K/k is a finite, separable field extension, then for every field extension k'/k, the ring $K\otimes_k k'$ is reduced.*

Proof. Since K/k is finite and separable, it follows from the Primitive Element theorem that there is an element $u\in K$ such that $K=k(u)$. Moreover, separability implies that if $f\in k[x]$ is the minimal polynomial of u, then all roots of f in some algebraic closure of k are distinct. The isomorphism $K\simeq k[x]/(f)$ induces an isomorphism

$$K\otimes_k k'\simeq k'[x]/(f).$$

If $g_1,\ldots,g_r$ are the irreducible factors of f in $k'[x]$, any two of them are relatively prime (otherwise f would have multiple roots in some algebraic closure of k). It then follows from the Chinese Remainder theorem that we have an isomorphism

$$K\otimes_k k'\simeq\prod_{i=1}^{r}k'[x]/(g_i).$$

Since each factor on the right-hand side is a field (the polynomial g_i being irreducible), the product is a reduced ring. □

Lemma 6.10. *If k is a perfect*[4] *field and K/k is a finitely generated field extension, then there is a transcendence basis $x_1,\ldots,x_n$ of K over k such that K is separable over $k(x_1,\ldots,x_n)$.*

Proof. Of course, the assertion is trivial if $\mathrm{char}(k)=0$, hence we may assume that $\mathrm{char}(k)=p>0$. Let us write $K=k(x_1,\ldots,x_m)$. We may assume that $x_1,\ldots,x_n$ give a transcendence basis of K/k, and suppose that $x_{n+1},\ldots,x_{n+r}$ are not separable over $K':=k(x_1,\ldots,x_n)$, while $x_{n+r+1},\ldots,x_m$ are separable over K'. If $r=0$, then we are done. Otherwise, since x_{n+1} is not separable over K', it follows that there is an irreducible polynomial $f\in K'[T]$ such that $f\in K'[T^p]$ and such that $f(x_{n+1})=0$. We can find a nonzero $u\in k[x_1,\ldots,x_n]$ such that $g=uf\in k[x_1,\ldots,x_n,T^p]$.

We claim that there is $i\leq n$ such that $\frac{\partial g}{\partial x_i}\neq 0$. Indeed, otherwise we have $g\in k[x_1^p,\ldots,x_n^p,T^p]$, and since k is perfect, we have $k=k^p$, hence $g=h^p$ for some $h\in k[x_1,\ldots,x_n,T]$; this contradicts the fact that f is irreducible.

After relabeling the variables, we may assume that $i=n$. The assumption on i says that x_n is (algebraic and) separable over $K'':=k(x_1,\ldots,x_{n-1},x_{n+1})$. Note that since x_n is algebraic over K'' and K is algebraic over $k(x_1,\ldots,x_{n-1},x_n)$, it follows that K is algebraic over K'', and since all transcendence bases of K over k have the same number of elements, we conclude that $x_1,\ldots,x_{n-1},x_{n+1}$ is a transcendence basis of K over k. We may thus switch x_n and x_{n+1} to lower r. After finitely many steps, we obtain the conclusion of the lemma. □

[4]Recall that a field k is perfect if $\mathrm{char}(k)=0$ or $\mathrm{char}(k)=p$ and $k=k^p$. Equivalently, a field is perfect if every finite extension K/k is separable.

Proposition 6.11. *If k is a perfect field, then for every field extensions K/k and k'/k, the ring $K \otimes_k k'$ is reduced.*

Proof. We may assume that K is finitely generated over k. Indeed, we can write

$$K = \varinjlim_i K_i,$$

where the direct limit is over all $k \subseteq K_i \subseteq K$, with K_i/k finitely generated. Since we have an induced isomorphism

$$K \otimes_k k' \simeq \varinjlim_i K_i \otimes_k k',$$

and a direct limit of reduced rings is reduced, we see that it is enough to prove the proposition when K/k is finitely generated.

In this case we apply Lemma 6.10 to find a transcendence basis $x_1, \ldots, x_n$ of K/k such that K is separable over $K_1 := k(x_1, \ldots, x_n)$. We have

$$K \otimes_k k' = K \otimes_{K_1} K_1 \otimes_k k'.$$

Since $K_1 \otimes_k k'$ is a ring of fractions of $k'[x_1, \ldots, x_n]$, we have an injective homomorphism

$$K_1 \otimes_k k' \hookrightarrow K_2 := k'(x_1, \ldots, x_n).$$

By tensoring with K, we get an injective homomorphism

$$K \otimes_k k' \hookrightarrow K \otimes_{K_1} K_2.$$

Since K/K_1 is a finite separable extension, we deduce from Lemma 6.9 that $K \otimes_{K_1} K_2$ is reduced, hence $K \otimes_k k'$ is reduced. □

We can now prove our result about the coordinate ring of the product of two affine varieties.

Proof of Proposition 6.8. We will keep using the fact that the tensor product over k is an exact functor. Note first that we may assume that X and Y are irreducible. Indeed, let $X_1, \ldots, X_r$ be the irreducible components of X and $Y_1, \ldots, Y_s$ the irreducible components of Y. Since $X = X_1 \cup \ldots \cup X_r$, it is clear that the canonical homomorphism

$$\mathcal{O}(X) \to \prod_{i=1}^{r} \mathcal{O}(X_i)$$

is injective. Similarly, we have an injective homomorphism

$$\mathcal{O}(Y) \to \prod_{j=1}^{s} \mathcal{O}(Y_i)$$

and we thus obtain an injective homomorphism

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow \prod_{i,j} \mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j).$$

The right-hand side is a reduced ring if each $\mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j)$ is reduced, in which case $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. We thus may and will assume that both X and Y are irreducible.

We know that in this case $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are domains and let $k(X)$ and $k(Y)$ be the respective fraction fields. Since k is algebraically closed, it is perfect, hence $k(X) \otimes_k k(Y)$ is a reduced ring by Proposition 6.11. The inclusions

$$\mathcal{O}(X) \hookrightarrow k(X) \quad \text{and} \quad \mathcal{O}(Y) \hookrightarrow k(Y)$$

induce an injective homomorphism

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow k(X) \otimes_k k(Y),$$

which implies that $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. □

We now give another application of Lemma 6.10. We first make a definition.

Definition 6.12. A hypersurface in $\mathbf{A}^n$ is a closed subset of the form

$$\{u \in \mathbf{A}^n \mid f(u) = 0\} \quad \text{for some} \quad f \in k[x_1, \ldots, x_n] \smallsetminus k.$$

Proposition 6.13. *Every irreducible variety is birational to an (irreducible) hypersurface in an affine space* $\mathbf{A}^n$.

Proof. Let X be an irreducible variety, with function field $K = k(X)$. By Lemma 6.10, we can find a transcendence basis $x_1, \ldots, x_n$ of K/k such that K is separable over $k(x_1, \ldots, x_n)$. In this case, it follows from the Primitive Element theorem that there is $u \in K$ such that $K = k(x_1, \ldots, x_n, u)$. If $f \in k(x_1, \ldots, x_n)[t]$ is the minimal polynomial of u, then

$$K \simeq k(x_1, \ldots, x_n)[t]/(f).$$

It is easy to see that after multiplying u by a suitable nonzero element of $k[x_1, \ldots, x_n]$, we may assume that $f \in k[x_1, \ldots, x_n, t]$ and f is irreducible. In this case, we see by Theorem 5.18 that X is birational to the affine variety $V(f) \subseteq \mathbf{A}^{n+1}$. □

We end this section with some exercises about linear algebraic groups. We begin with a definition.

Definition 6.14. A *linear algebraic group* over k is an affine variety G over k that is also a group, and such that the multiplication $\mu\colon G \times G \to G$, $\mu(g, h) = gh$, and the inverse map $\iota\colon G \to G$, $\iota(g) = g^{-1}$ are morphisms of algebraic varieties. If G_1 and G_2 are linear algebraic groups, a *morphism of algebraic groups* is a morphism of affine varieties $f\colon G_1 \to G_2$ that is also a group homomorphism.

Linear algebraic groups over k form a category. In particular, we have a notion of isomorphism between linear algebraic groups: this is an isomorphism of affine algebraic varieties that is also a group isomorphism.

Exercise 6.15.
i) Show that $(k, +)$ and $(k^*, \cdot)$ are linear algebraic groups.
ii) Show that the set $\mathrm{GL}_n(k)$ of $n \times n$ invertible matrices with coefficients in k has a structure of linear algebraic group.
iii) Show that the set $\mathrm{SL}_n(k)$ of $n \times n$ matrices with coefficients in k and with determinant 1 has a structure of linear algebraic group.

iv) Show that if G and H are linear algebraic groups, then the product $G \times H$ has an induced structure of linear algebraic group. In particular, the (algebraic) *torus* $(k^*)^n$ is a linear algebraic group with respect to component-wise multiplication.

Definition 6.16. Let G be a linear algebraic group and X a quasi-affine variety. An *algebraic group action* of G on X is a (say, left) action of G on X such that the map $G \times X \to X$ giving the action is a morphism of algebraic varieties.

Exercise 6.17. Show that $\mathrm{GL}_n(k)$ has an algebraic action on $\mathbf{A}^n$.

Exercise 6.18. Let G be a linear algebraic group acting algebraically on an affine variety X. Show that in this case G has an induced linear action on $\mathcal{O}(X)$ given by

$$(g \cdot \varphi)(u) = \varphi\big(g^{-1}(u)\big).$$

While $\mathcal{O}(X)$ has in general infinite dimension over k, show that the action of G on $\mathcal{O}(X)$ has the following finiteness property: every element $f \in \mathcal{O}(X)$ lies in some finite-dimensional vector subspace V of $\mathcal{O}(X)$ that is preserved by the G-action (Hint: consider the image of f by the corresponding k-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(G) \otimes_k \mathcal{O}(X)$).

Exercise 6.19. Let G and X be as in the previous problem. Consider a system of k-algebra generators $f_1, \ldots, f_m$ of $\mathcal{O}(X)$, and apply the previous problem to each of these elements to show that there is a morphism of algebraic groups $G \to \mathrm{GL}_N(k)$, and an isomorphism of X with a closed subset of $\mathbf{A}^N$, such that the action of G on X is induced by the standard action of $\mathrm{GL}_N(k)$ on $\mathbf{A}^N$. Use a similar argument to show that every linear algebraic group is isomorphic to a closed subgroup of some $\mathrm{GL}_N(k)$.

Exercise 6.20. Show that the linear algebraic group $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ has an algebraic action on the space $M_{m,n}(k)$ (identified to $\mathbf{A}^{mn}$), induced by left and right matrix multiplication. What are the orbits of this action ? Note that the orbits are locally closed subsets of $M_{m,n}(k)$ (as we will see later, this is a general fact about orbits of algebraic group actions).

7. Affine toric varieties

In this section we discuss a class of examples of affine varieties that are associated to semigroups.

Definition 7.1. A *semigroup* is a set S endowed with an operation $+$ (we will use in general the additive notation) which is commutative, associative and has a unit element 0. If S is a semigroup, a *subsemigroup* of S is a subset $S' \subseteq S$ closed under the operation in S and such that $0_S \in S'$ (in which case, S' becomes a semigroup with the induced operation). A map $\varphi\colon S \to S'$ between two semigroups is a *semigroup morphism* if $\varphi(u_1 + u_2) = \varphi(u_1) + \varphi(u_2)$ for all u_1 and u_2, and if $\varphi(0) = 0$.

Example 7.2. i) Every Abelian group is a semigroup.
ii) The field k, endowed with the multiplication, is a semigroup.
iii) The set $\mathbf{N}$ of non-negative integers, with the addition, is a semigroup.
iv) The set $\{m \in \mathbf{N} \mid m \neq 1\}$ is a subsemigroup of $\mathbf{N}$.

v) If S_1 and S_2 are semigroups, then $S_1 \times S_2$ is a semigroup, with component-wise addition.

Given a semigroup S, we consider the semigroup algebra $k[S]$. This has a basis over k indexed by the elements of S. We denote the elements of this basis by χ^u, for $u \in S$. The multiplication is defined by $\chi^{u_1} \cdot \chi^{u_2} = \chi^{u_1+u_2}$ (hence $1 = \chi^0$). This is a k-algebra. Note that if $\varphi\colon S_1 \to S_2$ is a morphism of semigroups, then we get a morphism of k-algebras $k[S_1] \to k[S_2]$ that maps χ^u to $\chi^{\varphi(u)}$.

Example 7.3. We have an isomorphism

$$k[\mathbf{N}^r] \simeq k[x_1, \dots, x_r], \quad \chi^{e_i} \to x_i,$$

where e_i is the tuple that has 1 on the ith component and 0 on all the others. We similarly have an isomorphism

$$k[\mathbf{Z}^r] \simeq k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}].$$

Example 7.4. In general, if S_1 and S_2 are semigroups, we have a canonical isomorphism

$$k[S_1 \times S_2] \simeq k[S_1] \otimes_k k[S_2].$$

We will assume that our semigroups satisfy two extra conditions. First, we will assume that they are *finitely generated*: a semigroup S satisfies this property if it has finitely many generators $u_1, \dots, u_r \in S$ (this means that every element in S can be written as $\sum_{i=1}^r a_i u_i$, for some $a_1, \dots, a_r \in \mathbf{N}$). In other words, the unique morphism of semigroups $\mathbf{N}^r \to S$ that maps e_i to u_i for all i is surjective. Note that in this case, the induced k-algebra homomorphism

$$k[x_1, \dots, x_r] \simeq k[\mathbf{N}^r] \to k[S]$$

is onto, hence $k[S]$ is finitely generated.

We will also assume that S is *integral*, that is, it is isomorphic to a subsemigroup of a finitely generated, free Abelian group. Since we have an injective morphism of semigroups $S \hookrightarrow \mathbf{Z}^r$, we obtain an injective k-algebra homomorphism $k[S] \hookrightarrow k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$. In particular, $k[S]$ is a domain.

Exercise 7.5. Suppose that S is the image of a morphism of semigroups $\varphi\colon \mathbf{N}^r \to \mathbf{Z}^m$ (this is how semigroups are usually described). Show that the kernel of the induced surjective k-algebra homomorphism

$$k[x_1, \dots, x_r] \simeq k[\mathbf{N}^r] \to k[S]$$

is the ideal

$$\big(x^a - x^b \mid a, b \in \mathbf{N}^r, \varphi(a) = \varphi(b)\big).$$

We have seen that if S is an integral, finitely generated semigroup, then $k[S]$ is a finitely generated k-algebra, which is also a domain. Therefore it corresponds to an irreducible affine variety over k, uniquely defined up to canonical isomorphism. We will denote this variety[5] by $\mathrm{TV}(S)$. Its points are in bijection with the maximal ideals in $k[S]$,

[5]This is not standard notation in the literature.

or equivalently, with the k-algebra homomorphisms $k[S] \to k$. Such homomorphisms in turn are in bijection with the semigroup morphisms $S \to (k, \cdot)$. Via this bijection, if we consider $\varphi\colon S \to (k, \cdot)$ as a point in $\mathrm{TV}(S)$ and $\chi^u \in k[S]$, then

$$\chi^u(\varphi) = \varphi(u) \in k.$$

Given a morphism of finitely generated, integral semigroups $S \to S'$, the k-algebra homomorphism $k[S] \to k[S']$ corresponds to a morphism $\mathrm{TV}(S') \to \mathrm{TV}(S)$.

The affine variety $TV(S)$ carries more structure, induced by the semigroup S, which we now describe. First, we have a morphism

$$TV(S) \times TV(S) \to TV(S)$$

corresponding to the k-algebra homomorphism

$$k[S] \to k[S] \otimes_k k[S], \quad \chi^u \to \chi^u \otimes \chi^u.$$

At the level of points (identified, as above, to semigroup morphisms to k), this is given by

$$(\varphi, \psi) \to \varphi \cdot \psi, \quad \text{where} \quad (\varphi \cdot \psi)(u) = \varphi(u) \cdot \psi(u).$$

It is clear that the operation is commutative, associative, and has an identity element, given by the morphism $S \to k$ that takes constant value 1.

Remark 7.6. If $S \to S'$ is a morphism between integral, finitely generated semigroups, it is clear that the induced morphism of affine varieties $\mathrm{TV}(S') \to \mathrm{TV}(S)$ is compatible with the operation defined above.

Example 7.7. If $S = \mathbf{N}^r$, then the operation that we get on $\mathrm{TV}(S) = \mathbf{A}^r$ is given by

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n).$$

In particular, note that $\mathrm{TV}(S)$ is *not* a group.

Example 7.8. With the operation defined above, $\mathrm{TV}(\mathbf{Z})$ is a linear algebraic group isomorphic to $(k^*, \cdot)$. In general, if M is a finitely generated, free Abelian group, then the above operation makes $\mathrm{TV}(M)$ a linear algebraic group. In fact, we have $M \simeq \mathbf{Z}^r$, for some r, and therefore $\mathrm{TV}(M)$ is isomorphic, as an algebraic group, to the torus $(k^*)^r$ (see Exercise 6.15 for the definition of the algebraic tori). It follows from the lemma below that we can recover M from $\mathrm{TV}(M)$, together with the group structure, as

$$M \simeq \mathrm{Hom}_{\mathrm{alg-gp}}\big(\mathrm{TV}(M), k^*\big).$$

Lemma 7.9. *For every finitely generated, free Abelian groups M and M', the canonical map*

$$\mathrm{Hom}_{\mathbf{Z}}(M, M') \to \mathrm{Hom}_{\mathrm{alg-gp}}\big(\mathrm{TV}(M'), \mathrm{TV}(M)\big)$$

is a bijection.

Proof. A morphism of algebraic groups $\mathrm{TV}(M') \to \mathrm{TV}(M)$ is given by a k-algebra homomorphism $f\colon k[M] \to k[M']$ such that the induced diagram

$$\begin{tikzcd} k[M] \arrow[r, "f"] \arrow[d, "\alpha_M"] & k[M'] \arrow[d, "\alpha_{M'}"] \\ k[M] \otimes k[M] \arrow[r, "f\otimes f"] & k[M'] \otimes k[M'], \end{tikzcd}$$

is commutative, where α_M and $\alpha_{M'}$ are the k-algebra homomorphisms inducing the group structure. Given $u \in M$, we see that if $f(\chi^u) = \sum_{u' \in M'} a_{u,u'} \chi^{u'}$, then

$$\sum_{u' \in M'} a_{u,u'} \chi^{u'} \otimes \chi^{u'} = \sum_{u',v' \in M'} a_{u,u'} a_{u,v'} \chi^{u'} \otimes \chi^{v'}.$$

First, this implies that if u', $v' \in M'$ are distinct, then $a_{u,u'} \cdot a_{u,v'} = 0$. Therefore there is a unique $u' \in M'$ such that $a_{u,u'} \neq 0$ (note that $\chi^u \in k[M]$ is invertible, hence $f(\chi^u) \neq 0$). Moreover, for this u' we have $a_{u,u'}^2 = a_{u,u'}$, hence $a_{u,u'} = 1$. This implies that we have a (unique) map $\varphi\colon M \to M'$ such that f is given by $f(\chi^u) = \chi^{\varphi(u)}$. Since f is a ring homomorphism, we see that φ is a semigroup morphism. This shows that the map in the lemma is bijective. □

Exercise 7.10. Given an integral semigroup S, show that there is an injective semigroup morphism $\iota\colon S \hookrightarrow S^{\mathrm{gp}}$, where S^{gp} is a finitely generated Abelian group, that satisfies the following universal property: given any semigroup morphism $h\colon S \to A$, where A is an Abelian group, there is a unique group morphism $g\colon S^{\mathrm{gp}} \to A$ such that $g \circ \iota = h$. Hint: if $S \hookrightarrow M$ is an injective semigroup morphism, where M is a finitely generated, free Abelian group, then show that one can take S^{gp} to be the subgroup of M generated by S. Note that it follows from this description that S^{gp} is finitely generated (since M is) and S^{gp} is generated as a group by S.

Suppose now that S is an arbitrary integral, finitely generated semigroup. The semigroup morphism $\iota\colon S \to S^{\mathrm{gp}}$ induces a k-algebra homomorphism $k[S] \to k[S^{\mathrm{gp}}]$ and correspondingly a morphism of affine algebraic varieties $j\colon \mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$.

Lemma 7.11. *With the above notation, the morphism $j\colon \mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$ is an isomorphism onto a principal affine open subset of $\mathrm{TV}(S)$.*

Proof. Suppose that $u_1, \ldots, u_r$ is a finite system of generators of S. In this case S^{gp} is generated as a semigroup by $u_1, \ldots, u_r$, and $-(u_1 + \ldots + u_r)$. This shows that we can identify the homomorphism $k[S] \to k[S^{\mathrm{gp}}]$ with the localization homomorphism of $k[S]$ at $\chi^{u_1 + \ldots + u_s}$. □

Since the morphism $\mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$ is compatible with the operations on the two varieties, we conclude that in particular, the action of the torus $\mathrm{TV}(S^{\mathrm{gp}})$, considered as an open subset of $\mathrm{TV}(S)$, extends to an action of $\mathrm{TV}(S^{\mathrm{gp}})$ on $\mathrm{TV}(S)$. We are thus led to the following

Definition 7.12. An affine toric variety is an irreducible affine variety X, together with an open subset T that is (isomorphic to) a torus, such that the action of the torus on itself extends to an action of T of X.

We note that in the literature, it is common to require an affine toric variety to be normal, but we do not follow this convention. For the definition of normality and for the description in the context of toric varieties, see Definition 7.26 and Proposition 7.30 below.

We have seen that for every (integral, finitely generated) semigroup S, we obtain a toric variety $\mathrm{TV}(S)$. The following proposition shows that, in fact, every affine toric variety arises in this way.

Proposition 7.13. *Let X be an irreducible affine variety, $T \subseteq X$ an open subset which is a torus such that the action of T on itself extends to an action on X. Then there is a finitely generated, integral semigroup S and an isomorphism $X \simeq \mathrm{TV}(S)$ which induces an isomorphism of algebraic groups $T \simeq \mathrm{TV}(S^{\mathrm{gp}})$, and which is compatible with the action.*

Proof. Let $M = \mathrm{Hom}_{\mathrm{alg-gp}}(T, k^*)$, so that we have a canonical isomorphism $T \simeq \mathrm{TV}(M)$. The dominant inclusion morphism $T \to X$ induces an injective k-algebra homomorphism $f\colon \mathcal{O}(X) \to \mathcal{O}(T) = k[M]$, hence we may assume that $\mathcal{O}(X)$ is a subalgebra of $k[M]$. The fact that the action of T on itself extends to an action of T on X is equivalent to the fact that the k-algebra homomorphism

$$k[M] \to k[M] \otimes_k k[M], \quad \chi^u \to \chi^u \otimes \chi^u$$

induces a homomorphism $\mathcal{O}(X) \to k[M] \otimes_k \mathcal{O}(X)$. In other words, if $f = \sum_{u \in M} a_u \chi^u$ lies in $\mathcal{O}(X)$, then $\sum_{u \in M} a_u \chi^u \otimes \chi^u$ lies in $k[M] \otimes_k \mathcal{O}(X)$. This implies that for every $u \in M$ such that $a_u \neq 0$, we have $\chi^u \in \mathcal{O}(X)$. It follows that if $S = \{u \in M \mid \chi^u \in \mathcal{O}(X)\}$, then $\mathcal{O}(X) = k[S]$. It is clear that S is integral and since $k[S]$ is a finitely generated k-algebra, it follows easily that S is a finitely generated semigroup. In order to complete the proof of the proposition, it is enough to show that $M = S^{\mathrm{gp}}$.

It follows from Exercise 7.10 that we may take S^{gp} to be the subgroup of M generated by S. By hypothesis, the composition

$$TV(M) \xrightarrow{g} \mathrm{TV}(S^{\mathrm{gp}}) \xrightarrow{h} X = \mathrm{TV}(S)$$

is an isomorphism onto an open subset of X. Since we also know that h is an isomorphism onto an open subset of X, it follows that g gives is an isomorphism onto an open subset of $\mathrm{TV}(S^{\mathrm{gp}})$. In particular, this implies that g is injective. We now show that $M = S^{\mathrm{gp}}$.

Since M is a finitely generated, free Abelian group, we can find a basis $e_1, \ldots, e_n$ of M such that S^{gp} has a basis given by $a_1 e_1, \ldots, a_r e_r$, for some $r \leq n$ and some positive integers $a_1, \ldots, a_r$. In this case g gets identified to the morphism

$$(k^*)^n \to (k^*)^r, \quad (t_1, \ldots, t_n) \to (t_1^{a_1}, \ldots, t_r^{a_r}).$$

Since g is injective, we see that $r = n$. Moreover, if $a_j > 1$ for some j, then $\mathrm{char}(k) = p > 0$ and for every i we have $a_i = p^{e_i}$ for some nonnegative integer e_i. It is easy to see that in this case g is surjective (cf. Exercise 4.26). Since we know that it gives an isomorphism of

$\mathrm{TV}(M)$ with an open subset of $\mathrm{TV}(S^{\mathrm{gp}})$, it follows that g is an isomorphism. However, this implies $a_i = 1$ for all i. Therefore we have $S^{\mathrm{gp}} = M$. $\square$

We now turn to the description of toric morphisms. Suppose that X and Y are affine toric varieties, with tori $T_X \subseteq X$ and $T_Y \subseteq Y$.

Definition 7.14. With the above notation, a *toric morphism* $X \to Y$ is a morphism of algebraic varieties $f\colon X \to Y$ that induces a morphism of algebraic groups $g\colon T_X \to T_Y$.

Remark 7.15. Note that if $f\colon X \to Y$ is a toric morphism as above, then f is a morphism of varieties with torus action, in the sense that

$$f(t \cdot x) = g(t) \cdot f(x) \quad \text{for every } t \in T_X, x \in X.$$

Indeed, this follows by Lemma 5.7 from the fact that we have this equality for $(t, x) \in T_X \times T_X$.

If $\varphi\colon S_1 \to S_2$ is a semigroup morphism between two integral, finitely generated semigroups, we get an induced group morphism $S_1^{\mathrm{gp}} \to S_2^{\mathrm{gp}}$. We then obtain an induced morphism $f\colon \mathrm{TV}(S_2) \to \mathrm{TV}(S_1)$ that restricts to a morphism of algebraic groups $\mathrm{TV}(S_2^{\mathrm{gp}}) \to \mathrm{TV}(S_1^{\mathrm{gp}})$; therefore f is a toric morphism. The next proposition shows that all toric morphisms arise in this way, from a unique semigroup homomorphism.

Proposition 7.16. *If S_1 and S_2 are finitely generated, integral semigroups, then the canonical map*

$$\mathrm{Hom}_{\mathrm{semigp}}(S_1, S_2) \to \mathrm{Hom}_{\mathrm{toric}}\big(\mathrm{TV}(S_2), \mathrm{TV}(S_1)\big)$$

is a bijection.

Proof. By definition, a toric morphism $\mathrm{TV}(S_2) \to \mathrm{TV}(S_1)$ is given by a k-algebra homomorphism $k[S_1] \to k[S_2]$ such that the induced homomorphism $f\colon k[S_1^{\mathrm{gp}}] \to k[S_2^{\mathrm{gp}}]$ gives a morphism of algebraic groups $\mathrm{TV}(S_2^{\mathrm{gp}}) \to \mathrm{TV}(S_1^{\mathrm{gp}})$. It follows from Lemma 7.9 that we have a group morphism $\varphi\colon S_1^{\mathrm{gp}} \to S_2^{\mathrm{gp}}$ such that $f(\chi^u) = \chi^{\varphi(u)}$ for every $u \in S_1^{\mathrm{gp}}$. Since f induces a homomorphism $k[S_1] \to k[S_2]$, we have $\varphi(S_1) \subseteq S_2$, hence φ is induces a semigroup morphism $S_1 \to S_2$. This shows that the map in the proposition is surjective and the injectivity is straightforward. $\square$

Remark 7.17. We can combine the assertions in Proposition 7.13 and 7.16 as saying that the functor from the category of integral, finitely generated semigroups to the category of affine toric varieties, that maps S to $\mathrm{TV}(S)$, is an anti-equivalence of categories.

Example 7.18. If $S = \mathbf{N}^r$, then $\mathrm{TV}(S) = \mathbf{A}^r$, with the torus $(k^*)^r \subseteq \mathbf{A}^r$ acting by component-wise multiplication.

Example 7.19. If $S = \{m \in \mathbb{N} \mid m \neq 1\}$, then $S^{\mathrm{gp}} = \mathbb{Z}$. If we embed X in $\mathbb{A}^2$ as the curve with equation $u^3 - v^2 = 0$, then the embedding $T \simeq k^* \hookrightarrow X$ is given by $\lambda \to (\lambda^2, \lambda^3)$. The action of T on X is described by $\lambda \cdot (u, v) = (\lambda^2 u, \lambda^3 v)$.

Exercise 7.20. Show that if X and Y are affine toric varieties, with tori $T_X \subseteq X$ and $T_Y \subseteq Y$, then $X \times Y$ has a natural structure of toric variety, with torus $T_X \times T_Y$. Describe the semigroup corresponding to $X \times Y$ in terms of the semigroups of X and Y.

Exercise 7.21. Let S be the sub-semigroup of $\mathbb{Z}^3$ generated by e_1, e_2, e_3 and $e_1 + e_2 - e_3$. These generators induce a surjective morphism $f\colon k[\mathbb{N}^4] = k[t_1, \dots, t_4] \to k[S]$. Show that the kernel of f is generated by $t_1t_2 - t_3t_4$. We have $S^{\mathrm{gp}} = \mathbb{Z}^3$, the embedding of $T = (k^*)^3 \hookrightarrow X$ is given by $(\lambda_1, \lambda_2, \lambda_3) \to (\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2/\lambda_3)$, and the action of T on X is induced via this embedding by component-wise multiplication.

The following lemma provides a useful tool for dealing with torus-invariant objects. Consider $X = \mathrm{TV}(S)$ and let $T = \mathrm{TV}(S^{\mathrm{gp}})$ be the corresponding torus. As in the case of any algebraic group action, the action of T on X induces an action of T on $\mathcal{O}(X)$ (see Exercise 6.18). Explicitly, in our setting this is given by

$$\varphi \cdot \chi^u = \varphi(u)^{-1}\chi^u \quad \text{for all} \quad u \in S, \varphi \in \mathrm{Hom}_{\mathrm{gp}}(S^{\mathrm{gp}}, k^*).$$

Lemma 7.22. *With the above notation, a subspace $V \subseteq k[S]$ is T-invariant (that is, $t \cdot g \in V$ for every $g \in V$) if and only if it is S-homogenous, in the sense that for every $g = \sum_{u \in S} a_u\chi^u \in V$, we have $\chi^u \in V$ whenever $a_u \neq 0$.*

Proof. We only need to prove the "only if" part, the other direction being straightforward. By definition, V is T-invariant if and only if for every group morphism $\varphi\colon S^{\mathrm{gp}} \to k^*$ and every $g = \sum_{u \in S} a_u\chi^u \in V$, we have

$$\sum_{u \in S} a_u\varphi(u)^{-1}\chi^u \in V.$$

Iterating, we obtain

$$\sum_{u \in S} a_u\varphi(u)^{-m}\chi^u \in V \quad \text{for all} \quad m \geq 1. \tag{3}$$

Claim. Given pairwise distinct $u_1, \dots, u_d \in S$, we can find $\varphi \in T$ such that $\varphi(u_i) \neq \varphi(u_{i'})$ for $i \neq i'$. Indeed, let us choose an isomorphism $S^{\mathrm{gp}} \simeq \mathbf{Z}^n$, so that each u_i corresponds to $(a_{i,1}, \dots, a_{i,n})$. After adding to each $(a_{i,1}, \dots, a_{i,n})$ the element $(m, \dots, m)$ for $m \gg 0$, we may assume that $a_{i,j} \geq 0$ for all i and j. Since each polynomial

$$Q_{i,i'} = \prod_{j=1}^n x_j^{a_{i,j}} - \prod_{j=1}^n x_j^{a_{i',j}}, \quad \text{for} \quad i \neq i'$$

is nonzero, it follows that the open subset $U_{i,i'}$ defined by $Q_{i,i'} \neq 0$ is a nonempty subset of $\mathbf{A}^n$. Since $\mathbf{A}^n$ is irreducible, it follows that the intersection

$$(k^*)^n \cap \bigcap_{i \neq i'} U_{i,i'}$$

is nonempty, giving the claim.

By applying the claim to those $u \in S$ such that $a_u \neq 0$, we deduce from (3) and from the formula for the Vandermonde determinant that $\chi^u \in V$ for all u such that $a_u \neq 0$. $\square$

In the next two exercises we describe the torus-invariant subvarieties of $\mathrm{TV}(S)$ and the orbits of the torus action. We begin by defining the corresponding concept at the level of the semigroup.

Definition 7.23. A face F of a semigroup S is a subsemigroup such that whenever u_1, $u_2 \in S$ have $u_1 + u_2 \in F$, we have $u_1 \in F$ and $u_2 \in F$.

Note that if F is a face of S, then $S \smallsetminus F$ is a subsemigroup of S. Moreover, if S is generated by $u_1, \ldots, u_n$. then a face F of S is generated by those u_i that lie in F. In particular, if S is an integral, finitely generated semigroup, then S has only finitely many faces, and each of these is an integral, finitely generated semigroup.

Exercise 7.24. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T \subset X$. A subset Y of X is *torus-invariant* if $t \cdot Y \subseteq Y$ for every $t \in T$.

i) Show that a closed subset Y of X is torus-invariant if and only if each irreducible component of Y is torus-invariant.
ii) Show that the torus-invariant irreducible closed subsets of X are precisely the closed subsets defined by ideals of the form
$$\bigoplus_{u \in S \smallsetminus F} k\chi^u,$$
where F is a face of S.
iii) Show that if Y is the closed subset defined by the ideal in ii), then we have $\mathcal{O}(Y) \simeq k[F]$, hence Y has a natural structure of affine toric variety.

Exercise 7.25. Let $X = \mathrm{TV}(S)$ be an affine toric variety, with torus $T_X \subseteq X$.

i) Show that if $M \hookrightarrow M'$ is an injective morphism of finitely generated, free Abelian groups, then the induced morphism of tori $\mathrm{TV}(M') \to \mathrm{TV}(M)$ is surjective.
ii) Show that if F is a face of S with corresponding closed invariant subset Y, then the inclusion of semigroups $F \subseteq S$ induces a morphism of toric varieties $f_Y \colon X \to Y$, which is a retract of the inclusion $Y \hookrightarrow X$. Show that the torus O_F in Y is an orbit for the action of T_X on X.
iii) Show that the map $F \to O_F$ gives a bijection between the faces of S and the orbits for the T_X-action on X.

We now discuss normality for the varieties we defined. Recall that if $R \to S$ is a ring homomorphism, then the set of elements of S that are integral over R form a subring of S, the *integral closure* of R in S (see Proposition 2.2 in Review Sheet 1).

Definition 7.26. An integral domain A is *integrally closed* if it is equal to its integral closure in its field of fractions. It is *normal* if, in addition, it is Noetherian. An irreducible, affine variety X is *normal* if $\mathcal{O}(X)$ is a normal ring.

Remark 7.27. If A is an integral domain and B is the integral closure of A in its fraction field, then B is integrally closed. Indeed, the integral closure of B in K is integral over A (see Proposition 2.3 in Review Sheet 1), hence it is contained in B.

Example 7.28. Every UFD is integrally closed. Indeed, suppose that A is a UFD and $u = \frac{a}{b}$ lies in the fraction field of A and it is integral over A. We may assume that a and b are relatively prime. Consider a monic polynomial $f = x^m + c_1 x^{m-1} + \ldots c_m \in A[x]$ such that $f(u) = 0$. Since
$$a^m = -b \cdot (c_1 a^{m-1} + \ldots c_m b^{m-1}),$$

it follows that b divides a^m. Since b and a are relatively prime, it follows that b is invertible, hence $u \in A$.

In particular, we see that every polynomial ring $k[x_1, \ldots, x_n]$ is integrally closed.

Definition 7.29. An integral, finitely generated semigroup S is *saturated* if whenever $mu \in S$ for some $u \in S^{\mathrm{gp}}$ and some positive integer m, we have $u \in S$.

Proposition 7.30. *If S is an integral, finitely generated semigroup, the variety* $\mathrm{TV}(S)$ *is normal if and only if S is saturated.*

Proof. The rings $k[S] \subseteq k[S^{\mathrm{gp}}]$ have the same fraction field, and $k[S^{\mathrm{gp}}] \simeq k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ for some n, so $k[S^{\mathrm{gp}}]$ is normal, being a UFD. Therefore $k[S]$ is normal if and only if it is integrally closed in $k[S^{\mathrm{gp}}]$.

Suppose first that $k[S]$ is normal. If $u \in S^{\mathrm{gp}}$ and if $mu \in S$, then $(\chi^u)^m \in k[S]$ and $\chi^u \in k[S^{\mathrm{gp}}]$. As $k[S]$ is integrally closed in $k[S^{\mathrm{gp}}]$, it follows that $\chi^u \in k[S]$, so $u \in S$.

Conversely, let us assume that S is saturated, and let R be the integral closure of $k[S]$ in $k[S^{\mathrm{gp}}]$. It is clear that R is a torus-invariant linear subspace of $k[S^{\mathrm{gp}}]$, hence it follows from Lemma 7.22 that it is S^{gp}-homogeneous. In order to show that $R = k[S]$ it is thus enough to check that for every $\chi^u \in R$, we have $u \in S$. By assumption, χ^u satisfies an equation of the form

$$(\chi^u)^m + a_1(\chi^u)^{m-1} + \ldots + a_m \chi^{v_m} = 0,$$

for a positive integer m and $a_1, \ldots, a_m \in k[S]$. By only considering the scalar multiples of χ^{mu}, we may assume that in fact $a_i = c_i \chi^{v_i}$ for some $c_i \in k$ and $v_i \in S$. It follows that $v_i + (m-i)u = mu$ if $a_i \neq 0$, hence $iu = v_i$. Since some a_i must be nonzero, we have $iu \in S$ for some $i \geq 1$, and because S is saturated we deduce $u \in S$. □

Exercise 7.31. We have seen in Exercise 7.24 that if X is an affine toric variety and Y is a torus-invariant irreducible subset, then Y has a natural structure of toric variety. Show that if X is normal, then every such Y is normal.

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