Chapter 9

Developing Meanings for the Operations

This chapter is about helping children connect different meanings, interpretations, and relationships to the four operations of addition, subtraction, multiplication, and division so that they can effectively use these operations in real-world settings.

The main thrust of this chapter is helping children develop what might be termed operation sense, a highly integrated understanding of the four operations and the many different but related meanings these operations take on in real contexts.

As you read this chapter, pay special attention to the impact on number development, basic fact mastery, and computation. As children develop their understanding of operations, they can and should simultaneously be developing additional ideas about number and ways to think about basic fact combinations. Story problems for operations meaning are also a method of developing computational skills.

Big Ideas

1. Addition and subtraction are connected. Addition names the whole in terms of the parts, and subtraction names a missing part.
2. Multiplication involves counting groups of like size and determining how many are in all (multiplicative thinking).
3. Multiplication and division are related. Division names a missing factor in terms of the known factor and the product.
4. Models can be used to solve contextual problems for all operations and to figure out what operation is involved in a problem regardless of the size of the numbers. Models also can be used to give meaning to number sentences.

Mathematics

Content Connections

The ideas in this chapter are most directly linked to concepts of numeration and the development of invented computation strategies.

- Number Development (Chapter 8): As children learn to think about number in terms of parts and missing parts, they should be relating these ideas to addition and subtraction. Multiplication and division require students to think about numbers as units. In $3 \times 6$ each of the three sixes is counted as a unit.

- Basic Facts (Chapter 10): A good understanding of the operations can firmly connect addition and subtraction so that subtraction facts are a natural consequence of having learned addition. A firm connection between multiplication and division provides a similar benefit.

- Whole-Number Place Value and Computation (Chapters 11 and 12): Students work with and develop ideas about the base-ten number system as they solve story problems involving larger numbers. It is reasonable to have children invent strategies for computing with two-digit numbers as they build their understanding of the operations.

- Algebraic Thinking (Chapter 14): Representing contextual situations in equations is at the heart of algebraic thinking. This is exactly what students are doing as they learn to write equations to go with their solutions to story problems.

- Fraction and Decimal Computation (Chapters 16 and 17): These topics for the upper elementary and middle grades depend on a firm understanding of the operations.

Addition and Subtraction Problem Structures

We begin this chapter with a look at four categories of problem structure for additive situations (which include both addition and subtraction) and later explore four problem structures for multiplicative situations (which include both multiplication and division). Although these categories are not knowledge that students are expected to master, teachers are expected to learn these categories as
part of pedagogical content knowledge (PCK) (Shulman, 1986), which is the deep understanding that teachers need to effectively organize and support students' mathematics learning. Teachers who are not aware of the variety of situations and structures may randomly offer problems to students without the proper sequencing to support students' full grasp of the meaning of the operations, thus not preparing students for the variety of real-world contexts they will encounter. By knowing the logical structure of these problems you will be able to help students interpret a variety of mathematical situations. Again, students will not need to identify a problem with a “join” or “separate” classification by name, but as a teacher you will need to present a variety of problem types as well as recognize which structures cause the greatest challenges for students.

Researchers have separated addition and subtraction problems into categories based on the kinds of relationships involved. These include join problems, separate problems, part-part-whole problems, and compare problems (Carpenter, Carey, & Kouba, 1990; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Gutstein & Romberg, 1995). The basic structure for each of these four types of problems is illustrated in Figure 9.1. Each structure involves a number “family” such as 3, 5, 8. A different problem type results depending on which of the three quantities in the situation is unknown.

**Examples of the Four Problem Structures**

The number family 4, 8, 12 is used in each of the story problems that follow and can be connected to the structure in Figure 9.1. These drawings are not intended for students but to help you as a teacher. Also note that the problems are described in terms of their structure and interpretation and not as addition or subtraction problems. Contrary to what you may have thought, a joining action does not always mean addition, nor does separate or remove always mean subtraction.

**Join Problems.** For the action of joining, there are three quantities involved: an initial or starting amount, a change amount (the part being added or joined), and the resulting amount (the total amount after the change takes place). In Figure 9.1(a) this is illustrated by the change being “added to” the initial amount. Any one of these three quantities can be unknown in a problem as shown here.

**Join: Result Unknown**

Sandra had 8 pennies. George gave her 4 more. How many pennies does Sandra have altogether?

**Join: Change Unknown**

Sandra had 8 pennies. George gave her some more. Now Sandra has 12 pennies. How many did George give her?

**Join: Initial Unknown**

Sandra had some pennies. George gave her 4 more. Now Sandra has 12 pennies. How many pennies did Sandra have to begin with?
Separate Problems. Notice that in the “separate” problems, the initial amount is the whole or the largest amount, whereas in the “join” problems, the result is the whole. In “separate” problems the change is that an amount is being removed from the initial value. Again, refer to Figure 9.1(b) as you consider these problems.

Separate: Result Unknown
Sandra had 12 pennies. She gave 4 pennies to George. How many pennies does Sandra have now?

Separate: Change Unknown
Sandra had 12 pennies. She gave some to George. Now she has 8 pennies. How many did she give to George?

Separate: Initial Unknown
Sandra had some pennies. She gave 4 to George. Now Sandra has 8 pennies left. How many pennies did Sandra have to begin with?

Part-Part-Whole Problems. Part-part-whole problems involve two parts that are combined into one whole as in Figure 9.1(c). The combining may be a physical action, or it may be a mental combination where the parts are not physically combined.

There is no meaningful distinction between the two parts in a part-part-whole situation, so there is no need to have a different problem for each part as the unknown. For each possibility (whole unknown and part unknown), two problems are given here. The first is a mental combination where there is no action. The second problem involves a physical action.

Part-Part-Whole: Whole Unknown
George has 4 pennies and 8 nickels. How many coins does he have?

George has 4 pennies and Sandra has 8 pennies. They put their pennies into a piggy bank. How many pennies did they put into the bank?

Part-Part-Whole: Part Unknown
George has 12 coins. Eight of his coins are pennies, and the rest are nickels. How many nickels does George have?

George and Sandra put 12 pennies into the piggy bank. George put in 4 pennies. How many pennies did Sandra put in?

Separate Problems. Compare problems involve the comparison of two quantities. The third amount does not actually exist but is the difference between the two amounts. Figure 9.1(d) illustrates the comparison problem type. There are three ways to present compare problems, corresponding to which quantity is unknown (smaller, larger, or difference). For each of these, two examples are given: one problem where the difference is stated in terms of more and another in terms of less.

Compare: Difference Unknown
George has 12 pennies and Sandra has 8 pennies. How many more pennies does George have than Sandra?

George has 12 pennies. Sandra has 8 pennies. How many fewer pennies does Sandra have than George?

Compare: Larger Unknown
George has 4 more pennies than Sandra. Sandra has 8 pennies. How many pennies does George have?

Sandra has 4 fewer pennies than George. Sandra has 8 pennies. How many pennies does George have?

Compare: Smaller Unknown
George has 4 more pennies than Sandra. George has 12 pennies. How many pennies does Sandra have?

Sandra has 4 fewer pennies than George. George has 12 pennies. How many pennies does Sandra have?

Pause and Reflect
Go back through all of these examples and match the numbers in the problems with the components of the structures in Figure 9.1. For each problem, do two additional things. First, use a set of counters or coins to model (solve) the problem as you think children in the primary grades might do. Second, for each problem, write either an addition or subtraction equation that you think best represents the problem as you did it with counters.

In most curricula, the overwhelming emphasis is on the easier join and separate problems with the result unknown. These become the de facto definitions of addition and subtraction: Addition is “put together” and subtraction is “take away.” The fact is, these are not the definitions of addition and subtraction.

When students develop these limited put-together and take-away definitions for addition and subtraction, they
often have difficulty later when addition or subtraction is called for but the structure is other than put together or take away. It is important that children be exposed to all forms within these four problem structures.

**Problem Difficulty.** The various types of problems are not at all equal in difficulty for children. The join or separate problems in which the initial part is unknown are among the most difficult, probably because children modeling the problems directly do not know how many counters to put down to begin with. Problems in which the change amounts are unknown are also difficult.

Many children will solve compare problems as part-part-whole problems without making separate sets of counters for the two amounts. The whole is used as the large amount, one part for the small amount and the second part for the difference. Which method did you use? There is absolutely no reason this should be discouraged as long as children are clear about what they are doing.

As students begin to translate the variety of story problems in the previous pages into equations to solve, they may be challenged in creating a matching equation that emphasizes the corresponding operation. This is particularly important as students move into explorations that develop algebraic thinking. The structure of the equations also may cause difficulty for English language learners who may not initially have the flexibility in creating equivalent equations due to reading comprehension issues with the situation described in the story. Therefore, we need to look at how knowing about computational and semantic forms of equations will help you help your students.

**Computational and Semantic Forms of Equations.** If you wrote an equation for each of the problems as just suggested, you may have some equations where the unknown quantity is not isolated on one side of the equal sign. For example, a likely equation for the join problem with initial part unknown is \( \square + 4 = 12 \). This is referred to as the *semantic* equation for the problem since the numbers are listed in the order that follows the meaning of the problem. Figure 9.2 shows the semantic equations for the six join and separate problems on the previous pages. Note that the two result-unknown problems place the unknown alone on one side of the equal sign. An equation that isolates the unknown in this way is referred to as the *computational* form of the equation. When the semantic form is not also the computational form, an equivalent equation can be written. For example, the equation \( \square + 4 = 12 \) can be written equivalently as \( 12 - 4 = \square \). The computational form is the one you would need to use if you were to solve these equations with a calculator. Students need to see that there are several ways to represent a situation in an equation. As numbers increase in size and children are not solving equations with counters, they must eventually learn to see the equivalence between different forms of the equations.

![Table](image)

**Figure 9.2** The semantic equation for each of the six join and separate problems on pages 146–147. Notice that for results-unknown problems the semantic form is also the computational form. The computational form for the other four problems is an equivalent equation that isolates the unknown quantity.

### Teaching Addition and Subtraction

So far you have seen a variety of types of story problems for addition and subtraction and you probably have used some counters to help you understand how these problems can be solved by children. Combining the use of contextual problems and models (counters, drawings, number lines) is important in helping students construct a rich understanding of these two operations. Let's examine how each approach can be used in the classroom. As you move through this section, note that addition and subtraction are taught at the same time.

### Contextual Problems

There is more to think about than simply giving students problems to solve. In contrast with the rather sterile story problems in the previous section, consider the following problem.

*Yesterday we were measuring how tall we were. You remember that we used the connecting cubes to make a big train that was as long as we were when we were lying down. Dion and Rosa were wondering how many cubes long they would be if they lay down head to foot. Dion had measured Rosa and she was 84 cubes long. Rosa measured Dion and she was 102 cubes long. Let's see if we can figure out how long they will be end to end, and then we can check by actually measuring them.*

Fosnot and Dolk (2001) point out that in story problems, children tend to focus on getting the answer. "Context
problems, on the other hand, are connected as closely as possible to children's lives, rather than to 'school mathematics.' They are designed to anticipate and to develop children's mathematical modeling of the real world" (p. 24). Contextual problems might derive from recent experiences in the classroom, a field trip, a discussion you have been having in art, science, or social studies, or from children's literature.

Lessons Built on Context or Story Problems. The tendency in the United States is to have students solve a lot of problems in a single class period. The focus of these lessons seems to be on how to get answers. In Japan, however, a complete lesson will often revolve around one or two problems and the related discussion (Reys & Reys, 1995).

What might a good lesson for second graders that is built around word problems look like? The answer comes more naturally if you think about students not just solving the problems but also using words, pictures, and numbers to explain how they went about solving the problem and why they think they are correct. Children should be allowed to use whatever physical materials they feel they need to help them, or they can simply draw pictures. Whatever they put on their paper should explain what they did well enough to allow someone else to understand their thinking (allow at least a half page of space for a problem).

The second-grade curriculum of Investigations in Number, Data, and Space places a significant emphasis on connecting addition and subtraction concepts. In the excerpt shown on page 150, you can see an activity involving word problems for subtraction. Take special note of the emphasis on students' visualizing the situation mentally and putting the problem in their own words.

Choosing Numbers for Problems. Even pre-K and kindergarten children should be expected to solve story problems. Their methods of solution will typically involve using counters or actual experiments in a very direct modeling of the problems. This is what makes the join and separate problems with the initial parts unknown so difficult. For these problems, children initially use a trial-and-error approach (Carpenter, Fennema, Franke, Levi, & Empson, 1999).

Although the structure of the problems will cause the difficulty to vary, the numbers in the problems should be in accord with the number development of the children. Pre-K and kindergarten children can use numbers as large as they can grasp conceptually, which is usually to about 10 or 12.

Second-grade children are also learning about twodigit numbers and are beginning to understand how our base-ten system works. Rather than waiting until students have learned about place value and have developed techniques for computing numbers, word problems are a problem-based opportunity to learn about number and computation at the same time. For example, a problem involving the combination of 30 and 42 has the potential to help students focus on sets of ten. As they begin to think of 42 as 40 and 2, it is not at all unreasonable to think that they will add 30 and 40 and then add 2 more. As you learn more about invented strategies for computation in Chapter 12, you will develop a better understanding of how to select numbers for the problems you use in your lessons to aid in computational development.

The Standards authors make clear the value of connecting addition and subtraction. “Teachers should ensure that students repeatedly encounter situations in which the same numbers appear in different contexts. For example, the numbers 3, 4, and 7 may appear in problem-solving situations that could be represented by 4 + 3, 3 + 4, or 7 – 3, or 7 – 4. . . . Recognizing the inverse relationship between addition and subtraction can allow students to be flexible in using strategies to solve problems" (p. 83).

Introducing Symbolism. Very young children do not need to understand the symbols +, –, and = to learn about addition and subtraction concepts. However, these symbolic conventions are important. When you feel your students are ready to use these symbols, introduce them in the discussion portion of a lesson where students have solved story problems. Say, “You had the whole number of 12 in your problem and the number 8 was one of the parts of 12. You found out that the part you did not know was 4. Here is a way we can write that: 12 – 8 = 4.” The minus sign should be read as “minus” or “subtract” but not as “take away.” The plus sign is easier since it is typically a substitute for “and.”

Some care should be taken with the equal sign. The equal sign means “is the same as.” However, most children come to think of it as a symbol that tells you that the “answer is coming up.” It is interpreted in much the same way as the = on a calculator. That is, it is the key you press to get the answer. An equation such as 4 + 8 = 3 + 9 has no “answer” and is still true because both sides stand for the same quantity. A good idea is to often use the phrase “is the same as” in place of or in conjunction with “equals” as you read equations with students.

Another approach is to think of the equal sign as a balance; whatever is on one side of the equation “balances” or equals what is on the other side. This will support algebraic thinking in future grades if developed early (Knuth, Stephens, McNeil, & Alibali, 2006). (See Chapter 14 for a more detailed look at teaching the equal sign as “is the same as” rather than “give me the answer.”)