31

Agreeing to Disagree: Developing Sociable Mathematical Discourse

MAGDALENE LAMPERT
PEGGY RITTENHOUSE
CAROL CRUMBAUGH

All students have beliefs about what to do in school in order to learn. These beliefs—their "folk learning theories"—are an expression of what our culture believes about school knowledge and how it is acquired. As Jerome Bruner (1990) has noted:

> All cultures have as one of their most powerful constitutive instruments a folk psychology, a set of more or less connected, more or less normative descriptions about how human beings "tick," what our own and other minds are like, what one can expect situated action to be like, what are possible modes of life, how one commits oneself to them, and so on. We learn our culture's folk psychology early, learn it as we learn to use the very language we acquire and to conduct the interpersonal transactions required in communal life. (p. 35)

The beliefs that make up our folk psychology of learning are not necessarily explicit, but they shape norms of interaction in school settings. One component of students' folk learning theories is a set of beliefs about what the subject under study is and how knowledge of it might be acquired. Elsewhere, Lampert has written about how common assumptions about the nature of mathematical knowledge and its acquisition conflict with theories of practice in the discipline of mathematics (Lampert, 1990, 1992; Putnam, Lampert, and Peterson, 1990). This conflict is problematic for educational reformers who wish to bring classroom practice closer to what people who invent and use mathematics do.

In this chapter, we examine another potentially problematic aspect of student's folk learning theories: the potential conflict between the kind of discourse that scholars and reformers believe will foster learning in school and common beliefs about disagreeing in public. Although we agree with others in the field that certain cultural norms privilege some students and disenfranchise others in the performance of traditional school tasks (see Heath, 1983), we propose that this dichotomy does not apply as directly to the performance of the kinds of school tasks that students are being asked to perform in "reformed" classrooms. In particular, the dichotomy between home norms and school norms seems inadequate as we try to understand students' disagreeing with peers about ideas. We assume that school classrooms draw on diverse communities and have multicultural populations; classrooms are places where people who come from different family backgrounds work and learn together. In connecting communication and discourse practices with understanding and knowing, scholars and reformers are advocating the creation of norms of public debate about ideas in school lessons that may have no readily recognizable counterparts in most of the home cultures from which the school population is drawn. The differences between folk theories of public disagreement and theories of teaching and learning that hinge on classroom discussion and debate have significant implications for how we think about what teachers need to do to teach mathematics and what students need in order to talk mathematics—or any other subject—in school.

Mathematical Talk in One Classroom

We begin with a story told by Lampert about a disagreement that occurred in a whole class discussion[1] in her fifth-grade mathematics class.[2] Such disagreements were common in her classroom, and in fact, she designed the problems that students were to work on so that discrepancies would be likely to arise in both strategies and solutions. In whole class discussion, she monitored the presentation of discrepant perspectives on problems and their solutions. She also attempted to provide students with some ideas about what kind of language might be appropriate for their arguments with peers. We present this case of student disagreement about a mathematical idea to provide an image of the kind of teaching and learning practices that scholars and reformers currently advocate. The role in social and mathematical discourse that Lampert assumes as the teacher requires the establishment of a different set of "sociomathematical norms" (Cobb, Wood, and Yackel, 1993).

Here Lampert describes a part of the lesson:[3]

On April 6, 1990, the class had been working in groups of four on an exercise in which they were given several sets of ordered pairs and their task was to find the relationship between the pairs in each set (Willoughby, Bereiter, Hilton, and Rubenstein, 1990). During this introductory lesson on functions, I made the transition from small group work to large group discussion by calling the class's attention to one of the exercises that a number of students had said was the "hardest" on that page. "Number six" was a cluster of ordered pairs which many students had been discussing with their groups, and there had been considerable controversy about how to state the "rule" for getting from the numbers in the first column to their corresponding "outputs" in the second column. I asked the class how one might characterize the relationship between the number in the first column and its corresponding value in the second column (see Figure 31.1). Ellie was the first to speak.

ELLIE: Um, well, there were a whole bunch of—a whole bunch of rules you could use, use, um, divided by two—And you could do, um, minus one-half.

I responded to Ellie's second assertion with a question: "And eight minus a half is?" To which she answered: "Four."

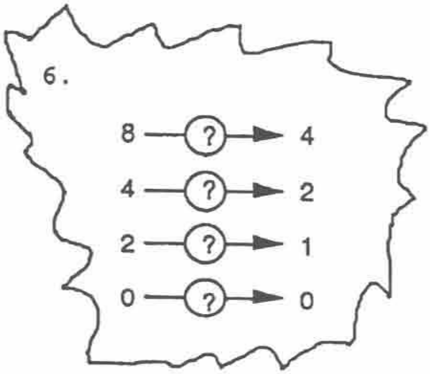


Figure 31.1

At this point a gasp arose from the class and several other students made bids to enter the conversation. I had solicited from Ellie a confirmation of her interpretation of "minus a half" and several of her classmates were surprised by what she said.

As the teacher in this situation, what I thought I needed to do first was to protect Ellie's right to practice mathematics in this fifth-grade classroom by monitoring the discourse so that she would have the opportunity to explain her thinking and justify her assertion. This kind of teaching move would make it possible for Ellie and the others to do a particular kind of learning move: trying to express their thinking in language or symbols, trying to understand what other people were thinking, and coming to some agreement on a usable definition for the operation "minus a half." So, before calling on any of the students who were eager to argue with Ellie, I set the terms of the conversation: You can express an idea that is different from Ellie's, but you also need to make an attempt to take her position seriously. Many of Ellie's classmates believed that she had asserted that eight minus a half is four. Most of the following discussion centered around the counter assertion that eight minus a half is seven and a half.

To set the terms of discussion, I asked for more ideas from the class and justified my move to get different ideas out on the table while also asking those who might disagree to treat Ellie's assertion with respect:

> You think that would be four? What does somebody else think? I, I started raising a question because a number of people have a different idea about that. So let's hear what your different ideas are and see if you can take Ellie's position into consideration and try to let her know what your position is. Enoyat?

Several students raised their hands and I chose Enoyat to speak first.

My reasons for structuring the discourse in this way were both mathematical and social. According to stated norms of the mathematical community, the people who invent and use mathematics, the legitimacy of an assertion is not judged without considering the assumptions and the reasoning that are supplied to justify the assertion. In the course of trying to explicate the assertion that "eight minus one half is four" or the counterassertion that "eight minus one half is seven and one-half," the students and I together became clearer about the assumptions and definitions that underlie Ellie's assertion. Ellie's assertion that the rule could be either "divide by two" or "minus one half" might be thought of as the result of a mathematical intuition. She "saw" both relationships in the set of ordered pairs that had been given in the exercise, but she was struggling to find a way to talk about what she saw. Based on the kinds of computations I had seen her do on other occasions, I did not think that the problem was that Ellie could not correctly perform the subtraction of one half from eight. Rather I proceeded as if the students were struggling with a big new mathematical idea: that a number could be both a *quantity* and an *operator on quantities*. (In this case, for example, "one-half" could mean either the quantity "one half of one whole" or the operation "take one half of".) This idea is fundamental to knowing how to interpret and to use rational numbers and confusion about this idea often gets in the way of students being able to work successfully with fractions. Even more significant perhaps is that this work occurs on the boundary between arithmetic and algebra. I used these exercises to engage students in thinking about the idea of a variable and how two variables are related. Ellie was struggling with finding a way to respond to her classmates that expressed her understanding of the relational nature of functions while many of them were focused on making a mathematical statement that would fit the constraints of one particular set of ordered pairs in the exercise.

On the social front, what I thought the students needed to learn was how to disagree respectfully, how to express the evidence for their disagreement, and how to revise their own thinking while saving face among their peers. In order to accomplish the goal of equitable participation, I could not let Ellie "go it alone"—some of her classmates would take pride in shooting down her idea, and then she would not again venture to participate for some time. I also knew that a few of the students would need special encouragement to enter the conversation, and that they might do little more than restate arguments that had already been made.

Mathematically, I expected that the students' talk would be a sort of "first draft" attempt to speak about the complex distinction between numbers as operators and numbers as quantities.[4] I sparingly tried to help them edit their talk as I asked for further clarification and justification. One after another, students expressed ideas about the meaning of one half and experimented with how to talk about it in relation to the other numbers.

As the following transcript excerpt shows, the students expressed not only disagreement, but conditions under which they might agree. I called on Enoyat first after Ellie's assertion because I knew that he would be gentle in his response, even if he disagreed.

ENOYAT: Well, see, I agree with Ellie because you can have eight minus one half and that's the same as eight divided by two or eight minus four.

LAMPERT: Eight divided by two is four, eight minus four is four? Okay, so Enoyat thinks he can do all of those things to eight and get four. Okay? Charlotte?

CHARLOTTE: Um, I think eight minus one half is seven and a half because—

LAMPERT: Why?

CHARLOTTE: Um, one half's a fraction and it's half of one whole and so when you subtract you aren't even subtracting one whole number so you can't get even a smaller number that's more than one whole. But I see what Ellie's doing, she's taking half the number she started with and getting the answer.

LAMPERT: So, you would say one half of eight? Is that what you mean?

CHARLOTTE: Yeah, one half of eight equals four.

LAMPERT: How do you know that?

CHARLOTTE: Because, um, eight and one half is um, eight and half of eight is four, so if you have two groups of four you would, is eight.

At this point, I decided to check in with Ellie:

LAMPERT: Ellie, what do you think?

ELLIE: Um, I still think, I mean, one half, it would be eight minus one half, they would probably say oh, eight minus one half equals four. Um.

LAMPERT: Who would say that?

ELLIE: I don't know. Well, well if if I saw something like that, like if we were having something and the answer was missing.

LAMPERT: Um. . .hm.

ELLIE: Um, and it was eight minus one half I would probably say four.

I wondered if more talk would help clarify the alternatives:

LAMPERT: What do other people think? Sam?

SAM: Um, I agree with Charlotte and um I don't agree with Ellie. Because um, like one half is not even one, so if, so when Ellie said that people that people would like, um a really good mathematician would probably, like, would probably write seven and a half, not four because they would have to know what the one half was meaning, half of a number to um, to understand it.

LAMPERT: You know when Charlotte was talking she said that she thought one half meant half of a whole. And it sounds like that's the way you are interpreting it. But Ellie might be interpreting one half to mean something else. Lev, what do you think?

LEV: I think um, I would agree with Ellie if she had said eight minus one half of eight because half of eight would be four because four plus four would be eight.

LAMPERT: So, in your case, you're saying one half, if Ellie meant one half was half of eight wholes, then it would work. Okay, Tyrone?

TYRONE: I agree with Charlotte and Sam and I disagree with Ellie and like I think Ellie meant like because four is half of eight, like one half would be a half, but, and I agree with Lev when he said if she meant one half uh equals, wait, eight, equals half of eight and I agree with Sam and uh Charlotte because, um if, if, uh, four is not, uh eight equals half of four is not right because it's seven and a half because half of like, eight is the whole and um, one number away from that is seven and plus a half would be seven and half.

LAMPERT: Uh huh. So um, that reminds me of some of the discussion that we were having yesterday, which is that if you, if you use addition, it helps you to understand what it means to take away something on the other side. Okay? Let's hear from Shahroukh.

SHAHROUKH: I would agree with Ellie if she had added something else to her explanation, if she said one half of the amount that you have to divide by two.

LAMPERT: Okay. You guys are on to something really important about fractions, which is that a fraction is a fraction of something. And we have to have some kind of agreement here if it's a fraction of eight or if it's a fraction of a whole. Let's, let's just hear from a couple more people Ellie, and then I'll come back and hear from you, okay? Er uh Darota?

DAROTA: Well, I think, um, I disagree with Ellie because if she means that one half of the whole you would get seven and then if you add that half on again you would get seven and a half.

LAMPERT: Okay. Well, that's quite similar to what Tyrone was saying that addition is a way of helping me think about the meaning of subtraction. But again, both you and Tyrone are assuming that one half means one half of one. Okay? Anthony.

ANTHONY: [Inaudible]. . .if you are assuming it is one half of the original number it would be different than if it is half of a whole. So it depends on what your point of view is.

LAMPERT: Right. And that, that would be very important to clarify. Okay.

My "Okay." was meant to bring some closure to the discussion, and I turned again to Ellie:

ELLIE: Um, well, I agree with Shahroukh and, um, when Charlotte said um, she thought that um, it should be one half of eight, um, instead of just plain one half, I don't agree with her because not all of them are eight. Not all of the problems are eight.

LAMPERT: Okay. Let's um, one of the things that is kind of a convention in mathematics is that when we just talk about numbers and we don't associate them with any objects or groups of objects, that this symbol means half of one whole. So if, if you were gonna communicate with the rest of the world who uses mathematics, they would take this [pointing to the expression "$8-{}^{1}/_{2}$" on the chalkboard] to mean eight wholes minus one half of a whole. Okay? Ellie?

ELLIE: Um, well, I—I think that eight if, you had, all you—all these numbers are that are going into the um.

LAMPERT: Function machine.

ELLIE: Um on number six, they're all, they can all be divided into halves and um four minus um well two is one half of four.

LAMPERT: Okay, so the number that comes out is one half of the number that went in. Okay. And in this case is that true?

ELLIE: Um.

LAMPERT: Is one one-half of two? Is zero one half of zero?

ELLIE: Um, yes.

LAMPERT: So, what do you think about that? We could write this in words, you know, we don't have to use these equations, but it's more efficient. You, you feel that—

ELLIE: One half is—

LAMPERT: If, if you said that the number that comes out is half the number that goes in, it would be easier for you to understand?

ELLIE: That's what I meant, but I just couldn't put it in there, but that's what was in my mind.

LAMPERT: Okay. But I think you raised a lot of interesting questions by your idea of taking away a half.

In this class, students are engaged in doing mathematics with one another and with their teacher. They are learning about mathematical forms of discourse by inventing ways to talk about quantitative relationships, making conjectures about patterns in relationships among numbers, justifying those conjectures with logic and mathematical evidence, and considering the reasonableness of the assertions made by others who seem to disagree. Working on the question of what makes an assertion true or acceptable for use is central to both the pedagogy and the mathematics in this class. As they try to figure out how "one-half" can mean something other than the quantity half of one whole, this teacher and her students are participating in a fifth-grade version of mathematical argument—a conversation whose purpose is to determine the mathematical legitimacy of a chain of reasoning and the appropriateness of applying that reasoning to a problem situation. They are also intentionally engaged in what we have come to call "the social construction of knowledge."[5]

But it is not only mathematical content knowledge that is being constructed here. And the students are not the only participants in this constructive activity. As teacher and students work together, they are inventing new forms of activity and interaction for learning mathematics in school. In order to be judged as successful learners of mathematics in this classroom, the students need to demonstrate that they are reasoning collaboratively toward their conclusions rather than accepting answers on the authority of the teacher or a textbook or a student who they have identified as "smart in math." The students' learning involves accepting the goal of "making sense" as a legitimate purpose for classroom work. In order for the teacher to judge herself successful at getting students to change their mathematical thinking, such a change must be effected through reasoned argument by teachers or other students rather than by simple exercise of authority.

Schooling Mathematical Intentions

Many of the students in Lampert's class in 1989–90 came from more traditional fourth-grade classrooms, where their experience of teacher-student interaction during mathematics lessons was quite different.[6] In the traditional mathematics classroom, the result of Ellie's answer of "four" to the teacher's question: "Eight minus a half is?" would likely be for students and teacher to judge Ellie to be wrong. Students might even be heard to mutter, "That's a dumb answer!" or to give nonverbal signals that they did not consider Ellie to be very bright. The teacher might be more polite, but equally straightforward in her judgment that Ellie was deficient. The teacher's job would be to explain to Ellie why she is wrong or to use some other means to get her to answer "Seven and one-half" the next time she is asked what eight minus a half is. Students and teacher would be likely to judge the students who were anxious to speak in reaction to Ellie's error as "smarter at math" than Ellie, and they might imagine that learning would occur if Ellie would listen to someone the teacher would call on to give the "right" answer. But they might also feel sorry for Ellie, imagining that this would be an embarrassing moment for her.[7] In most cases, there would be nothing like a discussion and no expectation on the part of students that they should know how to "agree" or "disagree" with Ellie and how to convince others their position makes sense.

In most school mathematics lessons, teacher and students make one assertion after another about what is true or false, right or wrong. In conventional teaching, what makes an assertion true or right is the authority of the teacher or the textbook or a "smart" student. In contrast, in the classroom we have just looked in on, all students are called upon to justify the correctness of their own assertions with mathematical evidence. They are asked to explain discrepancies between how they think about something and how someone else might think about it by examining their assumptions and defining their terms of discourse.

The teacher of the class in which this discussion took place engages students in such mathematical conversation in order to teach them what it means to do mathematics.[8] By posing interpretable problems and encouraging disagreement, the teacher sets the stage for students to clarify their thinking and relate thought to communication. Teaching and learning activities during lessons are related to the practice of doing mathematics by the teacher and her students raising questions about how one another's mathematical assertions are justified. In the process of justifying what makes sense to themselves and their peers, students struggle with constructing a mathematical language and a set of relationships among its terms that is internally consistent.

The teaching in this discussion reflects current trends toward thinking about learning and knowing as *social* as well as individual activities, both in the disciplines and in school classrooms. Acquiring knowledge is not thought to be a private interaction between knower and subject matter or a one-to-one interaction between teacher and individual student, but it is understood as a broadly social practice engaged in with peers and more knowledgeable others. These epistemological trends are reflected in the set of new "standards" for teaching mathematics established recently by the National Council of Teachers of Mathematics (NCTM). For example, in a section on "students' role in discourse," the *Professional Standards for Teaching Mathematics* say:

> The teacher of mathematics should promote classroom discourse in which students:
>
> - listen to, respond to, and question the teacher and one another;
> - use a variety of tools to reason, make connections, solve problems, and communicate;
> - initiate problems and questions;
> - make conjectures and present solutions;
> - explore examples and counterexamples to investigate a conjecture;
> - try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers;
> - rely on mathematical evidence and argument to determine validity.
>
> (National Council of Teachers of Mathematics, 1991, p. 45)

As with reform documents in other disciplines, this set of new standards puts a strong emphasis on changing the nature of classroom discourse to include talk in the spirit of disciplinary work. It is expected that students will consider and challenge one another's assertions, and even challenge the teacher, and presumably by using mathematical evidence, convince others of the reasonability of their claims. The *Teaching Standards* are premised on the assumption that the nature of classroom discourse is a significant influence on what students learn about mathematics. "Major shifts" in the environment of mathematics classrooms are advocated so that schools can "move from current practice to mathematics teaching for the empowerment of students."

> We need to shift—
>
> - toward classrooms as mathematical communities—away from classrooms as simply a collection of individuals;

- toward logic and mathematical evidence as verification—away from the teacher as the sole authority for right answers;
- toward mathematical reasoning—away from merely memorizing procedures;
- toward conjecturing, inventing, and problem solving—away from an emphasis on mechanistic answer-finding;
- toward connecting mathematics, its ideas, and its applications—away from treating mathematics as a body of isolated concepts and procedures.

(NCTM, 1991, p. 3)

"Reasoning" here is something that is fundamentally linked to communication: Making conjectures means putting out tentative ideas for consideration by oneself and others. What it means for something to be true is to be determined in social exchanges where mathematical evidence is used to verify an assertion. The social nature of learning is emphasized in the view that classrooms should be "mathematical communities" rather than collections of individual learners. Such communities would establish norms and patterns of discourse, not only for how to talk but for what counts as evidence and therefore as knowledge. This view of knowledge production is grounded in the discourse ideals of the discipline, and it is consonant with contemporary psychological theories about the role of discourse and disagreement in individual knowledge growth (Strike and Posner, 1985; Wiser and Carey, 1983; Hatano and Inagaki, 1991).

By using the word "promote" to describe the pedagogical activity of the teacher who seeks to have such things happen in the classroom, NCTM sidesteps the question of what exactly teachers need to teach and students need to learn for this kind of talk to be seen as an appropriate mode of public interaction between schoolchildren and their teacher (Ball, 1991). At the same time, NCTM emphasizes that it is setting standards for mathematics teaching and learning that are to apply to "every child":

> Goals such as learning to make conjectures, to argue about mathematics using mathematical evidence, to formulate and solve problems—even perplexing ones—and to make sense of mathematical ideas are not just for some group thought to be "bright" or "mathematically able." Every student can—and should—learn to reason and solve problems, to make connections across a rich web of topics and experiences, and to communicate mathematical ideas. (NCTM, 1991, p. 21)

Our experience with school and university teaching and with communication among researchers suggests that this kind of discourse does not come easily, either to school learners or more advanced scholars. In the rest of this chapter, we will examine this problem.

Folk Theories about Appropriate Forms of Public Debate

How might we think about the cultural milieu into which we have dropped educational ideals like the NCTM *Teaching Standards?* How are we to understand what students bring with them from outside of school in the way of knowledge about how to disagree about ideas, how to argue with peers? "Arguing" is an activity most people engage in privately and publicly inside and outside of school. Even in sociable conversation, it is likely that participants will disagree. Although we come from different cultural backgrounds, we learn how to respond to such disagreements in ways that accomplish multiple goals, including preserving relationships with people who make assertions that we believe to be unreasonable. If maintaining such relationships contributes to other social and personal goals, we find ways to avoid or to ameliorate disagreement. We learn when it is acceptable to simply ignore an assertion with which we disagree, and when we can assert physical or political power to force agreement. From experience, we acquire a model of what it means to disagree in public and how to do it successfully.[9]

In *Roget's Thesaurus* we find that in its social sense, *reasoning* can be used synonymously with words like *commenting, discussing*, and *inquiring*. In arcane usage, one might use the phrase *torture a question* as a synonym for reasoning. More commonly, *reasoning* is also thought to be synonymous with *agitating, quarreling*, and *wrangling*. Somewhere in the middle between these extremes are more socially neutral synonyms for reasoning like *arguing* and *disagreeing*.

In the academic world, arguing about ideas is supposed to be our stock-in-trade, although in fact we rarely engage in doing it face to face. We believe that knowledge in a field grows from challenges to existing ideas and revisions of inadequate theories. In our ideal academic scenario, the quest for knowledge begins with conjecturing, and ideas get refined as conjectures get revised. Ideally, we avoid quarreling or agitating those with whom we have differences by following acceptable canons of academic discourse, such as reviewing manuscripts "blindly" and writing in the third person (Swales, 1990).

In the world of schoolchildren, arguing and disagreeing are closer to agitation and quarreling—not something you would do to a friend, or even to someone you knew you were going to have to work with every day for the rest of the week or month or year (Isaacs, 1971). Schoolchildren certainly modify their disagreement behaviors when they enter the classroom from the playground or the lunchroom, but when left to their own devices, they are not likely to carry on what we would recognize as an academic argument when someone says something in public that they believe is wrong. Children do not readily separate the quality of ideas from the person expressing those ideas in judging the veracity of assertions (Olson and Astington, 1993). It is also plausible that they would not naturally distinguish between norms for disagreeing about who should have the first turn on the swing and norms for disagreeing about who has expressed the most adequate formulation of a mathematical relationship.

What Some Fifth Graders Have to Say about Mathematical Disagreements in School

In the same way that we study how children think about number or spatial relations to help us figure out how to teach them arithmetic or geometry, it seems appropriate to try to examine their ways of understanding intellectual debate to help us think about how they might learn to engage in productive social encounters in school. One way to begin to study school learners' folk learning theories would be to look at both what they say about disagreement and what they do about it at a point in the school year when the teacher would not yet have had much influence on their ways of interacting and when the teacher is not present to moderate their activity (Spradley, 1972; Holland and Quinn, 1987). Data that have been collected in Lampert's classroom offer us these opportunities. First, there was a discussion in the classroom about the subject of "disagreement" that occurred a few days after the lesson we looked at above. Ellie, the girl with whom many students disagreed about the meaning of "eight minus a half," had a lot to say in that discussion about disagreement, and we will look particularly at her views. There were also several occasions at the beginning of the year when students worked in small groups to develop solutions to problems without much intervention from the teacher. After we look at what students had to say about disagreement, we will look in on a discussion that occurred early in the year among four other students in the class. In this discussion, the students disagreed about how to work on a problem relating time, speed, and distance. One of the central figures in the debate was Sam, a boy who had joined the class just the week before and was trying to figure out how to get his peers to see the problem his way.

A few days after the discussion of "eight minus a half" reported above, a colleague visited Lampert's classroom who was interested in the pedagogy of Socratic dialogue and how it might work in an elementary mathematics classroom. Lampert asked the students in her class to spend a few minutes at the beginning of math period telling the visitor about their discussions in math, and what they thought about having discussions in math: "What you like about them, what you don't like about them, what you think you learn from them." These students had been participating in whole class discussions about mathematical disagreements since early October, and it was now April. This was the first time they had publicly talked about their experience. Their "discussion of discussions" went on for the whole class hour.

The first student who spoke said, simply, "I don't like discussions when you are wrong." Lampert asked how that made him feel, but before he could answer, another student jumped in and said, "I think it's a way of trying to experience different people's point of views and what other people think, how they solve problems and things." The third student to speak said, "I don't like discussion because, because you have to give, you always have to give your reasoning and sometimes you just think of a problem and you don't think of why you said that." And another student disagreed: "I disagree with Karim because if you want to like give reasonings and if you are just thinking the problem, and you solve the problem for other people, they wouldn't know how they got the answer, and if you tell them how you got the answer they might not want to do the work."

Two of these students seem to be equating reasoning with conventional ideas about academic argument, expressing dispositions that are consonant with what we currently think is good learning practice in schools: find out how other people think about something, and do not be satisfied only with an answer if the answer has not been justified. The other two are somewhat less disposed toward the activity of discussing mathematics, one because he doesn't like to be wrong in public and the other because he is not always prepared to articulate why he thinks what he thinks. After this somewhat balanced exchange of opinions, the students' responses took a turn toward expressing almost entirely negative feelings about participating in mathematical discussions. Here we focus primarily on Ellie's contributions because of the role she played in the disagreement about how to interpret "eight minus a half."[10]

The first time Ellie spoke, she expressed a dislike for disagreement in the form of being the object of someone else's reasoned argument:

ELLIE: Um, I don't like reasoning because whenever you have a wrong answer people try so hard to prove you're wrong.

Her friend Saundra agreed with her and said that being in that position made Saundra feel embarrassed, even mortified:

SAUNDRA: Yeah, I agree with Ellie because you know it can get sort of embarrassing at times, because like everybody else, like you say something and everybody will raise their hand and want to say something different or they all disagree with you. And it makes you sort of feel like you want to crawl into a hole and die.

Ellie then commented on the persistence of those who might disagree, and echoed Saundra's social discomfort with being involved in this kind of exchange:

ELLIE: Um, when, when you do realize that you have the answer wrong they still want to prove it to you that it's wrong. Then um, like if you're going, like you have one lesson and you're wrong, and then the next lesson people are still raising their hands to prove that it is wrong. And saying, "Oh yeah this, and I think this because," um and you want to just crawl under your desk.

But Ellie had an intellectual concern as well. She was worried about her own capacity to hold on to her thinking in the face of disagreement:

ELLIE: Um, sometimes I don't like discussions because when you're trying to prove something it just turns into something else and you don't get to say what you think.

After she said this, several students in the class muttered agreement.

Later in the discussion, Ellie commented on still another aspect of classroom discourse that made her uncomfortable: the ownership of ideas. In a setting where collaboration was valued and rewarded by the teacher, both in math class and in other classes as well, the issue of individual productivity and recognition was clearly on students' minds. Ellie referred to a situation in which several students might work together to come up with the "answer" but only one gets to say what they think in the whole class discussion. What she says here suggests that there are consequences to such actions that go beyond feeling bad—if one "takes" someone else's answer, one risks being temporarily ostracized, perhaps by someone who had previously been counted as an ally.

ELLIE: Um, I um, I agree with Shahroukh and I sort of don't because if you were the only person that had their hand raised, of a couple people it's fun to have an answer that's right, but if everybody is raising their hand and um, you're called on, you feel really bad, if, if um, somebody really wanted that, to um, give that answer. And sometimes people say, "Oh, you took my answer," and they don't talk to you for a while.

As the discussion went on Ellie returned to her concern about the fact that she did not always get to say what she thought when she disagreed with someone, referring to an occasion when she had to leave a class discussion to go to an appointment. Again she reminds us that there are public consequences (teasing) beyond personal discomfort when students do not agree:

ELLIE: Well, um, when you were doing the fractions, then we did all those percents on that board back there and the eights and we tried to see what the differences were, before that we were working on something and everybody disagreed with me, and I didn't get to say what I thought.

LAMPERT: And um, say more about that. Why is that a problem?

ELLIE: Um, because you don't have a chance to tell them what you think and they and if their answer is wrong and maybe if you tell them your reasoning and why you think and maybe they'll think that um, maybe they won't tease you anymore.

She elaborates as she talks about disagreements in small group, rather than whole class discussions, claiming that teasing is a way for students to save face in a disagreement, so that when they find out they are wrong, "they don't feel bad":

ELLIE: Um, there is one other thing that I really don't like about small group. One is that like if you're right, and one person thinks that you're wrong, it turns out that you're right and then um, when they find out that they're wrong, they make fun of you. And they um, so they don't feel bad, if they didn't, that they weren't right.

After Ellie spoke about not liking discussion in small groups, she was countered by another student, Sam. What Sam likes about small group discussions compared with whole class discussions is that there are fewer people involved with whom one might have to engage in the repair of a relationship when the lesson is over:

SAM: Um, I like small discussions, small group discussions because like when Ellie said that the um, whole class discussion will prove you wrong, like [in small group] the most you can get is usually about five people trying to prove you

> wrong. Because other people, um, the rest of the class is all in their groups, some people well it all depends on who, how many people are absent or not and like the most is usually five.

These utterances indicate that what one does in mathematics class has both social and personal repercussions. How you handle a disagreement, even whether you participates in disagreements at all, can affect how you feel about yourself and how others feel about you and treat you. We might imagine, after listening to such comments, that a synonym these fifth graders would choose for "discussion" is "torture." The torture is milder, Sam reminds us, if fewer people are involved in doing it. We do not interpret his comments, or Ellie's, to suggest that these fifth graders are overtly mean to one another, either in small group or large group discussion. Rather it seems to be the case that these students experience "people trying to prove you are wrong" as a personal assault, even in situations where the teacher insists that assertions be justified with mathematical evidence.

What Some Fifth Graders *Do* about Mathematical Disagreements in School

We now turn to a study of student disagreement in a setting where students were not supervised by the teacher, yet had been asked to work with a small group to figure out a mathematics problem. (See Blunk, 1995.) This exchange was one of several that were videotaped in an attempt to collect a large sample of data about student interactions during "small group time." When we looked at a selection of these tapes, this one stood out because all four of the students in the group seemed to be actively engaged over a considerable period of time although there was no teacher intervention during the time they were on camera. We also chose to analyze this conversation because it was possible to hear most of the conversation among these four students even though the rest of the class was similarly engaged in argument.

It is early November in Lampert's fifth-grade mathematics class. Each day when the class comes into the room after recess, the problems to be discussed on that day are written on the blackboard (see Figure 31.2).[11]

In this lesson as in most every lesson during the year, about half of the one-hour class is spent in students working alone or with peers on the problems of the day. They talk and write in individual notebooks where they are expected to do "experiments" like drawing, figuring, and diagramming information, to make a conjecture about the solution to the problem, and to write their reasoning about why they think their solutions make sense. It is not required that the students reach a consensus in their small groups, but they are expected to understand the perspectives of others. The second part of class is a whole class discussion, like the one we looked at above, in which the teacher both participates in the mathematical conversation to model appropriate modes of discourse and speaks directly about how students are expected to interact with one another and use mathematical

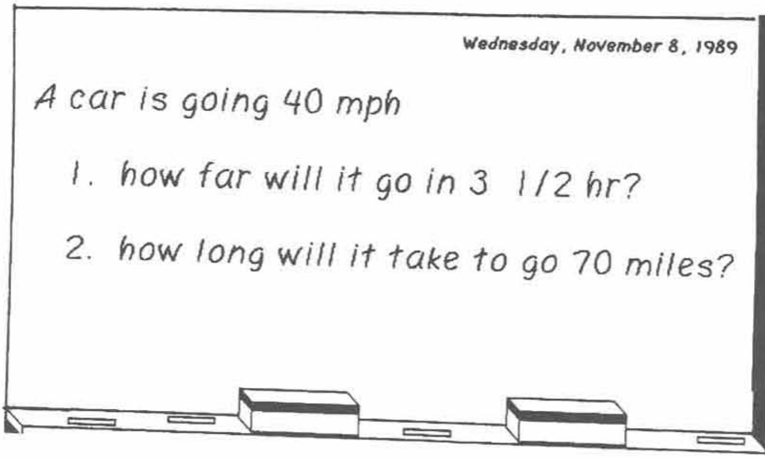


Figure 31.2

evidence to support their assertions. The teacher chooses the focus of the large group discussion based on her observations of students' work in the first part of the hour. During both parts of the lessons, students are expected to make and defend conjectures. They are expected to listen to, respond to, and question the teacher and one another. All assertions are open to investigation, and a frequently heard teacher response to any assertion is, "Why do you think that?"

Lampert provides some perspectives on how the problems she poses to her class might lead to discrepancies in students' thinking and disagreements about solutions. We include her reflections on the mathematics she expected students would be engaged in as students worked on these problems to provide a context for examining the nature of the students' disagreement:

> When these problems were posed, my class had been investigating time-speed-distance relationships for about two weeks. I had not given them a set of rules to follow to solve these kinds of problems, and they had not had any formal instruction about the concepts of rate and ratio. The mathematical work in which I intended the students to be engaged was not simply finding the answer to the time-speed-distance questions. If that had been my goal, I would have taught them a formula and given them the opportunity to practice using it. Instead, I encouraged them to reason about how changes in one of the variables (time) would affect the other (distance) given a constant rate of speed. Before this particular day, they had worked on problems similar to number one, above, where the speed is given and the problem is to find out how far vehicle will travel at that constant rate in a given time. At first, I assigned problems where the speed was given in miles per hour and the time was given in hours. As they worked on these problems, their reasoning involved something like: if the speed is 40 m.p.h. that means it goes 40 miles in one hour. They calculated the distance in journeys of more than one hour using successive addition or multiplication. Students talked about these relationships and drew diagrams to illustrate them.
>
> Then I gave them problems like problem 1 above, where the time was not a simple multiple of one hour, but included portions of an hour. In these problems, I expected that

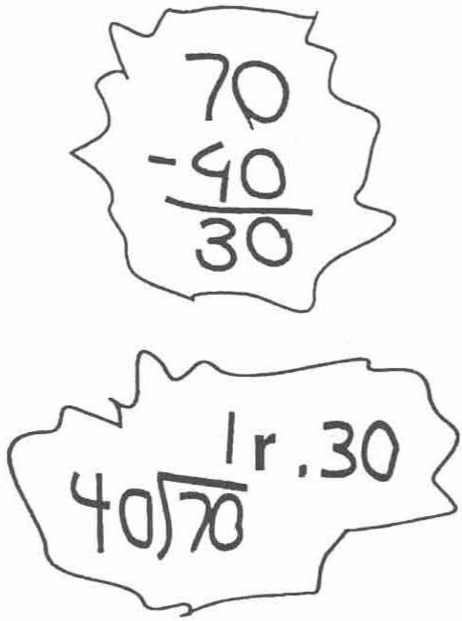


Figure 31.3

three operations would be needed to find the distance: first they would need to find how far a vehicle would go in three hours given that they knew about how far it would go in one hour; then they would have to find out how far it would go in half an hour given that they knew about how far it would go in one hour; and then they would need to add those two distances together. On the day of the discussion we look at here, I decided for the first time to give the class a problem where they knew the distance and they had to reason about the effect that it would have on the *time* the journey would take at a constant rate of speed.

So, what you know (in problem 2 above) is that the car goes 40 miles in one hour. And what you need to figure out is: How long will it take to go 70 miles? Using their by now familiar heuristic (i.e., that 40 m.p.h. means the car goes 40 miles in one hour), I expected that they might reason that it would take at least an hour for this journey, and that after an hour there would still be 30 *miles* left to go.

In terms of arithmetic operations, they could represent this as either taking 40 miles away from 70 miles and having 30 miles left, or dividing 70 miles by 40 miles per hour and having 30 left. But in either case they would face the question: Thirty what? How is that remainder of "30" to be related to the given information that the car goes 40 miles per hour? Based on what I had seen of their approaches to other problems, I expected to see students' first step in doing this problem look something like one or the other of the computations shown here (see Figure 31.3). Having done this computation, the students face a problem of mathematical interpretation: How does the difference of "30" in the subtraction or the "r. 30" in the division relate to the time-speed-distance relationship in the task they are working on? What does the remainder mean when you take 40 away from 70 and have 30 "left over." What is it that is left over? Would it be 30 minutes? Or 30 miles? Or is 30 a number on which another operation needs to be performed? That is the problem I expected them to struggle with on this day.

The relationship among 30, 40, 70, and 1 in this problem is not a simple one: The "1" in the quotient refers to one *hour*. The 30 in the "r. 30" is 30 *miles* and what is important here is that 30 miles is three-fourths of 40 miles. The next step is to figure out how to use that information to find how much more *time* the journey will take. Proportional reasoning is required to find the solution: because 30 miles is three-fourths of 40 miles, it will take *three-fourths of an hour* to complete the rest of the journey. After the proportional reasoning, another computation is required if one wants to express the time in minutes: how many minutes in three-quarters of an hour? And finally, one needs to remember that the car has already gone an hour before these extra 30 miles were under consideration. So, an hour for the first 40 miles, three-quarters of an hour for the next 30 miles, 1 hour and 45 minutes all together for the journey.

The tasks they were assigned on November 8 were deliberately designed to take me and my students on an interesting and worthwhile mathematical journey. I anticipated that there would be several ways in which discrepancies might arise in how students would think about this problem, especially in how they thought about the "30" that would be left. I was attempting to push their thinking, not only by giving distance and asking for time, but by giving a distance that involved their grappling with fractions of an hour in ways they had not done before.

As I introduced the task for the day, I sought from the students their interpretations of the difference between the two parts on this problem:

LAMPERT: Okay, this problem has two parts. It has one condition. The condition is: A car is going 40 miles per hour. And that condition applies to both problem 1 and problem 2. Problem 1 is: How far will it go in three and one-half hours? And problem 2 is: How long will it take to go 70 miles? Before you start, can you see the difference between problem 1 and problem 2? Raise your hand if you can see the difference.

After some discussion of the relationships among time, speed, and distance and the units in which each of these quantities might be measured, the class did not reach closure on the question of what units the solutions to each of these problems would be expressed in. I set the students to working on the problems with the hope that this would become clearer as they experimented with drawings and calculations and discussed their work with their small groups:

LAMPERT: Okay, I'd like everybody to work on this problem, and after you work on it, then we'll try and decide whether it should be miles per hour for the answer, miles for the answer, or hours for the answer. We need a number to go here and I want you to think about what all this means, and figure it out. And I want to see in your notebooks, I want to see the problem, I want to see experiments, and I want you to write your reasoning.

I was being deliberately unclear. I expected that as students talked with one another in their groups of four, they would grapple with the meaning of the numbers in this problem and how they should be related to one another. I also expected students to flounder as they explored what it would be like to try to get their mathematical ideas across, but I wanted to establish "talking with peers" as a norm for coming up with sensible interpretations and solutions. I also thought about such occasions as an opportunity to learn about students' capacities for communication about mathematics and to deliberate on what and how to teach them.

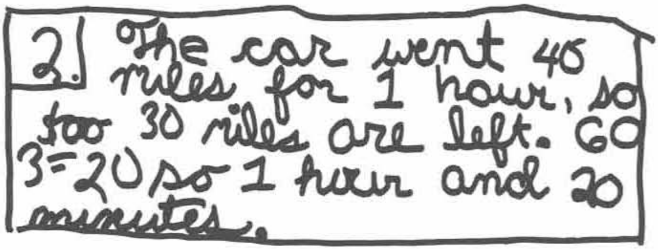

2) The car went 40
miles for 1 hour, so
too 30 miles are left. 60
3=20 so 1 hour and 20
minutes.

Figure 31.4

Having heard the teacher's perspective on the purposes of this lesson, we now look in on a group of students whose independent work on problem 2 (above) can be seen on videotape.

In the back of the room, during small group work time, four students are talking loudly and gesturing toward one another and their notebooks. The tone is contentious with intermittent attempts at reconciliation. They are working on the second part of the problem: How long will it take a car going 40 miles per hour to go 70 miles? First we see Sam and Enoyat, the two boys in the group, talking to one another. As Enoyat tries to figure out what to put in his notebook, he wants to know why Sam and Connie disagree, and he draws Connie into the conversation. There is some talk about the problem of figuring out the time, but they are distracted from their disagreement on this by the question of whether the "answer" is miles or minutes, and the disagreement here seems to be based on a misreading of what Connie has written in her notebook. In this exchange, Sam and Connie try to change one another's minds while also trying to convince another member of the group that their own approach is correct.

What Sam has written in his notebook under the heading "reasoning" is shown in Figure 31.4. This is the only clue we have to his reasoning besides what he says in the discussion while this group is on camera.

SAM: Sixty divided by 3 is gonna be 20 minutes. So it's 1 hour and 20 minutes, right? One hour and 20 minutes, not 120 minutes.

ENOYAT: One hour and 20 minutes?

SAM: Yes. One hour is going to be the 40 miles, there's 30 miles left. Sixty divided by 3 is 20. So 1 hour and 20 minutes. Do you understand?

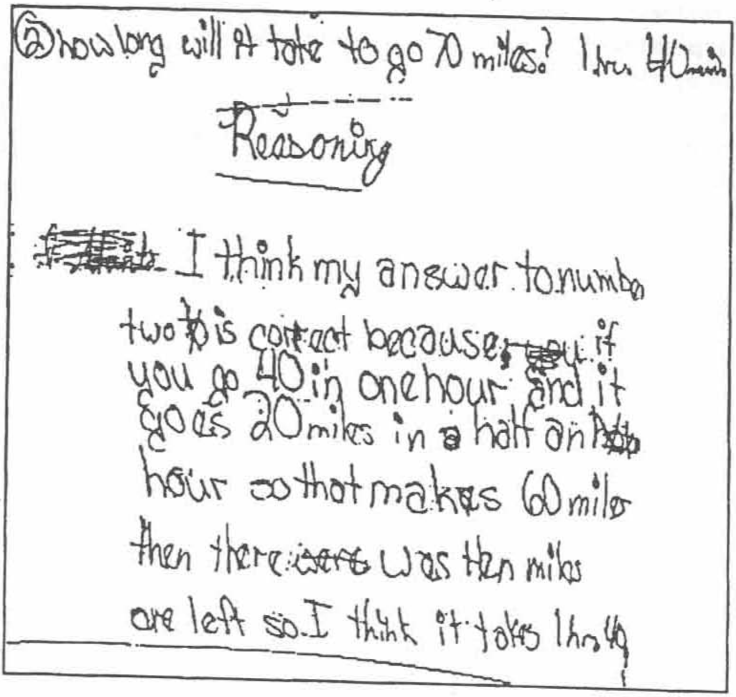
ENOYAT: Now I do. Why doesn't, why doesn't she agree? [Pointing to Connie across the table.]

SAM: I don't know, she says, um, she thinks the answer is 1 hour and 40 minutes.

CONNIE: No, I *know* what it is. I think the answer is 1 hour and 40 minutes.

SAM: Well, you didn't figure that out though, you just guessed. You said it had to be over my number.

Connie's notebook is shown in Figure 31.5. Connie has accounted for 40 plus 20 (or altogether 60) of the miles using proportional reasoning. She has 10 miles left



(2) how long will it take to go 70 miles? 1 hr. 40 min.

Reasoning

I think my answer to number two is correct because, ~~you~~ if you go 40 in one hour and it goes 20 miles in a half an hour so that makes 60 miles then there ~~were~~ was ten miles are left so I think it takes 1 hr 40

Figure 31.5

to account for, and it seems that what she does is add 10 minutes on to the time to account for the extra 10 miles.

ENOYAT: [Referring to both problems 1 and 2.] How can they both be 140 minutes?

SAM: No, no, one is talking about miles, one is talking about minutes.

ENOYAT: So, it can't be miles, it has to be minutes. It has to be. . .

CONNIE: I know! One hour and 40 minutes, not miles.

SAM: [Tapping aggressively on Connie's notebook.] You put miles.

CONNIE: [Pointing to her own writing.] M-I-N-period.

Connie seems to be in sort of a defend-my-answer mode, as she makes several attempts to explain her reasoning, her assumptions, even her penmanship. Sam's intonation, countenance, and notebook tapping, indicate that he is criticizing Connie. He seems to be trying to reduce her credibility with the others in the group, especially when he accuses Connie of "guessing" rather than "figuring it out." Sam's attempt to get Enoyat to "understand" involves repeating the calculations he did; he does not explain why he divided 60 by 3, or why the result of this calculation is 20 *minutes*. Connie's strong statement of her solution is not accompanied by an explanation of why she is so sure she is correct, either. Instead she exerts rhetorical power, emphasizing that she "knows" rather than "thinks" she is right, possibly believing that this will have a stronger effect on convincing her peers to agree.

After his disagreement with Connie about the time the journey takes is established, Sam turns to Catherine and tells her what he thinks. Catherine seems to be following his assertions when they are interrupted by Connie who offers a reasoned challenge to Sam's conclusion.

SAM: Okay, Catherine, have you heard what she told you, what she thought it was?
CATHERINE: Yes.
SAM: Now I think it's 1 hour and 20 minutes, okay? Cause the car is going 40, right? And that takes 1 hour.
CATHERINE: Uh-huh.
SAM: So there's, what is it, 30 miles left. Right?
CONNIE: Yeah.
SAM: Sixty divided by 3 is 20, right?
CATHERINE: Yeah.
SAM: So, 1 hour and 20 minutes.

Connie has been listening to this exchange, even assenting to the first part of Sam's reasoning about his solution. But when Sam states his conclusion, she strongly objects again, and tries to reason with Sam:

CONNIE: But NOOOOO! That's not right. How come it only goes 20 miles in half an hour and you're trying to get 30 so—
SAM: It doesn't go 20 miles, yes it does.

Marking her disagreement with a resounding "But NOOOOO!" Connie starts to ask an open-ended question, which points out the flaw in Sam's assumptions, and in his method and subsequent solution. Connie uses a technique familiar to mathematicians in her argument with Sam: She assumes that Sam's conclusion is true, and then reasons backward to show that his conclusion implies an assumption that is contradictory to the situation described in the problem. If the car goes 40 miles in 1 hour, and the length of the journey is 70 miles, it still has 30 miles to go after the first hour. Connie goes along with Sam's solution procedure up to this point.

But when Sam repeats the calculation he did—"Sixty divided by 3 is 20, right?"—Catherine agrees, and Sam goes on to conclude, "So 1 hour and 20 minutes." Here Connie strongly disagrees and draws attention to the weakness in Sam's thinking, which he starts to acknowledge. He listens to Connie explain that after 1 hour there are 30 miles left to travel, so if the car is going 40 miles per hour, it must take longer than 20 minutes to go that distance. Therefore, Connie reasons, the answer must be more than Sam's often-repeated "1 hour and 20 minutes." Perhaps because he is beginning to see her point of view, he wavers in his disagreement, saying, "It doesn't go 20 miles, yes it does."

If this discussion were going along as the teacher might have liked, what might happen at this point is that Connie would invoke stronger kinds of evidence to prove to Sam and to the group that it has to take more than 20 minutes for the car to go 30 miles. What happens instead is that reasoning and social negotiation become intermingled. In a mélange of social and mathematical moves, the students struggle to figure out how to both maintain their relationships and do what the teacher has asked.

As the discussion continues, all four students are struggling with the meaning of the "30" that is left over when 70 is divided by 40. They do some proportional reasoning, following Connie's lead, but it does not seem to result in any sort of mathematical resolution. Even Connie retracts her earlier certainty.

CONNIE:	So, it should be more, cause you gotta try and get 30, 30 more.
CATHERINE:	You gotta get 75.
CONNIE:	It's gotta go 70 miles. You got 30 left.
SAM:	You go 30.
CONNIE:	Then in half an hour it would go 20 miles.
ENOYAT:	Half an hour it would go 30.
CATHERINE:	Huh?
ENOYAT:	It's a half an hour.
SAM:	No, it wouldn't. It would go 20 miles.
ENOYAT:	Oh, okay. I thought you said half an hour is 20 minutes. I said no.
CONNIE:	I don't know.

In this somewhat confusing exchange, Connie first says, "It's gotta go 70 miles. You got 30 left." This suggests that what she thinks is left is 30 *miles*. Sam takes the "30" that Connie asserts is left and explicitly turns it into miles saying, "You *go* 30." But then Connie says, "Then in half an hour it would go 20 miles." Here, the "30" refers to minutes, it is "half an *hour*." Enoyat seems to be trying to combine both of these ideas, maintaining some sort of agreement with both Connie and Sam when he says "Half an hour it would go 30." At this point, it seems like what the students are doing is grappling with the problem that the teacher intended them to grapple with, trying to figure out the meaning of the "30" that is left over.

But this struggle gets left behind as the teacher moves the class from small group to large group discussion. We see Sam, in the next transcript excerpt, returning to his original position, saying he's "putting 1 hour 20." Connie immediately reasserts her stance and says she'll "put 1 hour 40." Here the students say what they are writing in their notebooks. Mathematically, the disagreement is not resolved. But socially, Enoyat comes up with a solution that may be an attempt to resolve the discomfort he feels in choosing between Connie and Sam:

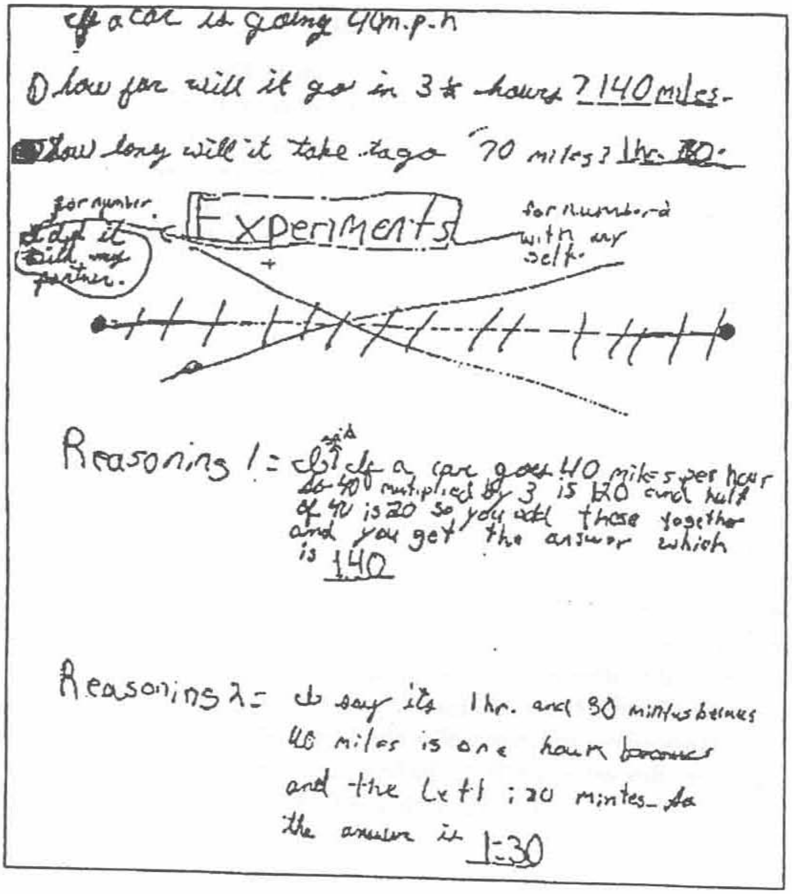
SAM: I'm just putting 1 hour 20.
CONNIE: I'll put 1 hour 40.
ENOYAT: What am I supposed to do, average you guys and write it down?
CONNIE: No, you write what you think.
SAM: Write 1 hour 30. I don't care. You write what you think the answer would be. Like, you either, you could either do it one of our ways or you could do it your way.
CONNIE: I'm going to try this.
ENOYAT: I like both of your ways.
CONNIE: I'll try it like this.
ENOYAT: I'll just average both of you guys out.

When Enoyat says, "I like both of your ways," he seems to be making a statement that has little to do with mathematical reasoning. Instead of arguing, he reminds Connie and Sam that he does not want to choose between them; he wants them to agree. In the next part of the conversation, Sam seems to be reminding everyone that they are supposed to have "experiments" in their notebooks—or perhaps he is using *his* experiments as a justification for what he is writing down as an answer. Connie has returned to her earlier conviction. Enoyat is still looking back and forth between Connie and Sam, and Catherine plays a comforting role by reassuring Connie that her handwriting is readable.

SAM: [Pointing to his notebook.] Experiments.
CONNIE: Yeah. Yeah, but I still think this is the answer.
ENOYAT: A car is going 40 miles per hour. How long will it take to go 70 miles? One hundred—[to Connie] what are you, what, what are you writing 1 hour and 40 minutes? [To Sam] 'Cause she wrote 140 minutes.
CONNIE: Look. ONE hour period dot, I mean 1 hour point 40 minutes.
ENOYAT: I thought that was a 1 and that was a 4 and a 0.
CONNIE: Nobody can ready my writing. I write too sloppy.
CATHERINE: No you don't.
[Catherine is defending Connie's handwriting.]
STUDENT: One hundred and thirty.
STUDENT: One hundred and forty.
ENOYAT: Let me write my reasoning down first, okay?

What Enoyat has written in his notebook for "number two" is indeed an average of Connie and Sam's answers, but he also adds a mathematical justification which was not part of the conversation at all. For him, reasoning in this situation is a private matter, done after listening to the views of his peers. He says he thinks the answer is 1 hour and 30 minutes because "30 minutes" are "left over." His notebook entry is shown in Figure 31.6.

What Catherine winds up with in her notebook (see Figure 31.7) is an answer to problem 2 that is the same as Connie's. She has written no "reasoning" to explain why she thinks that answer is correct.



If a car is going 40m.p.h

① how far will it go in 3½ hours? 140 miles-

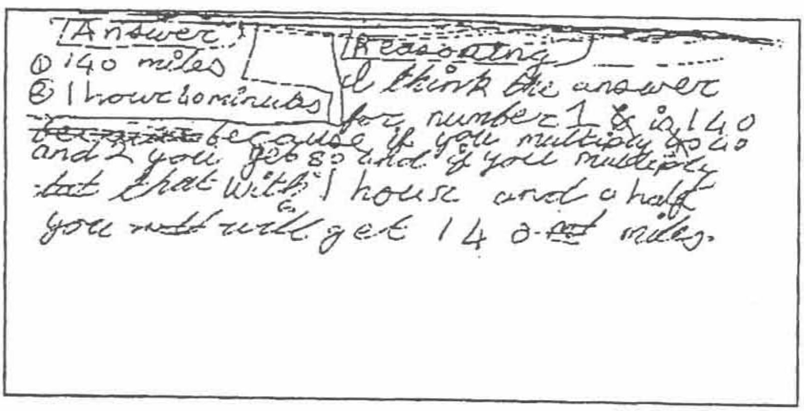
② How long will it take to go 70 miles? 1hr. 30·

for number 1 I did it with my partner. Experiments for number 2 with my self.

Reasoning 1 = I said if a car goes 40 miles per hour so 40 multiplied by 3 is 120 and half of 40 is 20 so you add these together and you get the answer which is 140

Reasoning 2 = I say its 1hr. and 30 mintus becaus 40 miles is one hour and the Left :30 mintes so the answer is 1:30

Figure 31.6



Answer

① 140 miles

② 1 hour 30 minutes

Reasoning

I think the answer for number 1 is 140 because if you multiply 40 and 2 you get 80 and if you multiply that with 1 hour and a half you will get 140 miles.

Figure 31.7

Reaching Agreement about Disagreement

In this discussion of a mathematics problem, Connie, Sam, Enoyat, and Catherine present evidence about their "theory in action" of what it means to disagree. How do they think they are supposed to act in a classroom when their answers to a math problem are different? As the teacher had hoped, the problem provoked some cognitive conflict, potentially productive of increased understanding—both about what computations are relevant and about how to label the answer. But the students experience discomfort and they believe it is important to smooth it over rather than quarreling. In this classroom where students have been told that "thinking" is valued, the conflict gets smoothed over by everyone "writing what they think." Their joint activity is not just an expression of what they bring to this conversation by way of beliefs about how to disagree—they are shaping those beliefs dynamically as they interact (Mead, 1964; Blumer, 1977; Davidson, 1986).

Connie, Sam, Enoyat, and Catherine are from different cultural backgrounds: African-American urban, white suburban, Muslim African urban, and Christian African rural. But they have all been attending an American school in a small city for several years. In addition to whatever they have learned about disagreement from their families, they have learned cultural routines for going to school, including learning what to do when they get a different answer to a math problem than their peers. We assume that they have refined their models of how to interact in such circumstances on the basis of what kinds of behaviors are functional for achieving a complex set of goals, including pleasing the teacher and maintaining friendships, or at least for avoiding getting teased or ostracized on the playground.

There is little evidence that anyone but Connie believes that what they are supposed to be doing as they "argue" mathematics is convincing the others, using mathematical evidence that an answer makes sense. Sam does try to convince others that he is right and Connie is wrong. His repetition of his calculations with increasing volume and gestural force might be interpreted as an expression of the belief that if the others would just listen to what he *did* they would change their minds, perhaps because *he* did it, perhaps because the computation is done correctly. He does not seem to believe that he needs to justify the need to do that particular calculation—it seems obvious to him, and presumably should be obvious to the others. Enoyat and Catherine act as if they believe that the most important thing to do in a mathematical argument is to maintain congeniality, letting everyone hang on to his or her idea, in the end, unchallenged. Although Sam's actions initially suggest that he believes that this conversation is supposed to be about convincing his peers to agree with him, in the end, his mathematical intention also gets confounded with a social one as he seems satisfied with everyone "writing what you think the answer would be."

The theory of how to manage a discrepancy in solutions and strategies here seems to be that any way is as good as any other way if someone "thinks" it. As it is played out in this exchange, the fifth graders' folk learning psychology seems to be relativistic. It seems important that everyone be able to "save face" and for Sam at least, this cannot be accomplished by letting himself be convinced that a peer's approach to the problem is more reasonable.

These students are interactionally defining the meaning of public argument among diverse peers as they argue in the classroom. They act in ways that take into account the actions of others, and because they have shared the culture of going to school, they can coordinate their actions and organize their responses on well-grounded assumptions about what the actions of others mean in this context. On the one hand, these students act as if it is appropriate to listen to others and try to communicate their understandings to them. (Whether they do this to learn or because they were directed to do so by their teacher is not clear.) On the other hand, they seem to feel that disagreement is hard and uncomfortable and that they have a responsibility not to make it too difficult on one another. They fail to resolve their disagreement in a way that draws on mathematical evidence and reasoning. At the end of the time allotted, Connie does not seem to have convinced anyone to think about the problem in a way that is true to the given conditions. But these students have solved another problem: They have invented moves for interaction that could be judged successful if the goal is for everyone in the group to feel socially comfortable at the point where the class turns from small group to large group work.

What happened in this conversation could be interpreted as an expression of competing conceptions of public disagreement. But, the practices of these students as they come to know mathematics collaboratively are not the ones that psychologists imagine should be in place if teachers embrace their theories about how learning happens; neither are they the practices celebrated in much of the mathematics reform literature. As in situations where learners are observed accommodating their prior beliefs to a new scientific theory to explain an observed physical phenomenon,[12] what we have seen here are temporary advances toward instantiations of the "new" theory of public argument put forth by the teacher and retreats to the more familiar norms that students bring with them in order to understand what they are supposed to do when someone has a different answer.

How Do We Interpret What These Fifth Graders Say and Do?

From the perspective of a teacher or an educational reformer who believes that mathematical discussion has the potential to produce understanding, there are several ways of interpreting what happened in this group and complementary ways of thinking about how to teach students to focus on the task of convincing others of the mathematical sense of their assertions. We might see these events as a characteristic of the children's stage of social development, imagining that older children or adults, less focused on finding an identity among their peers, would be more rationally assertive and persistent. In this case, we would simply need to be patient and wait until they "grow out of" their self-consciousness about disagreement. Or we might imagine that these fifth graders were ignorant of effective methods of group interaction, or that they had not been paying attention when lessons about how to develop consensus were taught, or that they simply need more practice. In this case we would want to reinforce the use of standard rules and roles, making clear to the students what counts as evidence and what sorts of challenges are allowed among peers. A third way to think about this case, and the one we will argue for here, is that these students were reflecting and reinventing cultural beliefs about how people are supposed to interact in public, protecting both their personal and political interests in a way that also enabled them to incorporate something of the practice of mathematical reasoning. What these students did *socially* made sense to them as a solution to the problem of the discrepancies in their answers to the problems. Their behavior could very well be described as functional, particularly in light of the kinds of interpretations of public disagreement illustrated by Ellie's comments above.

And their behavior is not atypical. Lynn Michel Brown and Carol Gilligan (1993) document several cases, drawn from a population of students in a good private school, of girls who as young children naturally seem to disagree and challenge one another or adults but starting around the age of nine or ten learn to value "being nice" and "studying the textbooks" over expressing their ideas because they want to avoid being negatively judged by their peers. As these girls link argument with losing relationships, they become better at figuring out how others think and at the same time they decide that accepting and pleasing others is safer than disagreeing. One girl refers to this development as "forgetting her mind" and focusing instead on "what's being shoved into her brain" (Brown and Gilligan, 1993, pp. 93, 138).

Catherine and Connie, Enoyat and Sam were in a classroom where they were expected to interact with one another and use mathematical evidence to support their assertions. Left to their own devices, what they did instead was to try to achieve agreement by repeating their positions, and by gathering social and political support (except for Connie). When these strategies failed to produce agreement, they agreed to disagree. The students' talk and actions suggest that, like the students studied by Brown and Gilligan, they believed it was at least as important to maintain relationships as it was to argue mathematics. The comments in the "discussion of discussions" by Ellie and other members of the class suggest that they are struggling with the social discomforts involved in figuring out the ways that others think and see. They do not entirely resist participation in discussions where disagreement occurs and is encouraged, like the discussion of "eight minus a half," but they do not wholeheartedly embrace this approach to learning mathematics, either.

Is this problem with maintaining social relations while talking about discrepancies in ideas only a problem for students at the upper elementary level? Is there any evidence that the problem "goes away" as people become more secure with their social positions? There are no formal studies of adult "folk theories" of public disagreement, but we can look at two sources for some hints about what their practice suggests such theories might say. Since we have written here about mathematical disagreement, we will look first at analyses of mathematical practice for evidence about what practitioners believe about debate and disagreement. Then we will look at what experience we might have as adults in this culture of discussing ideas with diverse groups of people in which disagreement and argument are expected, in or out of academia.

Although reform efforts like those reflected in the NCTM *Teaching Standards* are designed to bring classroom practice closer to practice in the discipline of mathematics, as we look at what adults in academic circumstances actually do, we recognize that what is being expressed in the *Standards* are the ideals of practice rather than reality (Lampert, 1992). In commentaries about mathematicians who try to grapple with disagreements, references to name-calling and social shaming are not unusual. In *Proofs and Refutations*, for example, in which Imre Lakatos portrays an interaction among mathematicians (including himself) arguing about a geometrical theorem, one way to manage the pain of disagreement is called "the Stoic theory of error": anyone who makes a mathematical mistake is considered by the others in the conversation to have a "sick mind" (Lakatos, 1976, p. 32). Further on in the discussion, Lakatos suggests to his fellow mathematicians that the person who was initially labeled by his peers as a "sick mind" instead might be thought of as an ally in the search for mathematical truth because his error turns into an insight. Here, he is expressing the sort of ideal that seems to be expressed in our thinking about what should happen in classrooms—students should expose their own and one another's mistakes and learn from discussing these mistakes about both the content and the discourse of mathematics. But in the discussion that Lakatos recounts, one of the participants responds to his idealism in a way that is reminiscent of what the fifth graders said about discussions with their peers: "Alpha" exclaims that "Opponents are less *embarrassing* than allies" (Lakatos, 1976, p. 33), suggesting that if one makes an error, the feeling of discomfort is not alleviated by knowing that someone else has made the same error. In their sociological analysis of mathematical rhetoric, Philip Davis and Reuben Hersch (1986, pp. 65–73) describe similar social norms governing actual disagreements among mathematicians. They list various methods that mathematicians are well known for using to convince their peers to agree, such as "proof by intimidation," "proof by eminent authority," and "proof by exhaustion (of the participants)."

The difficulties that arise from mixing social interaction with the refinement of ideas are characteristic of work outside of mathematics as well. Although we hold the ideal view that knowledge grows by revision and modification, it is risky to be the one whose idea gets revised or modified, even by oneself.[13] Considering all assertions to be tentative and open to reasoned challenge from one's peers—an approach to study that Peter Elbow (1986) calls "the doubting game" and Georg Polya (1968) calls "the inductive attitude"—seems to go hand in hand with damaged egos and feelings that one is being treated in a "mean" way. If it did not, Polya would not have found it necessary to speak of intellectual *courage* and *wise restraint* as the virtues required to be a good scientist. These are moral commitments that govern the way we regard others and their ideas (Scheffler, 1960; Peters, 1959). These mores are not based on the rules of logic. They do not replace the rules of logic, but they make it possible for human beings to disagree with less pain than they might otherwise experience.

And what of public disagreement about ideas outside of the academic workplace? We have all been in situations as children and adults where "might makes right" or something comes to be the accepted truth because no one has the courage to disagree or because the process of challenging ideas in a community is tedious and often even boring. We also know of situations where everyone agrees to play by the rules to avoid confrontation or embarrassment even when the rules are meaningless. How many people have ever participated in a discussion in which they have listened to, responded to, and questioned the teacher or one another? Or tried to convince themselves and others of the validity of their representations, solutions, conjectures, or answers? Or relied on mathematical evidence and argument to determine validity? Although we imagine that these practices should happen with some frequency in academic research, they are not ordinary practices for working out public disagreements. They are not the activities that most people think of when they imagine trying to learn something.

Experience of academic argument as an amicable mode of interaction in our culture is rare. More likely, in the nonacademic community, adults and children justify their solutions to problems by reference to answer books or reference books or newspapers or television. It is considered appropriate to call upon authorities like teachers or parents or bosses to sanction the validity of our approaches to problems. Many people believe that "memory" or "natural genius" account for being able to get the right answer in a math problem (Schoenfeld, 1985; Stigler and Baranes, 1988; Dweck, 1983), and so by implication, there is nothing to discuss when someone gets the wrong answer. Since the rise of scientific problem solving at the beginning of this century, there has been an increasing tendency to believe that professional "experts" are responsible for coming up with solutions to social as well as physical problems, and the rest of us simply need to be kept informed of their results.

Both inside and outside of academic settings, people also have beliefs about appropriate ways to interact in public settings that are likely to bear on the choices they make about what they will do with one another to learn, in or out of school. "Arguing" is thought to be something that people do when they are being impolite, not getting along, or losing self control (Belenky, Clinchy, Goldberger, and Tarule, 1986). Disagreement, especially disagreement with a person in authority, is thought to be aggressive and even antisocial behavior. As a people, Americans are more disposed toward isolation than toward the reasoned determination of public consensus (Bellah, Madsen, Sullivan, Swidler, and Tipton, 1985). We do not have many cultural mores that support the conjunction of disagreement and civil social encounters. In the talk of adults outside of academic settings, there are few places where we can regularly witness debate among the kind of diverse populations found in many school classrooms. This form of talk has become popular on television and radio talk shows, and we sometimes see political figures engaged in debates. However, recent experience suggests that these interactions involve more name-calling and defamation of character than inquiry and reasoning.[14]

It is not surprising then, that students in a classroom, left to their own devices, would not choose to argue mathematics with their classmates without also giving some attention to smoothing over the social discomforts, even if this interferes with good reasoning. The school classroom is a place where friends are made and lost, where identity is developed, where pride and shame and caring and hurting happen to kids. What they learn from social interaction cannot be described simply in terms of the mathematics covered by hashing out logical conflicts between various approaches to a problem. Mustering evidence to prove that an assertion is right or wrong is not a decontextualized learning activity. In the classroom, mathematical argument is done with and to the same people one plays with, eats lunch with, lives next door to, or has a crush on.

In a recent study of the problems that arise for children when they are asked to evaluate one another's writing, Tim Lensmire (1994) observes:

> Children bring to the classroom, playground and cafeteria experiences, individual and collective histories in and out of school, that contribute to their evaluation of each other as friends and audiences. As teachers of any particular group of children, we have limited control over important aspects of peer relations. I am certainly not saying that we can do nothing to influence or enhance these relationships. Only that, at any given moment, children are working out their relations with each other, and they are doing it from their pasts, behind our backs, and outside the room, as well as within the situations we have greater access to and upon which we exert greater influence. (pp. 73–74)

Like Lensmire, we are "certainly not saying that we can do nothing to influence or enhance" the kinds of relationships that students can develop with one another in school. What we do argue is that bringing about the kind of social climate in schools that supports academic argument requires a major shift in cultural norms. Simply providing rules for the standardization of interactions in "math groups" will probably not be enough to reduce the students' discomfort and make change possible. If students are to "listen to, respond to, and question the teacher and one another. . . . make conjectures and present solutions,. . .[and] try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers" (NCTM, 1991), some way must be found to minimize the personal and social risks that does not interfere with serious argument.

Talking about communication in conjunction with understanding is a radical idea. Understanding used to be thought of as a function of individual minds, and teaching and learning as transactions between the teacher and individual learners, even when there are 30 of them in the room at one time. But school people are beginning to take into account the social construction of knowledge, the relations between thought and language, and the importance of collaboration to real problem solving. Our research suggests that the teacher is an important partner in the knowledge-building collaboration that occurs in classrooms. Without the teacher's intervention, the disagreement about the meaning of "eight minus a half" (described in the first part of this paper), for example, would likely have been very different. The teacher's knowledge of the fundamental mathematical structures behind the linguistic expressions that students were constructing in this discussion enabled her to focus the disagreement in ways that required the clarification of important conceptual differences (see also Hoyles and Noss, 1992).

But the teacher's role goes beyond the connection of students' work with big ideas in the discipline. Our analysis of how students "agree to disagree" when left to construct their own social interaction suggests that teacher intervention is also significant on the social front. Because common cultural beliefs about public argument run counter to the ideals of academic discourse, and those "folk theories" about how to learn in school are shared by students, the teacher must take on the task of modeling a different set of social norms and offering students safe mechanisms for expressing their thinking when it is different from that of their peers. Discourse-oriented reforms are not a simple matter of mandating different kinds of classroom structures like small-group problem solving or the discussion of discrepant ideas among peers. These reforms require not only a fundamental reconstruction of students' beliefs about what knowledge is and how it is acquired but also a re-forming of how everyone in the classroom thinks about what is appropriate in the way we talk with our friends and our enemies about ideas.

Acknowledgments

This chapter is based on a paper presented in 1994 at the annual meeting of the American Educational Research Association, in New Orleans. The authors wish to acknowledge the thoughtful comments of Peri Weingrad, Mary Catherine O'Connor, Susan Florio-Ruane, and David Cohen on earlier drafts of this chapter. We would also like to acknowledge the Spencer Foundation for funding the data analysis and the National Science Foundation for funding the data collection on which this chapter is based. The data presented, the statements made, and the views expressed are solely the responsibility of the authors.

Notes

1. In this paper, we use the terms "discussion" and "conversation" interchangeably to refer to the talk among students and teacher in Lampert's classroom. We do not intend to be making here a technical claim about the nature of this discourse.

2. During the 1989–90 school year, a research team collected extensive data to conduct an in-depth study of a year of teaching and learning practice in Lampert's fifth-grade mathematics classroom. They recorded daily classroom interactions on both video- and audiotape, kept field notes on all lessons, maintained daily logs of student work, interviewed students, and collected Lampert's thoughts about the lessons by having her keep a daily journal of plans and reflections and interviewing her on a regular basis. A portion of the data collected were used in the study reported in this chapter.
3. We are indebted to Nan Jackson for the close analysis she has done of this lesson segment and its relation to other occurrences in Lampert's class during the 1989–90 school year.
4. Douglas Barnes (1976) calls this "exploratory talk" and analyzes the difference between this kind of classroom discourse and the more "finished" talk that is conventionally employed in answers to teachers' questions.
5. We recognize that this is a problematic concept and we are not claiming to define "the social construction of knowledge" by our use of this term here. See Driver, Asko, Leach, and Scott (1994) and Cobb (1994) for a discussion of the multiple ways in which this term might be interpreted.
6. Although we do not have the space to report on this aspect of our research in detail, our claims here are substantiated by the analysis of interview data with the students collected before, during, and a year after their participation in Lampert's class.
7. See Stodolsky (1988) for a characterization of norms of interaction in American mathematics classes at the upper elementary level. For a contrast between this conventional set of American classroom norms and those found in Japan, see Stigler and Baranes' (1988) description of what happens to student "errors" in elementary-level mathematics classes.
8. It is not assumed that these students are becoming mathematicians. Rather they are being taught where mathematics fits in the lexicon of ways of knowing (Lampert, 1990). See also Kitcher (1984) for a discussion of the relationship between mathematical activity, mathematical language, and mathematical knowledge.
9. It is not assumed here that American culture is monolithic. Based on recognized differences in our own ethnic backgrounds, we are aware of the fact that families and communities range broadly in how they handle disagreement. What is at issue here is how people from different ethnic backgrounds behave when they disagree in the presence of others of similarly diverse backgrounds.
10. A paper in progress analyzes the entire "Discussion of Discussions" as well as interviews with Lampert's students after they have spent a year in more traditional sixth-grade classrooms.
11. Calling these "problems" does not mean to suggest that they were the mathematical problems that engaged students during their work together. They might more appropriately be called "tasks" or "questions." What we write here uses the everyday language of the classroom in which they were called "Problems of the Day." See Lampert (1990) for an analysis of the pedagogical difference between such "problems" and the mathematics that engages students as they work on them.
12. See for example diSessa's (1982, 1983) work on students learning Newtonian physics. We are indebted to Mary Catherine O'Connor for pointing out this analogy.
13. See Peter Elbow's essay on literary composition entitled "Closing my eyes as I speak: An argument for ignoring audience" and Pamela Richards' autobiographical reflections on writing for social scientists in Becker (1986). Richards says: "For me, sitting

down to write is risky because it means I have to open myself up to scrutiny. . . . Every piece of work can be used as evidence about what kind of sociologist (and person) you are. . .I cannot face the possibility of people thinking I'm stupid" (pp. 113–114).

14. This seemed to be particularly the case in the November 1994 elections in the United States when many congressional members, senators, and governors were up for election.

References

Ball, D. (1991). "What's all this talk about discourse?" *Arithmetic Teacher, 39* (3), 44–48.

Barnes, D. (1976). *From communication to curriculum*. Harmondsworth, England: Penguin Books.

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., and Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice, and mind.* New York: Basic Books.

Bellah, R. N., Madsen, R., Sullivan, W. M., Swidler, A., and Tipton, S. M. (1985). *Habits of the heart: Individualism and commitment in American life.* New York: Harper and Row.

Blumer, H. (1977). Sociological implications of the thought of George Herbert Mead. In B. R. Cosin, J. R. Dale, G. M. Esland, D. Mackinnon, and D. F. Swift (Eds.), *School and society: A sociological reader* (pp. 11–17). London: Routledge and Kegan Paul.

Blunk, M. (1995). "Communication in and about small group learning." Research report, Mathematics and Teaching through Hypermedia Project. Ann Arbor: University of Michigan.

Brown, J. M. and Gilligan, C. (1993). *Meeting at the crossroads.* Cambridge: Harvard University Press.

Bruner, J. (1990). *Acts of meaning.* Cambridge: Harvard University Press.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23* (7), 13–20.

Cobb, P., Wood, T., and Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In N. Minick, E. Forman, and A. Stone (Eds.), *Education and mind: Institutional, social, and developmental processes* (pp. 91–119). New York: Oxford University Press.

Davidson, D. (1986). A nice derangement of epitaphs. In E. LePore (Ed.), *Truth and interpretation* (pp. 433–446). New York: Basil Blackwell.

Davis, P. and Hersch, R. (1986). *Descartes dream: The world according to mathematics.* Orlando: Harcourt, Brace, Jovanovich.

diSessa, A. A. (1982). Unlearning Aristotelian physics: A study of knowledge-based learning. *Cognitive Science, 6*, 37–75.

diSessa, A. A. (1983). Phenomenology and the evolution of intuitions. In D. Gentner and A. L. Stevens (Eds.), *Mental models* (pp. 15–33). Hillsdale, NJ: Lawrence Erlbaum.

Driver, R., Asko, H., Leach, J., and Scott, P. (1994). Constructing scientific knowledge in the classroom, *Educational Researcher, 23* (7), 5–12.

Dweck, C. S. and Bempechat, J. (1983). Children's theories of intelligence: Consequences for learning. In S. G. Paris, G. M. Olson, and H. W. Stevenson (Eds.), *Learning and motivation in the classroom* (pp. 239–256). Hillsdale, NJ: Lawrence Erlbaum.

Elbow, P. (1986). *Embracing contraries: Explorations in teaching and leaning.* New York: Oxford University Press.

Hatano, G. and Inagaki, K. (1991). Sharing cognition through collective comprehension activity. In L. B. Resnick, J. M. Levine, and S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 331–348). Washington: American Psychological Association.

Heath, S. B. (1983). *Ways with words: Language life and work in communities and classrooms.* New York: Cambridge University Press.

Holland, D. and Quinn, N. (1987). *Cultural models in language and thought.* Cambridge: Cambridge University Press.

Hoyle, C. and Noss, R. (1992). *Learning mathematics and Logo.* Cambridge: MIT Press.

Isaacs, S. (1971). *The children we teach: Seven to eleven years.* New York: Schocken.

Kitcher, P. (1984). *The nature of mathematical knowledge.* Oxford, England: Oxford University Press.

Lakatos, I. (1976). *Proof and refutations: The logic of mathematical discovery.* J. Worrall and E. Zahar (Eds.). Cambridge, England: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27* (1), 29–64.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics in school. In F. Oser, A. Dick, and J.-L. Patry (Eds.), *Effective and responsible teaching: The new synthesis* (pp. 295–314). New York: Jossey-Bass.

Lensmire, T. (1994). *When children write: Critical re-visions of the writing workshop.* New York: Teachers College Press.

Mead, G. H. (1964). *On social psychology: Selected Papers.* Chicago: University of Chicago Press.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Olson, D. R. and Astington, J. W. (1993). Thinking about thinking: Learning how to take statements and hold beliefs. *Educational Psychologist, 28* (1), 7–23.

Peters, R. (1959). *Authority, responsibility, and education.* London: Allen and Unwin.

Polya, G. (1968). *Induction and analogy in mathematics.* Princeton: Princeton University Press.

Putnam, R., Lampert, M., and Peterson, P. (1990). Alternative perspectives on knowing mathematics in elementary schools. In C. Cazden (Ed.), *Review of research in education* (Vol. 16, pp. 57–150). Washington: American Educational Research Association.

Richards, P. (1986). Risk. In H. S. Becker (Ed.), *Writing for social scientists: How to start and finish your thesis, book, or article* (pp. 108–120). Chicago: University of Chicago Press.

Scheffler, I. (1960). *The language of education.* Springfield, IL: Thomas.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Spradley, J. P. (1972). *Culture and cognition: Rules, maps, and plans.* New York: Chandler.

Stigler, J. and Baranes, R. (1988). Culture and mathematics learning. In C. Cazden (Ed.), *Review of research in education* (Vol. 15, pp. 253–306). Washington: American Educational Research Association.

Stodolsky, S. (1988). *The subject matters: Classroom activity in math and social studies.* Chicago: University of Chicago Press.

Strike, K. and Posner, G. (1985). A conceptual change view of knowing and learning. In L. West and A. L. Pines (Eds.), *Cognitive structure and conceptual change* (pp. 211–232). Orlando, FL: Academic Press.

Swales, J. (1990). *Genre analysis: English in academic and research settings.* Cambridge, England: Cambridge University Press.

Willoughby, S., Bereiter, C., Hilton, P., and Rubenstein, J. H. (1990). *Real math: Level five.* LaSalle, IL: Open Court.

Wiser, M. and Carey, S. (1983). "When heat and temperature were one." In D. Gentner and A. L. Stevens (Eds.), *Mental models* (pp. 267–298). Hillsdale, NJ: Lawrence Erlbaum.