ANALYSIS OF ARITHMETIC FOR MATHEMATICS TEACHING

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CHAPTER

4

TEACHING AND LEARNING LONG DIVISION FOR UNDERSTANDING IN SCHOOL

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Contents

> I want to get through division with two-digit numbers because then I can really go with the flow of kids' ideas, be more creative in math. Kids put a lot of math sense on finishing long division. It seems to finish arithmetic for them.
>
> —Fifth-grade public school teacher

"Long" division is an important milestone in this teacher's work with his fifth-grade students. And he believes that the students themselves look at this topic as the culmination of their work on whole number arithmetic. Yet the topic is not associated, in his mind, with creativity or "the flow of kids' ideas." Mathematics educators who wish to reform curriculum and instruction would agree that long division, as it is now taught, has little to do with mathematical creativity, and may even get in the way of "the flow of kid's ideas." Teaching the long division algorithm often is used as an example of the misplaced emphasis on memorizing procedures in elementary arithmetic, and this emphasis is thought to be in opposition to understanding. Educators and mathematicians argue that the arithmetic procedures that dominate the school curriculum take students' and teachers' attention away from more essential mathematical ideas (Clemens, 1989; Hilton, 1989; Mathematical Sciences Education Board, 1989; National Council of Teachers of Mathematics, 1989).

What are the essential mathematical ideas underlying the long division procedure? Can the division of one large number by another be taught in ways that relate it to students' understanding of important concepts in mathematics? Is it possible to teach division in ways that support students' creative engagement with mathematical ideas? What might it look like for learners to "understand" the process of division and apply it to work with large numbers in school?

As we think about reforming the way we teach mathematics, we can decide to take the topic conventionally called "long division" out of the curriculum or we can decide to keep it in. One argument for keeping it is that it is familiar: to teachers, to parents, and to children. All of these groups consider it to be an important milestone in measuring mathematical accomplishment within the school culture. But this does not seem reason enough to teach long division, given that the technology for performing the procedure is now widely available. This chapter presents two other kinds of arguments for keeping long division in the curriculum: one derived from an analysis of the relationship between procedures for doing long division and important mathematical concepts, and the other based on the results of trying out some conceptual approaches to teaching long

division in a fourth-grade classroom.[1] In making this argument, the emphasis moves away from teaching the long division algorithm and toward teaching the relationships between the operation of division and important mathematical concepts and activities.

CHANGING TEACHER AND STUDENT ROLES IN SCHOOL MATHEMATICS

Although primarily addressing the question of what it might mean to teach long division for understanding in school, the research reported in this chapter also examines more general issues of mathematical pedagogy. In the case of teaching and learning long division that is examined here, students are expected to make mathematical assertions and verify the validity of those assertions within the class as a community of discourse about mathematics. Teaching mathematics in a way that engages learners in mathematical discourse means changing the roles that teacher and students play in classroom activities. The teacher poses problems and raises questions about solutions that lead students to examine, in public discussions during lessons, the reasonability of their own assertions and those of their peers. Students test their symbolic manipulations by mapping them on to parallel operations in more familiar domains. The teacher is a source of information about mathematical conventions and provides that information to students in response to their constructions of mathematical ideas.

If lessons on arithmetic operations such as long division are to be taught within the context of a mathematical discourse, they must be constructed to emphasize learning about why a particular move in the procedure is legitimate rather than simply focusing on which moves to make in which order (Lampert, 1986a, 1986b, 1990b). The goal of such arithmetic instruction would be to have students be able to explain why they do each step in a procedure in terms of its mathematical legitimacy. In addition to being able to explain the moves in the conventional procedures for addition, subtraction, multiplication, and division, students would be encouraged to

[1]The study reported here was part of a larger study, "Teaching Mathematics for Understanding and Understanding Mathematics Teaching," which the author conducted as a Spencer Fellow with the National Academy of Education. The author would like to acknowledge the assistance of Peggy Karns in collecting data about how long division is presented in conventional texts, the support of Jackie Frese and Nancy Arnold for recording observations of teaching and learning, the editorial suggestions of Gaea Leinhardt, Lauren Resnick, Orit Zaslavsky, Jack Smith, and Vicky Kouba, and the collaboration of Thom Dye in continuing research on the teaching of long division in school.

invent their own alternative procedures and expected to be able to explain the legitimacy of their inventions (Lampert, 1990b). The content of the curriculum thus would be deeply integrated with the mode of discourse and the social structure of the class (Lampert, 1988, 1990a). This approach to teaching arithmetic procedures is derived from an examination of the nature of mathematical knowledge: Because the evidence that makes mathematical assertions true and mathematical procedures legitimate is logical argument, students are expected to legitimate their answers with mathematical arguments appropriate to their level of expertise.[2]

The discipline-derived argument for teaching students to reason about arithmetic is complimented by arguments in psychology that suggest that if students can explain and invent legitimate procedures, then they understand the mathematical concepts that underlie those procedures (Greeno, Riley, & Gelman, 1984; Putnam, Lampert, & Peterson, 1989). No claim is made here that students' capacity to explain the mathematical legitimacy of the steps in arithmetic procedures will cause them to carry out those procedures faultlessly. Indeed, research on student learning thus far has failed to establish such a connection (Nesher, 1987; Resnick & Omanson, in press). The claim that is made is both more modest and more bold: A curriculum and a method of instruction can be invented that engages students in the activity of intentionally understanding authentic mathematical ideas in school—even when the topic is something as mathematically mundane as long division.[3]

THE MATHEMATICAL CONTENT IN LONG DIVISION

Most simply, long division could be defined as division that operates on large numbers. Because of the magnitude of the numbers, the operation requires the decomposition of the dividend and the divisor, such that the actual process of dividing occurs over several steps, and the quotient is assembled from their results. Finding the quotient in the most basic kind of division involves only one step: a simple reversal of the multiplication tables (e.g., 48 divided by 6 is 8 because 8 times 6 is 48). But these divisions become the first step in learning the symbolic routine for taking account of place value when division is carried out on large numbers. Beginning with

[2]See King and Brownell (1966) and Schwab (1978) for the argument that curriculum and instruction might be designed to reflect rules of discourse in a discipline.

[3]See Bereiter and Scardamalia (1989) for an explication of the notion that intentional learning might be related to school curriculum and instruction.

quotients that have only one digit, students are instructed to write the operation in its traditional form and "line up the ones" in the quotient with the "ones" in the dividend, for example:

$$\begin{array}{r} \text{tens ones} \\ 8 \\ 6\,\overline{)4\ \ 8} \end{array} \tag{1}$$

The traditional curriculum then moves on to divisions with one-digit divisors and larger numbers in the dividend, and students are taught to divide by moving from place to place, dividing when possible, "carrying" the remainder from one place to the next, and changing the way the units are grouped; the leftover thousands become hundreds, the leftover hundreds become tens, and the leftover tens become ones, for example:

$$\begin{array}{r} 1\ \ 4\ \ 2\ \ 3\ \text{R. } 4 \\ 6\overline{)8\ {}^{2}5\ {}^{1}4\ {}^{2}2} \end{array} \tag{2}$$

The dividend is decomposed by place values, and each place is operated on separately.

Breaking large numbers into parts to operate on them is not unique to the division procedure. All of the familiar arithmetic procedures work on multidigit numbers place by place. The possibility of operating on the digits one or two at a time rather than operating on the whole number at once is what makes our base-ten number system usable without reference to tables of equivalences; this feature of the Arabic number system is thought to be the basis for its rapid spread through the Mediterranean trading basin in the 15th century, when it replaced the more cumbersome Roman and Greek notations and made calculation available to a large segment of the population (Swetz, 1987).

In contrast to the school-taught algorithms for other operations, division begins with work on the left-hand side of the number rather than in the "ones" place. For example, in the short division in Equation 2, above, the conventional procedure begins with the operation "8 divided by 6," then goes on to "25 divided by 6," then "14 divided by 6," then "22 divided by 6." One does not operate on the original dividend as a whole quantity (eight thousand five hundred and forty-two) but on a succession of one- or two-digit numbers (eight, twenty-five, fourteen, twenty-two). The string of numbers being divided one after the other is difficult to relate to the magnitude of the dividend because as the "leftovers" from each place are carried to the right they are transformed into different kinds of units; thousands are traded for hundreds, hundreds for tens, tens for units.

In division by a two-digit divisor, the process is made even more complex by adding yet another step that takes the focus away from the magnitude of the numbers being operated on. For example, in

$$\begin{array}{r} 270 \text{ R. } 12 \\ 18\overline{)4872} \\ \underline{36} \\ 127 \\ \underline{126} \\ 12 \\ \underline{0} \\ 12 \end{array} \tag{3}$$

the task is to divide successive numbers by eighteen, but figuring out how many times eighteen "goes into" the successive parts of 4,872 would require knowing multiples of eighteen. Instead of expecting students to know the multiples of any possible divisor, we teach "estimation": Eighteen is close to twenty, and to divide a number like forty-eight by twenty, you need to "think": "two into four" or maybe "two into five." (Cf. Fennell, Reys, Reys, & Webb, 1987, p. 376, for example.) Why does it work to do that when the original number one wants to divide by eighteen is four thousand eight hundred and seventy-two? The mathematical justification that makes this estimation procedure (as well as the step-by-step attention to the dividend) legitimate is based on the relationship between division and the concepts of ratio and proportion.

What Does Long Division Mean in the Discipline of Mathematics?

Within the discipline of mathematics, the operation of division can be given meaning by seeing it as one of many "multiplicative structures" (Vergnaud, 1983, 1988). Those areas of the conventional curriculum that fall into the domain of multiplicative structures include multiplication, division, fractions, decimals, ratio, proportion, percent, and linear and nonlinear functions (Hiebert & Behr, 1988). Multiplicative structures are mathematical ideas that are used to analyze situations that can be modeled by proportions, and multiplicative problems are those that are solved by proportional reasoning. They are differentiated from additive structures, which are modeled by counting; additive problems are solved by joining

and separating quantities.[4] In proportions, a relationship between two numbers comes to be regarded as a number in and of itself; for example, the meaning of a number like $\frac{4}{5}$ is derived from the relationship between four and five. We say that the number $\frac{8}{10}$ is equivalent to the number $\frac{4}{5}$ because 8 and 10 have the same relationship to one another that 4 has to 5 and we call the equality $\frac{4}{5} = \frac{8}{10}$ a proportion. By contrast, when they are employed in additive structures, the numbers four, five, eight, and ten would be regarded as indications of quantity or size, and they would be variously grouped and counted.

To illustrate how the conventional procedure for carrying out long division instantiates the mathematical concept of multiplicative structures and engages its users in the relationships of ratio and proportion, we consider the division

$$73\overline{)1536} \tag{4}$$

In this division, the mathematical question being asked is: "What number multiplied by 73 will yield 1,536 as a product?" By the definition of multiplication, we come to be working with groups of 73 units, and we want to know how many such groups will be needed to make a total of 1,536 units: "If 73 is 1 group, 1,536 is how many groups?" Or, "What number will relate to 1,536 in the same way that 1 relates to 73?" In proportional terms, we have the relationship:

$$\frac{73}{1} = \frac{1536}{X} \tag{5}$$

where X is the quotient to be determined. This pair of equal ratios represents a linear function that maps a group of 73 units on to each single unit. Using conventional mathematical symbols, the function is written as:

$$\frac{73}{1} = \frac{f(X)}{X}$$

or

$$f(X) = 73X \tag{6}$$

Given this way of thinking about the problem, we can produce other ratios to fit the function, like

$$\frac{f(X)}{X} = \frac{146}{2} = \frac{292}{4} \tag{7}$$

[4]The distinction between additive and multiplicative structures is a theoretical one. There is no implication here that these types of reasoning would be easy to sort out in a particular instance of mathematical activity.

This way of thinking about the problem of division is what Vergnaud (1983) called a "dimensional model" involving proportional relationships among four terms (in the case of this division, 73:1::1536:X) and it is what Nesher (1988) called a "mapping rule"; that is, the function tells us how many times 73 will map on to a unit in the quotient.

It is this proportion or mapping rule that describes the kind of mathematical argument that is involved in justifying the first and subsequent steps of the conventional procedure for finding the answer to a long division problem. When we start to think about this division

$$73\overline{)1536} \tag{8}$$

by figuring out what multiple of 7 is close to 15, we are assuming that the quotient of

$$73\overline{)153} \tag{9}$$

will be about a tenth of the quotient of

$$73\overline{)1536} \tag{10}$$

because 153 is about a tenth of 1536, and the quotient of

$$7\overline{)15} \tag{11}$$

will be the same as the quotient of

$$73\overline{)153} \tag{12}$$

These assumptions are based on the place value and proportional relationships among these pairs of numbers.

In order to understand why 15 divided by 7 is a mathematically legitimate way to begin to find out the quotient to 1,536 divided by 73, one would need to think in terms of proportions or linear functions. That is, in order for this step in the procedure to be supported by mathematical reasoning (as opposed to being a mechanical procedure supported by a teacher's or a textbook's authority), one must consider the approximate equivalence between $\frac{153}{7}$ and $\frac{1536}{73}$, while at the same time recognizing the difference in order of magnitude between 153 and 1,536, and between 7 and 73.

In terms of functions, we have $f(X) = 73X$, and what we want to find is the value of X that will yield 1,536. This is the kind of thinking that goes into inventing a mathematically legitimate procedure for finding the quotient of these particular numbers. Reasoning proportionally, from 73 times 2,[5] one

[5]Doubling as a way to begin the proportional reasoning process here is somewhat arbitrary. It is productive in this case because of the particular numbers involved, but the choice of doubling as a place to begin has no particular mathematical justification.

might assert that the answer to the division is close to 20 because if 73 times 2 is 146, then 73 times 20 is 1,460:

$$\frac{73}{1} = \frac{146}{2} = \frac{1460}{20} = \frac{f(X)}{X} \tag{13}$$

The answer to the division problem will be obtained by continuing to produce equivalent fractions until one is found with a numerator of 1,536. Continuing with the process of successive approximation, 73 times 21 would be 73 more than 1,460 or 1,533, and 73 times 22 would be 73 more or 1,616. So we have

$$\frac{73}{1} = \frac{146}{2} = \frac{1460}{20} = \frac{1533}{21} = \frac{1616}{22} \tag{14}$$

from which we can conclude that the answer to the division is between 21 and 22, and very close to 21, because 1,533 is comparatively close to 1,536. This reasoning can also be represented in a function chart as follows:

X	$f(X)$
1	73
?	1536
2	146
20	1460
21	1533
22	1616

If X is doubled, $f(X)$ changes proportionally; it is also doubled. If X is multiplied by 10, so is $f(X)$. If X_1 is added to X_2 to get X_3, then $f(X_3)$ can be obtained by adding $f(X_1)$ and $f(X_2)$. Using this way of thinking, it is possible to build up to the value of X that will produce "1,536" by making combinations of known values. We get $f(22)$, for example, by adding $f(20)$ and $f(2)$. The chart illustrates that the value of X for which $f(X)$ equals 1,536 is somewhere between 21 and 22, closer to 21.

Continuing with this sort of reasoning, one could get a more and more exact answer: One-tenth multiplied by 73 would be 7.3, and so 73 × 21.1 would be 1,533 + 7.3 or 1,540.3. Too big. One-twentieth is half of one-tenth, so one-twentieth (or in decimal terms, .05) times 73 is half of 7.3 or 3.65. In chart form:

X	$f(X)$
1	73
?	1536
2	146
20	1460
21	1533
22	1616
.1	7.3
21.1	1540.3
.05	3.65

Now it is possible to figure out 21.05 × 73 by adding; 1,533 + 3.65 = 1,536.65—very close to the product I was aiming for. In proportional terms, we have:

$$\frac{73}{1} = \frac{1536.65}{21.05} \tag{15}$$

Depending on the accuracy required by the situation in which the division was constructed to solve a problem in the first place, one could continue working along these lines, producing a fraction with a numerator closer to 1,536.[6] The denominator would be the "answer" to the division problem.

Working through the division in this manner illustrates how the long division process is embedded in the domain of multiplicative structures. Because of the relationship between addition and multiplication, the division could also be done by successive subtractions: taking groups of 73 away from 1,536, and then counting the number of groups taken away. This procedure does not take advantage of the multiplicative nature of the operation, however. Division procedures are used to take away groups of groups, thereby going beyond the simple operation of counting into the domain of ratio and proportion.

[6]This judgment is not a matter of mathematical reasoning; deciding what degree of accuracy is required depends on the situation in which mathematics is being used. Some would argue that the interface between situation and reasoning should be a constant feature of mathematical activity in school (e.g., Schwartz, 1988), but others maintain that one must put the particularities of situations aside to fully appreciate the power of mathematics (e.g., Stewart, 1987).

STUDENTS' UNDERSTANDING OF MULTIPLICATIVE STRUCTURES

Little research has been done in psychology on the question of students' understanding of division per se (Hiebert & Behr, 1988), but the work that has been done on proportional reasoning and on students' thinking about rational numbers gives some clues about how students might think about the concepts that are involved in understanding the multiplicative concepts underlying the conventional long division procedure and why they might have difficulty with this procedure in school.

Numbers as Relationships Versus Numbers as Quantities

The proportional reasoning required to invent or understand a mathematically legitimate procedure for doing long division requires a significant change in how children think about the concept of "number" and this change does not build directly on their knowledge of how to use numbers in counting (Kieren, 1988; Schwartz, 1988; Steffe, 1988; Vergnaud, 1983, 1988). This does not mean that one cannot do division with large numbers using counting procedures, but such procedures are tedious and prone to error, and they distract from learning about other mathematical ideas that fall in the domain of multiplicative structures. In inventing more efficient procedures used for dividing one large number by another, one works with the concept of equivalent relationships between pairs of numbers, expressed as equal ratios or equivalent fractions (Lesh, Post, & Behr, 1988). This way of using number is significantly different from the way numbers are used to represent the process of combining and separating of quantities that underlies addition, subtraction, multiplication-as-repeated-addition, or division-as-repeated-subtraction. The need for such a "cognitive reorientation" in thinking about what the numbers mean in division and in fractions leads to the speculation "that there are not smooth continuous paths from early addition and subtraction to multiplication and division, nor from whole numbers to rational numbers. . . . The new concepts are not the sums of previous ones. Competence with middle school number concepts requires a break with the past, and a reconceptualization of number itself" (Hiebert & Behr, 1988, pp. 8–9). Cognitive researchers have found that as students approach the learning of multiplication and division and the work on fractions that accompanies learning how to transform the remainder, they persist in thinking of numbers as "counters" or as names for quantities of single units. This kind of thinking gets in the way of

their using numbers to count groups of groups or parts of units and using numbers to indicate a relationship between two quantities; it also hinders their ability to reason proportionally about the relationship between the dividend and the quotient (Behr, Lesh, Post, & Silver, 1983; Fischbein, Deri, Nello, & Marino, 1985; Hart, 1981; Nesher, 1986).

A large-scale survey assessment was administered in England in 1974–1979 to assess 12-year-old students' mathematical understanding (Hart, 1981). In the subsection of the test devoted to understanding addition, subtraction, multiplication, and division, it was found that the students tested and interviewed were able to solve problems and communicate about mathematical situations that involved addition and subtraction quite proficiently. They appropriately matched arithmetic symbols to problem situations that indicated addition and subtraction, and constructed procedures for solving the problems that illustrated their understanding of the operations. Many students were unable, however, to interpret the mathematical symbols associated with division in a meaningful and usable manner.

Expressions like

$$23\overline{)391}$$

and

$$391\overline{)23} \tag{16}$$

were called "the same" by about half of the students. A small but significant number of students had very little understanding of either multiplication or division as multiplicative rather than additive operations, but these students could solve multiplication and division problems to which the repeated addition or repeated subtraction strategies were appropriate (Hart, 1981).

In interviews where they were asked to relate arithmetic operations to problem contexts, many children in the British study used "divided by," "divided into," and "shared among" interchangably. When they were asked to interpret the division of a smaller number by a larger one, they often simply inverted the numbers. Correlations between these tests of understanding and a standardized test of computational skills showed that some children mastered the division algorithm without understanding the meaning of division, but those with hardly any understanding of the operation were not, in general, successful with the calculations. There were some children who performed well on the test of understanding and poorly on the test of algorithmic proficiency (Hart, 1981). Although their inability could be interpreted as a difficulty with language and symbols rather than with the concepts involved in doing division, they can also be taken more

broadly as evidence for the claim that many students never move beyond the association of operations on numbers with the simple act of counting.

Research on Teaching and Learning Long Division in School

We have known for a long time that the division algorithm is likely to cause trouble for students whose goal is to use the procedure to produce accurate answers. Buswell's (1926) research early in this century documented in great detail the nature and frequency of the sorts of errors students are likely to make in carrying out the many steps required. After documenting the habits that lead students to make errors in arithmetic procedures—not only in doing division, but in doing addition, subtraction, and multiplication as well—Buswell made two speculative suggestions for improving teaching: One was that simply drilling until answers were accurate was not going to be very effective, and the other was that teachers should try to understand the nature of our number system and how it is structured as a route toward understanding the difficulties students were having with computation (pp. 197–198). Buswell did not do research on teaching to examine the viability of practices based on these suggestions, but he certainly set the tone for more contemporary work in this area (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1988).

Most current instructional research on division begins from the dimensional model in Vergnaud's (1988) theory of multiplicative structures, using the sort of mapping table that was illustrated earlier (see also Nesher, 1988). The goal of these studies is to provide children with a tool that would encourage them to think multiplicatively rather than additively, that is to think of multiplication and division problems in terms of relationships among groups of groups, rather than as repeated addition. This mapping tool is thought to be useful on the basis of findings of the research on proportional reasoning in which children were found to revert to additive strategies and to use them inappropriately when the numbers in the problem did not have an obviously proportional relationship (Hart, 1988; Karplus, Pulos, & Stage, 1983). Teaching sixth-grade students to use mapping tables was found to be effective in encouraging students to think multiplicatively, even with complicated numbers. The mapping tables were also successful in guiding students in their decisions about when to multiply and when to divide. Other instructional research related to students' understanding of long division has been done beginning with problem situations in which students need to model the semantic relationships among numbers in a situation using arithmetic operations (Bell, Greer, Grimson, &

Mangan, 1989; Lesh et al., 1988; Nesher, 1987; Schwartz, 1988; Silver, 1988). This work is reviewed elsewhere in this volume in reports on "word problems" research.

TEXTBOOKS AS A CURRICULAR RESOURCE IN TEACHING LONG DIVISION FOR UNDERSTANDING

An analysis of the mathematical structure of long division, together with recognizing the difficulties students have in making the transition to thinking about numbers as symbols for proportional relationships, provides us with a framework for raising questions about what should be included in school curriculum and instruction if the goal is to have students appreciate the important mathematical ideas embedded in the long division procedure. In this section, we examine the extent to which textbooks take account either of the mathematical structure underlying the operation or of what researchers have argued about why the concepts behind the procedure might be difficult for children to understand.[7] Long division is introduced in most textbook series at the fourth-grade level. Students in this grade move from learning that division is the inverse of multiplication to learning about "short" divisions with remainders and then on to dividing numbers with multiple digits. The treatment of this transition in two textbook series is considered here in some detail. The two series were chosen because both are widely used in public schools and because both make claims to attend to number sense and estimation as the foundation for procedural competence (Fennell et al., 1987, pp. T10–T11, T16–T17; Willoughby, Bereiter, Hilton, & Rubenstein, 1987a, p. xviii).

Connecting Procedures and Stories, or Not

Fig. 4.1 is a reproduction of the page in the Teachers' Guide to the Grade 4 textbook in the *Mathematics Unlimited* series (Fennell et al., 1987, p. 172), where students first meet the idea of a quotient of more than one digit. The

[7]The author recognizes that the curriculum as enacted by teachers and students is not isomorphic with the material found in textbooks, but that textbooks do shape the decisions teachers make about what and how to teach as well as shaping the way teachers and students think about what it means to know a subject (Freeman et al., 1983; Schmidt, Porter, Floden, Freeman & Schwille, 1987; Talmage, 1972). In mathematics, teachers seem to rely on textbooks most heavily when it comes to teaching arithmetic operations (Freeman & Porter, in press; Stodolsky, 1988).

6 DIVISION

Objective: To divide 2-digit dividends by 1-digit divisors and obtain 2-digit quotients, with and without remainders

Warm-Up
Write these examples on the chalkboard.

6 ⟌640 9 ⟌280 4 ⟌330 5 ⟌160 2 ⟌150

Have students estimate each quotient. (80, 30, 80, 30, 70)

GETTING STARTED

Write *D M S C* on the chalkboard. Ask students to recall what each letter stands for in the division process. (divide, multiply, subtract, compare) Discuss each step in a division example.

TEACHING THE LESSON

The new step is the "bring down" procedure, which is a way to show regrouping. In the textbook example, 2 cannot be divided into the remainder 1 (tens) and result in a whole number. Thus it is necessary to "bring down" the 6 and divide 2 into 16 ones.

Emphasize the importance of estimating quotients to help students check on

- the number of digits that should be in the quotient.
- the reasonableness of their answers.

Checkpoint
The chart lists the common errors students make when dividing 2-digit dividends.

Correct Answers: 1b, 2c, 3d, 4c

Remediation

■ For this error, have students copy and complete these examples.

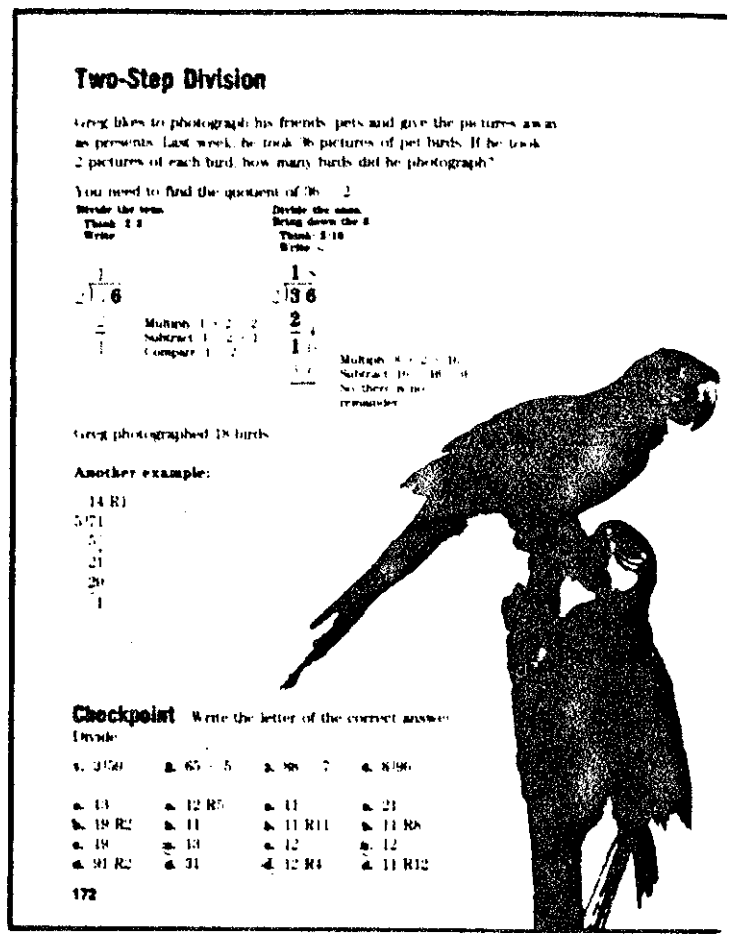
1 3	□ 6 R	□□ R
5⟌6 5	4⟌6 6	3⟌7 1
□	4	□
□ 5	□□	1 □
□□	□□	9
□	2	□

(13, 16 R2, 23 R2)

□ For this error, assign Reteach Master, p. 48.

○ For this error, assign both Follow-Up Activities on T.E. p. 173.

△ For this error, refer students to the examples on p. 170.



Two-Step Division

Greg likes to photograph his friends' pets and give the pictures away as presents. Last week, he took 36 pictures of pet birds. If he took 2 pictures of each bird, how many birds did he photograph?

You need to find the quotient of 36 ÷ 2.

Divide the tens. Think: 2 ÷ 2. Write 1.
Multiply: 1 × 2 = 2. Subtract: 3 − 2 = 1. Compare: 1 < 2.

Divide the ones. Bring down the 6. Think: 5 × 10. Write 2.
Multiply: 2 × 2 = 10. Subtract: 16 − 16 = 0. So there is no remainder.

Greg photographed 18 birds.

Another example:

14 R1
5⟌71
5
21
20
1

Checkpoint Write the letter of the correct answer.
Divide.

1. 3150 a. 65 b. 86 c. 7 d. 8196
a. 13 b. 12 R5 c. 11 d. 21
b. 19 R2 b. 11 b. 11 R11 b. 11 R6
c. 19 c. 13 c. 12 c. 12
d. 91 R2 d. 31 d. 12 R4 d. 11 R12

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COMMON ERRORS

Answer Choice	Type of Error
■ 1a 2b 3a	Divides tens and ones as separate examples
□ 1c 3c	Forgets to record the remainder
○ 2a 3b 4b 4d	Writes the remainder larger than or equal to the divisor
△ 1c 2d 4a	Records the digits of the quotient from the right to the left

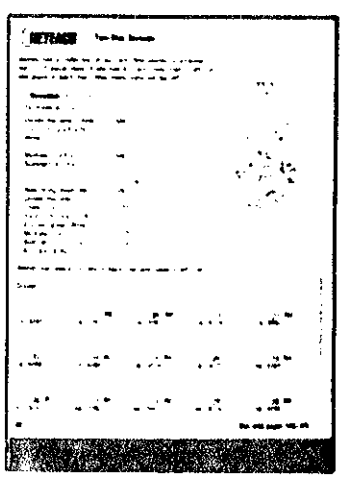

FIG. 4.1. Introducing long division in *Mathematics Unlimited* (from *Mathematics Unlimited*, Grade 4 by Francis Fennell, Barbara J. Reys, Robert E. Reys, and Arnold W. Webb, copyright © 1987 by Holt, Rinehart and Winston, Inc.; reprinted by permission of the publisher).

focus of this lesson is on the procedure to follow when division requires "two steps", that is, when you cannot just get the answer by reversing a multiplication fact. It is the figuring below the quotient that breaks the division into two steps and qualifies this lesson as an introduction to the procedure known as long division. At the top of the page, there is a story about dividing 36 pictures into groups of two. In the recommended procedure, the two digits in 36 are considered separately, and students are first directed to "Think $2\overline{)3}$" as they "Divide the tens." There is no indication that what they are doing here is figuring out how many groups there would be if they were dividing 30 pictures into groups of two. Although the story about dividing up pictures may serve in a general way to convey something about the meaning of division, it is not invoked as an aid to understanding what is happening at each step in the long division procedure, nor does it seem like a context in which it might actually make sense to go through the steps that students are directed to follow. As they consider the division of 3 by 2, students are directed to "Write 1" and the illustration shows that the "1" goes over the "3." Multiply, subtract, compare, and then on to the next digit, which is the "6" in the ones place. But what actually is now being divided is "16" because there were ten left over when thirty pictures were divided into two groups of ten. The authors of the series assert that they use stories to make topics "concrete and real—relating to students' own experience—to pave the way for future success in using estimation naturally and spontaneously in daily life" (p. T16). Although this intention addresses the understanding of the process of division as an operation, it does not illuminate the steps in the process or support their mathematical legitimacy. In actually doing the activity of dividing up the number of pictures, one might more sensibly proceed by dividing up thirty into fifteen groups of two and then dividing the other six pictures into three groups of two, making eighteen groups of two altogether. (Cf. Sowder, 1988.)

In the teachers' notes that accompany this lesson, teachers are reminded to "Emphasize the importance of estimating quotients to help students check on the number of digits that should be in the quotient [and] the reasonableness of their answers." Teachers also are informed of common errors students make when doing these kinds of divisions: "Divides tens and ones as separate examples, Forgets to record the remainder, Writes the remainder larger than or equal to the divisor, Records the digits of the quotient from the right to the left" (Fennell et al., 1987, p. 172). To help students who make the first three of these errors after being taught the lesson previously excerpted, the teacher is directed toward activities that focus on going over the mechanics of the algorithm in several different ways. For the fourth error, it is suggested that having students look back at a previous lesson on "estimating quotients" might help. That lesson also begins with a story, but like the lesson on "Two-Step Division" it only states the story and then refers back to it when stating the answer. The story is

not used as a context for guiding the procedure or even for evaluating the reasonableness of the estimate (Fennell et al., 1987).

The introduction of this level of division in the fourth-grade text in the *Real Math* series (Willoughby et al., 1987a, pp. 272–279) also uses a story but makes a very explicit connection between elements of the story and the mathematical justification for the procedure (Fig. 4.2). Like the lesson in *Mathematics Unlimited,* this lesson is intended to make the transition from division using simple reversals of the multiplication facts to division that involves place-value decompositions. The *Real Math* lesson also begins with a story about a collection of something to be divided up, but it does not begin with a two-digit number divided by a one-digit number, and by contrast with the *Mathematics Unlimited* approach, the story is integrated with the algorithm at every step along the way. The teacher is directed to actually carry out the divisions and trades with play money as a class demonstration, decomposing the dividend along place-value lines and carrying out the step-by-step operation both on the numbers and in the familiar domain of money. The authors of the *Real Math* series state that the purpose of this activity is "To provide a story that shows why the algorithm works" (Willoughby et al., 1987a, p. 272TE). Several more activities of this sort are provided (four lessons) before students are asked to practice working the algorithm without reference to money.

The authors of *Real Math* intentionally introduce the algorithm using numbers larger than students would be using in their first practice in order to convey the necessity for having the algorithm in the first place. They assert that "If the dividend in the story were too small, the characters would not need an algorithm, so the development would seem artificial" (Willoughby et al., 1987a, p. 273TE). A small amount of money easily could be dealt out to one child after the other or evenly divided by trial and error. The authors do not explain why they used a story about money to develop the algorithm, but one might speculate that the sorts of bills that the children found in the story lend themselves to the kind of place-value decompositions that are characteristic of arithmetic algorithms, whereas dividing up other kinds of things (like pictures of birds) does not. The emphasis here is on explaining, both why the algorithm works and why it makes sense to have an algorithm in the first place.

In the last chapter of the fourth-grade text in the *Mathematics Unlimited* series, division by two-digit divisors is introduced, following units on fractions, mixed numbers, decimals, measurement, geometry, and multiplication with two-digit divisors. The two-digit division unit begins with a lesson on "Mental Computation: Dividing Tens and Hundreds" in which students are instructed to pay attention to the nonzero digits in divisions like $30\overline{)1{,}800}$. In this example, the first direction is "Think $3\overline{)18}$." The number of zeros placed after the 6 in the quotient depends on getting the 6 in the right

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Dividing by 1-Digit Divisors (Long Form)

Student book pages 264, 265, 266, 267, 268, 269, 270, 271

Purposes

The purposes of this lesson are:

1. To introduce a division algorithm for 1-digit divisors.
2. To provide a story that shows why the algorithm works.

This is the first of 13 lessons on division with a 1-digit divisor. A checkpoint for assessing mastery is provided in lesson 94 and again in lesson 101. In this lesson, we develop a long form of the division algorithm (with the partial quotients) by showing it as a means of keeping track of a sum of money being divided equally among several people. In the next lesson (92), the students will practice using the algorithm (long form) to keep a record of how they split up a sum of play money. In lesson 93 we show a shorter form of the algorithm. By lesson 97, the students will have had many opportunities to understand, apply, and carry out the algorithm, so in that lesson we introduce the standard short form (the "shortest form"), which students may use from that point on.

By the end of the unit the students should be proficient with using some form of the algorithm to divide whole numbers by 1-digit divisors. They should also be proficient in determining when division is appropriate and in interpreting results. In addition to the checkpoints in lessons 94 and 101, ample opportunities are provided throughout the unit to assess each student's progress by observing students play the Four Cube Division Game (introduced in lesson 98) and by assessing their work on pages of division problems and on pages of mixed word problems (where only some of the answers require division).

Materials

For pages 264–269 Play money may be useful.

THE 7-WAY SPLIT

Rosa and 6 of her friends were out playing one day when they found an envelope.

Inside the envelope was money. There were eight $1000 bills, nine $100 bills, three $10 bills, and six $1 bills.

- How much money is that altogether? $8936
- What would you do if you found that much money? Answers will vary.

FIG. 4.2. Introducing long division in *Real Math*. (Reprinted from *Real Math Teacher's Guide Level 4* by Stephen S. Willoughby, Carl Bereiter, Peter Hilton, and Joseph H. Rubinstein, by permission of Open Court Publishing Company. Copyright © 1987, 1985, 1981 by Open Court Publishing Company.)

The children each took two $100 bills. Then they put that on their record.

```
     200
    1000
  7)8936
    7000
    1936
```

- How much money does each child have now? $1200
- How many $100 bills are left in the pile? 5
- How much money is left in the pile altogether? $536

The children also put on their record that they took fourteen $100 bills altogether, leaving $536 in the pile.

```
     200
    1000
  7)8936
    7000
    1936
    1400
     536
```

This is how much they just took altogether.
7 × 200 = 1400

This is how much they have left to divide.
1936 − 1400 = 536

They decided that the way to divide up the remaining five $100 bills was to exchange them for $10 bills.

- How many $10 bills should they get for five $100 bills? 50
- How many $10 bills will they have altogether? 50 from the exchange plus the 3 they had makes 53 in all.
- How many $10 bills should each child get? 7
- How many $10 bills will be left? 4

FIG. 4.2., continued

The 7 children took the money to the police station and gave it to the person at the lost-and-found department.

After 30 days, nobody had claimed the money. So the police gave the $8936 back to the children.

"How shall we divide the money?" asked Marvin.

"Let's each take a $1000 bill," replied Louis.

Each of the 7 children took one $1000 bill.

The children decided to keep a record of what they were doing. Because they wanted the $8936 to be divided into 7 equal amounts of money, they wrote the problem this way:

$$7\overline{)8936}$$

Each child took $1000. They kept track of this on the top of the record.

$$\begin{array}{r} 1000 \\ 7\overline{)8936} \end{array}$$

This is how much each child has taken so far

Now they had used up $7000, leaving $1936. They kept track of this at the bottom of the record.

$$\begin{array}{r} 1000 \\ 7\overline{)8936} \\ \underline{7000} \\ 1936 \end{array}$$

This is how much they have just taken altogether
7 × 1000 = 7000

This is how much they have left to divide.
8936 − 7000 = 1936

FIG. 4.2., continued

Mental arithmetic

Basic facts—inverse operations As in the mental arithmetic exercise in lesson 39, give the students problems with the basic facts: addition, subtraction, multiplication, and division. Focus special attention on the inverse relationship between the multiplication and division facts, as in these pairs of problems: 9 × 7 = (63), 63 ÷ 9 = (7); 8 × 7 = (56), 56 ÷ 8 = (7); and so on.

Student pages 264–269

Read this story with the class. It continues through page 269 and develops an algorithm for dividing a whole number by a 1-digit divisor. Stop at each question; have students give and discuss answers.

You may find it useful to actually "find" the $8936 somewhere in the classroom. Have the "bank" at your desk, or wherever play money is usually kept. You might also have 7 students play the roles of the children in the story.

When you get to page 265, begin keeping a record like the one shown on that page. (See comment for page 265.)

Note: The numbers used in the story in this lesson are intentionally larger than the numbers that the students will generally work with during the rest of this unit. There are several reasons for this:

1. If the dividend in the story were too small, the characters would not need an algorithm, so the development would seem artificial.
2. Unless the dividend were large enough, it would not be clear that the same series of steps is repeated again and again.
3. Working through a problem with large numbers (as a class, with the teacher) helps convince students that the algorithm works with any numbers, not just with simple or small ones.

Student page 265

When you get to this page, begin keeping a record like the one shown on the page. At each step, write what the figures mean. For example:

$$\begin{array}{r} 1000 \leftarrow \text{Amount each child has} \\ 7\overline{)8936} \\ 7000 \\ 1936 \leftarrow \text{Money left to be distributed} \end{array}$$

Make sure the students understand each transaction *before* you explain the record-keeping step.

The story continues on page 266.

Student pages 266–267
Continue going through this story with the class, as described in the comments for page 265.

Each child has taken a $1000 bill.

Now the 7 children have one $1000 bill, nine $100 bills, three $10 bills, and six $1 bills. "How shall we divide the rest of the money?" asked Kelli.

"We could each take a $100 bill," said Leonard.

"But what will we do with the $1000 bill?" asked Nora.

- What would you do? Answers will vary.

Kelli suggested they take the extra $1000 bill to the bank and change it for ten $100 bills.

- How many $100 bills will they have if they do this?
  The ten $100 bills from the exchange plus the nine $100 bills they had makes nineteen $100 bills in all.

At the bank, they changed the $1000 bill for ten $100 bills. Now they had nineteen $100 bills.

"We can each take two $100 bills," said Elaine.

"Yes, but we're going to have some left over," Rosa said.

- How many $100 bills will be left after each child has taken two of them? 5

FIG. 4.2., continued

place over the dividend. There are no directions given for how to decide where to place it, however, although students are directed to check the quotient by multiplying it by the divisor to see if the product equals the dividend. In a later lesson, on "Dividing by Multiples of 10," students are directed to divide $40\overline{)286}$ by following this procedure:

Divide the hundreds.	Think	$40\overline{)2}$. Not enough hundreds.
Divide the tens.	Think	$40\overline{)28}$. Not enough tens.
Divide the ones.	Think	$40\overline{)286}$, or $4\overline{)28}$.

Estimate 7.

$$\begin{array}{r} 7 \text{ R. } 6 \\ 40\overline{)286} \\ \underline{280} \\ 6 \end{array}$$

(Fennell et al., 1987, p. 324).

The teacher is directed to "Point out that because the divisor is greater than the first two digits, students must place the quotient digit over the ones place in the dividend" (Fennell et al., 1987, p. 324). There is no explanation of why $4\overline{)28}$ can be exchanged for $40\overline{)286}$ as a strategy for arriving at the quotient. Earlier in the page students are instructed to

Think: $20\overline{)80}$ is about the same as $2\overline{)8}$.

Look at the pattern.

$2\overline{)8}$ $\quad$ $20\overline{)80}$ $\quad$ $20\overline{)800}$

(Fennell et al., 1987, p. 324).

The fact that eighty is in the same relationship to twenty as eight is to two is not explained anywhere, to teacher or student, and the use of the word "about" in the directions for thinking here is curious. The "Reteach" worksheet that is recommended for students who have difficulty doing the procedure correctly contains the same language and the same sort of example.

The explanation of the sort of proportional relationship that enables us to estimate when dividing by a two-digit quotient is not given much more attention in the *Real Math* series, but there is some mathematical justifica-

tion given for the procedure. The "long division algorithm" is not introduced in this series until fifth grade, and it is not supposed to be taught until students have done considerable work on rates, ratio, equal proportions, decimal numbers, and functions.

In relation to the problem of dividing 414 by 24, the following directions are given:

> First find an approximate answer.
> Round 24 to 20. Then divide both numbers by ten.
>
> $20\overline{)414} \longrightarrow 2\overline{)41.4}$
>
> Now divide.
>
> $$\begin{array}{r} 20 \\ 2\overline{)41.4} \end{array}$$
>
> So the answer is about 20.

(Willoughby, Bereiter, Hilton, & Rubenstein, 1987b, p. 230)

The process of changing divisions with large numbers into divisions with smaller numbers by considering them as equal ratios is explained in relation to a story context (analogous to the one in the fourth-grade book) in which problems need to be solved in this way (Willoughby et al., 1987b, p. 203) and teachers are told that "The material in this lesson is essential for the work in subsequent lessons. If any students are having difficulty it would be wise to extend this lesson to two or more days, using the extra teaching outlined below" (Willoughby et al., 1987b, p. 230). The extra teaching that is recommended focuses on students understanding of place-value relationships, and the use of play money, base-ten materials, and place-value charts are recommended.

These two textbooks provide an interesting contrast in the attention that is given to the connection between the mathematical structure of long division and the conventional procedure. The justification for the steps in the procedure are modeled in the *Real Math* version by using trades among different denominations of money to represent the decomposition of the dividend. In the *Mathematics Unlimited* version, the decomposition is presented as a series of mechanical steps with no justification. *Real Math* provides more resources for the teacher who wants to teach long division as a mathematical idea, but it does not give teachers much guidance about the difficulties we know students have with this idea.

Mathematics, research on learning and thinking, and curriculum are all important considerations in examining what might be entailed in students' learning long division in a way that is related to authentic mathematical understanding. But even if taken together, they do not explicitly address the question of mathematical pedagogy. In his summary comments on the papers presented at the National Council of Teachers of Mathematics Research Agenda Project Conference on Number Concepts in the Middle Grades, Case (1988) reviewed the research on learning the number concepts that are taught at the upper elementary level, and found very little in that literature on teaching. He suggested that it would be appropriate for researchers to pursue the following questions at this time:

> How can we *teach* so that students' major conceptual errors are articulated, or at least 'headed off at the pass,' and so that partial understandings are redirected into more complete understandings?
>
> How can *teachers* best stay in tune with their students' mental models and processes and the changing nature of their partial understandings?
>
> What can *teachers* do to deal with the individual variability they encounter in examining children's mathematical representations and processes?
>
> (p. 269, emphasis added)

These are all questions about how teaching can be organized to pay attention to the way students think about numbers and number relationships. In their recent review of the research that has been done on mathematics education at the upper elementary and middle-school level, Hiebert and Behr (1988) also suggested that we need to find classroom-teaching strategies that will "facilitate students' constructions of meaning for written mathematical symbols [and] identify the support and development of central conceptual strategies, from their first appearance in an incomplete or intuitive form to more formal procedures that generalize to all structurally similar problems" (p. 15).

Research on Teaching from the Teacher's Perspective

The research on the teaching of long division reported here is intended to address these questions. It looks at student thinking and curricular innovation through the lens of teaching practice, describing the teacher's actions interactively with curriculum and student cognition. It addresses such questions as what a teacher might need to do in an ordinary classroom setting to make use of such tools as mapping tables to construct lessons

that are responsive to students' ways of thinking and what practices might be successful in altering the social structure of lessons to engage students in actively trying to figure out the meaning of division as an operation. The general program of research and development, of which the work on teaching long division was a part, was organized around the questions:

What sort of work is entailed in teaching that is designed to take account of how students think in the setting of ordinary classroom lessons?

What sort of work might teachers do to support the development of students' thinking in mathematically productive directions?

What constitutes mathematical understanding in the school classroom setting?

(Cf. Lampert, 1988.) This inquiry into teaching was conducted by the author from the teacher's perspective.[8] It is different from conventional instructional research in that I developed curriculum and instruction interactively with finding out about how students think about a topic, and in that it occurred as part of the regular instructional program for an entire class. It is also different from research on students' thinking that is carried out in clinical interviews by psychologists (e.g., Kieren, 1988) and research on children's mathematical competence carried out in problem-solving settings that arise outside of classrooms (e.g., Saxe, 1988) from projects that use the findings of such research to design instructional strategies (e.g. Carpenter et al., 1988).

The assumption here was that teaching practice can be designed situationally to express important mathematical principles in a way that is responsive to the particular knowledge about student thinking that can be obtained by a teacher in the course of a lesson, as well as making use of more general propositional knowledge derived from cognitive research (Lampert, 1988a). A related theoretical assumption is that students' understanding and teachers' understanding are contextual. What students can and will do in the classroom is different from what they can and will do in an interview setting and different from what they can and will do in solving a problem that arises in the course of daily living outside of either of these formats. What teachers can do and say in university settings is different from what they can do and say in classrooms. What learners and teachers can be said to "know" or "understand" in different settings is therefore functionally different. We know very little about how understanding "travels" among such different settings (Pea, 1988). Given these limitations,

[8]See Lampert (1988) and Lampert (1990a) for a more complete description of the relationship between research and practice that is assumed in this work and for a review of the methodological arguments for writing about teaching and learning from the teachers' perspective.

it seems problematic to define understanding based on research done outside of classrooms and then to assess whether classroom instruction is successful in producing that kind of understanding in the social setting of school activities, just as it is problematic to equate a teachers' knowledge of research with what he or she is able to do with that knowledge in a classroom context.

From some perspectives, the fact that the teacher inquires about student thinking in the classroom in the course of instruction might be considered a methodological problem. A teacher is invested in the outcomes of the interaction in a way that a researcher would not be. This problem was addressed in part by using data from many sources to document what occurred in lessons as a supplement to my own records. The decision not to use independent pre- and posttest data or interviews to examine students' thinking was based on the assumption that students' performance on tests or in interviews would not adequately measure what they were able to do during class while involved in mathematical activities with their peers and teacher. This research method is based on a definition of knowledge as a collaborative situational production of teacher and students (Greeno, 1986).

Research Methods and Setting

I have taught fourth- and fifth-grade mathematics during the past 6 years, collecting data on both teaching and learning during 3 of those years. For 1984–1985, these data included audio tapes of lessons for 6 months, video tapes of two units, records of speech and visual communication kept by an observer at least three times a week over the whole school year, notebooks in which students did their daily work including writing and drawing they do to represent their thinking, and students' homework papers. I also have kept my own detailed field notes on lessons, including descriptions of how lessons and units were planned and implemented and initial analyses of the planning process itself, the lessons as they were taught, and students' work. For this chapter, I have further analyzed the portion of these data pertaining to the teaching of division using triangulation among different data sources and constant comparison within and between lessons.

Exploratory lesson development in the area of long division was done by the author in a fourth-grade class in 1984–1985, and it is the subset of data that was collected during this period that is analyzed here. This class had been involved for several weeks in work on multidigit multiplication prior to the work on division (Lampert, 1986a, 1986b). Before any instruction took place that was directed toward learning about long division, there

were a few students who were proficient in carrying out the steps of the conventional algorithm, although none of them were competent to make decisions about what to do with the remainder. Before and during the instructional sequence on division, some students were taught to carry out the conventional algorithm at home by parents, tutors, or older siblings. At the beginning of the lesson sequence, those students who knew how to perform a conventional procedure used whatever version of the procedure they had learned without regard for the problem context. Other students, presumably those not proficient in any version of the conventional algorithm, invented procedures to solve problems like "How would you share 86 cakes among 5 restaurants equally?" These solutions did not involve the use of the conventional algorithm, and the students who used them were more likely to dispose of the remainder in ways that were appropriate to the problem context. Other students simply responded to questions like this by saying, "I don't know how to do it." or "We haven't had division yet."

Lesson Architecture

Lessons were conducted as whole-class activities with me at the board and students sitting at tables in groups of four or five. The verbal interaction was conducted primarily as a large-group discussion; I posed problems and students speculated about how the problems might be solved and responded to one another's propositions. I used the blackboard to represent salient mathematical features of the students' contributions and relate them to the intended outcomes of the lesson. What was drawn or written on the blackboard was intended to be a bridge between the common language that students used for talking about their mathematical ideas, and the more formal conceptual structure represented by the conventional symbols in mathematics as an academic discipline (Vergnaud, 1988). Legitimate representations or solution strategies invented by a student became part of the lesson for the whole class as they were taken up and drawn on the board by the teacher and subsequently refined in whole-class discussion.

In order to accommodate a wide range of abilities and backgrounds among the student participants, I often acted as a coach and discussion leader while students constructed solutions to problems collaboratively with me and their classmates. The "answers" thus obtained were not the production of any one student, and students were not expected to work independently. More advanced students and the teacher provided a scaffold whereby less advanced students could develop their thinking beyond the sorts of solutions they would be competent to produce alone (Vygotsky, 1978). When students were expected to produce written solu-

tions to problems or representations of operations, they used drawings and words as well as numerical symbols. All of these instructional strategies were designed to promote a culture of mathematical discourse in the classroom, whereby teacher and students could communicate about ideas that made sense to everyone involved. (Cf. Lampert, 1989, 1990b.)

In the fourth-grade unit on division, there were two kinds of activities, each designed to address a particular facet of the conceptual structure of division in relation to students' thinking about it. One kind of activity focused on proportional relationships among numbers, and engaged students in observing patterns and creating various kinds of groupings. In many of these activities, there was an emphasis on decomposing numbers along place-value lines. In these activities, numbers were considered as abstract symbols for quantities or groups of quantities, and no particular objects were used as representations of the operations or relationships. The other kind of activity was representing divisions with stories and pictures that referred to particular objects. Here the students were engaged in the problem of what to do with the remainder given different problem contexts. They did not begin with problem contexts, but rather created them to fit divisions that were symbolized abstractly by numbers.[9] Estimation strategies were discussed and critically evaluated in a variety of problem settings, both concrete and abstract. For large numbers, students used the multiplication capacity of their calculators to test and refine their estimates, again focusing on the proportional reasoning aspect of thinking about division. In the drawings that students produced, division was represented in many different forms. Although the issue of giving meaning to the remainder and the issue of distinguishing between partitive and quotative interpretations of division were not excluded from the series of lessons reported here, neither were they explored thoroughly. The purpose of this series of lessons was to examine whether students could become engaged, in regular mathematics lessons, in thinking about the proportional relationships within divisions of large numbers; the other topics were taken up in more detail later with this group of children in the fifth grade.

The descriptions of lessons that follow serve to document both the sort of teaching that might go on in a school classroom to engage students in

[9]The decision to begin in this way was not deliberate and I do not mean to suggest the starting with either numbers or story contexts is the "right" way to teach. The purpose of the description and analysis in this chapter is to examine what happened rather than to justify it on the basis of research on learning. In the case of this lesson segment, as with all the others described here, the intention is to conduct an investigation into teaching and learning processes, not to portray the direct translation of research into practice; teaching is not the sort of practice in which such translations are possible (Lampert, 1985). Where justifications for teaching decisions do appear, they are based on the teachers' understanding of the mathematical terrain.

understanding long division and how a particular group of fourth graders responded to this sort of teaching. In their responses, there is evidence that they have the capacity to reason proportionally about relationships among numbers. There is also some informal evidence that suggests that they can connect this reasoning with conventional computational activity, but the research was not designed to examine this connection using conventional methods.

THE LESSONS

The lessons that are described in this section of the chapter are not meant to represent a script for how to teach long division. They were constructed on the foundation of the mathematical concepts described earlier, and they might be conceived as a journey through the "long division region" in the territory of multiplicative structures. Many alternative journeys through this region are possible, and if teaching is constructed interactively with students, it is likely that a different class would take a different journey.[10]

An Exploration of Students' Thinking About Grouping and Counting

The first activity in the fourth-grade division unit that I taught in 1984–1985 was designed to engage students in thinking about the patterns and relationships that would emerge if a given number of units were partitioned into groups of different sizes. The major purpose of this lesson was for me to gather information about how the members of the class would think about some of the ideas underlying the operation of division when it was not presented in its conventional curricular form. The activity could be done using several different procedures ranging from simple counting to conventional long division. The activity was set up in such a way as to emphasize the "mapping rule" that would relate the divisor and the quotient, thereby giving students a tool for thinking about division as a proportional relationship. What I was looking for was the extent to which students would be able to recognize relevant patterns and relationships, and the

[10]See Spiro, Vispoel, Schmitz, Samarapungavan, & Boerger (1987) and Greeno (1990) for a theoretical exposition of this way of thinking about curriculum and instruction. See also Resnick (1989).

language they used to describe them. The observations made by the students after they did several "divisions" of the given number into different size groups established a way of talking about the concepts underlying division. More conventional terms and symbols could be introduced gradually in direct connection with the students' own expressions.

This lesson was the first in a series in which the mathematical structure underlying long division was constructed collaboratively and interactively by teachers and students. As the more knowledgeable adult in these discussions, I interjected new ideas and information when they seemed related to the students' thinking and questions. I provided representations of elements in the discussions that focused the activity on elements relevant to understanding division by being somewhat symbolic, but they also made use of the ordinary language that fourth graders use to talk about the operation of division. This practice was designed to establish a system of communication midway between students' informal language for expressing their understanding of mathematical relationships and the formalities of mathematics that I wanted them to learn.[11]

In the lesson, I used two visual representations to begin to make connections between student thinking and the mathematics I wanted them to learn. The first was groups of units represented by rows of Xs on the blackboard, and the second was a chart in which the patterns and relationships in divisions of different quantities would be more transparent than they are in ordinary language. The groups of Xs on the board were intended to illustrate the relationship between multiplication and division in situations with large numbers and "leftovers." The first lesson described here is comparable in scope and sequence to the first set of textbook lessons described earlier in that it turned the class's agenda toward divisions that were more complex than reversals of the "multiplication facts."

In the introduction to the activity, a class volunteer was asked to pick a number between 25 and 50 to indicate the number of objects to be divided into groups; she picked "45." Then other volunteers were asked to suggest numbers that might be tried in the experiment to see what would happen if 45 were divided into groups of that magnitude. The first student suggested we try groups of five. Another student said that would make fifty groups. A third student said, "No, you don't add, you multiply. Nine times five is forty-five, so it's nine in a group." Another concurred, "It's five things in every group with none left over."

[11]Vergnaud (1988) called the mathematical relationships that are taken into account in students' informal reasoning "theorems in action." These relationships are not formally asserted by students in discussion, but Vergnaud asserted that they "underlie students' behavior" and can be expressed in formal mathematical terms by the teacher as a way to make a bridge between students' thinking and the mathematics the teacher wants them to learn.

We were well on our way to figuring out how to communicate about what division might mean. To represent the results of this discussion, I drew on the board:

XXXXXXXXX
XXXXXXXXX
XXXXXXXXX
XXXXXXXXX
XXXXXXXXX

calling the drawing "five groups of nine." Then I asked, "What would happen if I took the same forty-five things and made them into groups of two?" Two was chosen because students have a variety of techniques for dividing by two, and it would be an example of what it would mean for something to be left over. A student answered, "Twenty-two groups altogether. I counted by two's and kept going up to numbers close to forty-five, and got forty-four with one left over." Another student said simultaneously, "The leftover will be one." The groups of two were shown on the aforementioned drawing by ringing pairs of Xs.

At this point I proposed making 45 into groups of eight. Groups of eight is more challenging because the eight times tables are less likely to be mastered by fourth graders. But thinking about groups of eight can build proportionately on thinking about groups of two because the groups of eight can be composed of four groups of two. An activity like this has the potential, therefore, to help students bridge between additive and multiplicative ways of understanding the operation of division. Several students worked on the question by drawing groups of eight Xs and adding or counting until they came to a number close to 45. The first student to respond said, "Five groups of eight." To elicit the student's thinking about leftovers without asking the question directly, I asked "And?" in a questioning tone of voice. Another student responded: "Five left." This was recorded in the chart on the blackboard, as another student asserted, "You could also put six groups of eight and three left over." In response to this assertion, I asked the class, "Can we make another group of eight?" Although it was unconventional, the student's thinking was treated as a serious speculation, reflecting the way he thought about dividing a large quantity up into groups.[12] I asked the student and the

[12]This sort of teacher thinking has been referred to as "giving a child reason"; it evolved as part of the culture of the Teacher Development Project at the Massachusetts Institute of Technology as the teacher and researcher participants sought to find ways of making connections between children's informal understanding and the formal knowledge they are expected to acquire in school (Duckworth, 1988; Lampert, 1984).

rest of the class to think about this solution while drawing

XXXXXXXX
XXXXXXXX
XXXXXXXX
XXXXXXXX
XXXXXXXX
XXXXX

on the board. The visual portrayal of the data was intended to help the students think about the difference between what was being assumed in the proposed solution, and the assumptions behind the solution "five groups of eight with five left over." I counted the Xs in the bottom row aloud, and asked, "How many more would we need?" The student who had proposed six groups said, "Three more. We could make six groups if we had three more."

This kind of discourse leaves the reasoning about mathematical legitimacy to the student, and removes the teacher from the role of judging whether answers are right or wrong. What the teacher does do here is to surface the differences in the assumptions behind different answers and make them part of the public conversation. The teacher makes explicit some of the reasoning processes that students are using to arrive at their conclusions, and builds instruction on the basis of what can be learned about these processes.

I then asked, "Any more ideas about groups of eight?" And then, "What about groups of ten?" Ten is an "easy number" to divide and multiply by, and it would serve to assess whether anyone would make the same kind of assertion as the student had made about groups of eight; that is, going higher than the number, and identifying the "leftovers" with how many you would need to get complete groups. No one did. There seemed to be a consensus in the group that the assumption that one could make the division "come out even" by adding more to the dividend was inappropriate, or at least unconventional.

I assumed that the student who responded early in the discussion by adding 45 and 5 to get fifty, and the student who asserted that you could get six groups of eight by adding three more were revealing connections in their thinking between the concepts of division and arithmetic. I did not take the role in either of these instances of judging their thinking against a formal standard. The purpose here was to find out what students thought rather than to make an assessment of whether they were thinking in a conventionally correct manner. I did not simply elicit and accept assertions, however. In the process of listening to and interpreting what the students were saying, I also guided the conversation so that students had the opportunity to test their ideas in mathematical discussions with their peers. The students' assertions were revised in the process of discussion

with me and other students, not because I told them they were "wrong." (Cf. Lampert, 1990a.)

During the discussion, I had begun to fill in a chart on the blackboard, at first without any comment. At the beginning of the discussion, the chart looked like this:

NUMBER OF OBJECTS TO BE DIVIDED INTO GROUPS: _____

If you make groups of:	How many will be in each group?	How many left over?

After the discussion of different groupings of 45 objects, the chart looked like this:

NUMBER OF OBJECTS TO BE DIVIDED INTO GROUPS: 45

If you make groups of:	How many will be in each group?	How many left over?
5	9	0
2	22	1
8	5	5
10	4	5

The numbers 5, 2, 8, and 10 had been chosen by the teacher to introduce the different sorts of relationships that might exist among number in a group, number of groups, and number of leftovers. What is important here is that the visual representation of the activity was constructed collaboratively by students and teacher. Such an approach requires a more active role in sense making on the part of students than simply looking at a diagram that has already been constructed in a book.

Next I invited students to choose the number that would go in the first column of the chart, given the same 45 objects to be divided up. This gave the students more opportunities to try their own hypotheses about what might happen when a given quantity was divided into groups. The first student to volunteer a number picked "30" and I asked the class, "How many groups of 30 are in 45?" There was a long pause, after which a student responded, "One. There's only one because you would have to have 60 to make two." Another student added, "There's a remainder of fifteen left over." This was the first time the conventional language usually used to talk about division in school had been used. The first student probably arrived at this assertion by adding 30 and 30, but his thinking provides a foundation for the teacher's expanding his thinking into other, more complex proportional relationships. The second student was making a verbal, and perhaps a conceptual, connection between the activity of finding groups and the formal arithmetic of division. The students' language for referring to the conceptual relationships in division—both formal and informal—set the tone for that way teacher and students would continue to talk about the operation. This talk evolved out of students' individual ways of making sense of the operation, the ways they had been taught to talk about it in school and at home, and the teacher's insertion of formal mathematical language into the conversation. Another student volunteered, "You could make 45 groups of one, and there would be none left over." And another added, quickly and playfully, "Or one group of 45." The legitimacy of the idea of "groups" of one, or one group, was being tested by these students. They were pleased to get their contributions registered on the chart, and they had made a discovery that was rehearsed by other students as part of this activity when it was done with different numbers on subsequent occasions.

The next student to speak said, "What about groups of 40?" I asked in return, "What about it?" and the student responded, "Ten, twenty, thirty, forty, four." Another said, "No, there's only one group of forty." Then I asked the student who had speculated that there might be four groups, "If I take forty away from forty-five once, how many are left?" He responded, "There's only five left." "So how many groups of forty in forty-five?" "Only one." This exchange allowed the student to rethink and revise his initial conjecture with help from me and from another student. It was also meant to make the relationship between division and subtraction that he was using explicit. His initial assertion may have come from a mismatch between his thinking

and the language he was using to express it. He may have been thinking "four groups of ten in forty five," that is, groups that would add up to forty, rather than one group of forty.

So far, twelve of the twenty-one students present had made a verbal contribution to this discussion. With the chart still up on the board, I passed out a similarly constructed worksheet and directed the students to "Pick your own number, and divide it into different sized groups." Most students, after working for about 10 minutes, filled the front with groupings and were working on the back of the page. (See Fig. 4.3.) Near the end of the class period, I asked the students what they had noticed from their work.

Michael: The number of leftovers is always less than the number in the group.

Carl: When it comes out even you can switch the numbers around. Like 3 groups of 15 or 15 groups of 3.

Sally: If you pick a number close to the number you are working with you'll always have one group, and you can count the remainder.

Charles: As you make the size of the groups bigger, the number of groups you can get gets smaller.

Alice: If you double the size of the group, you get half as many.

These statements indicate that the students who spoke were beginning to use proportional reasoning to analyse their list of groupings and produce more groupings. They did not work out each item separately, but they looked for patterns and constructed mathematical relationships appropriate to the concept of division to find solutions. Although many had begun with the primitive strategy of repeated subtraction to arrive at their construction of the quotient, their work evolved into looking for patterns in the results that carried them into the domain of multiplicative reasoning.

Extending Students' Thinking About Grouping into New Domains and Making Connections with Conventional Symbols

Variations on this activity of making groups continued for the rest of the week with teacher and students choosing different numbers to divide into groups, and students sharing and refining their observations of patterns and relationships. By the end of the week, there was general consensus in the class that if you wanted to make a group that was more than half of the original number, the number of groups would always be one, and you could subtract to get the leftovers. Several students also noted that if the size of the group doubled, the number of groups would be halved, and some extended this reasoning to tripling and quadrupling. Numbers were identi-

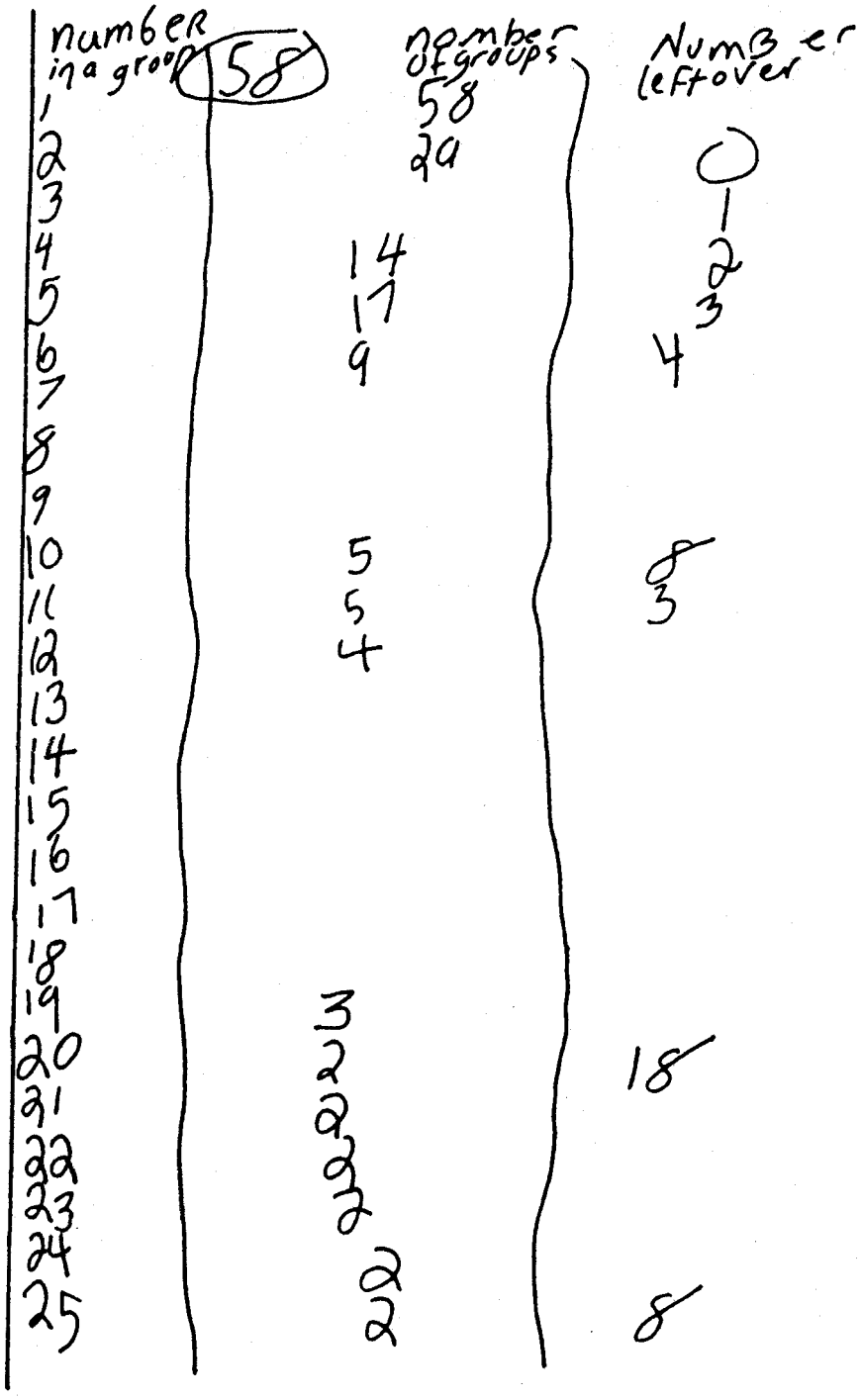


FIG. 4.3. Dividing a number into different sized groups.

fied that could be divided evenly in many ways (like 60 and 144) and these were contrasted with numbers that "never came out evenly, except if you used the whole thing as one group, or you made groups of one" (like 79 and 43). Some students proposed "breaking things in half to make the groups come out even" so that 13 for example could be made into two groups of 6½.

This line of thinking did not extend much beyond simple fractions, but it was a beginning for reasoning about what to do with the remainder. I chose not to take the journey off along that track at this time, however, and instead turned the agenda toward using the framework we had developed to work with larger numbers. This decision was driven in part by my intention to focus on place-value decomposition that would lead students toward an understanding of the steps in the conventional algorithm.

To begin the transition toward working with the conventional algorithm, the final class in this series was devoted to dividing up three, four, and five digit numbers into groups of 100, 1,000, or 10,000. The chart that was produced in this class looked like this:

How many units?	How many in a group?	How many groups?	How many left over?
43	100	0	43
98,999	100	989	99
999	100	9	99
8,632	100	86	32
7,591	100	75	91
7,591	1,000	7	591

Students defended their assertions using multiplication, with particular attention to the patterns that are evident in working with powers of ten. At the end of this work, students were asked to make speculations about what would happen to "any number" if it were made into groups of 100, 1,000, or 10,000. They made assertions and asked questions like:

The more zeros you add in the second column, the less places there will be in the third column.

I wonder what would happen if you just added zeros to the numbers in the first column, after the other digits.

If it was a ten [for how many in a group], the number left over was always less than ten. When it was a hundred [in a group], it was always less than 100 [left over], and when it was a thousand, what was left was less than 1,000. There was no way you could get bigger leftovers.

The chart that was used throughout these activities again provided a bridge between the students' natural language for talking about division

and the proportional character of multiplicative structures. It made use of enough common words to identify the relationships among the numbers in terms that suggest operations, but it also may have functioned to help students "'forget' non-essential features of the situation and concentrate on the relevant elements and relationships" of multiplicative structures (Vergnaud, 1988, p. 148).

The next step in the unit was to take three-digit numbers and divide them into a small number of groups, and to focus the discussion on what might be "done" with the leftovers to make the groups "come out even." The independent variable now would be the number of groups, and the problem would be to think about how the number of groups would affect the size of the groups and the leftovers. Ordinarily, this question is associated with the problem of what to do with the remainder after the division of the whole numbers is completed. But where we start thinking in terms of a remainder is arbitrary, determined by the level of accuracy required by the problem context. Do you need an answer to the nearest hundred, the nearest ten, unit, tenth, hundredth? In

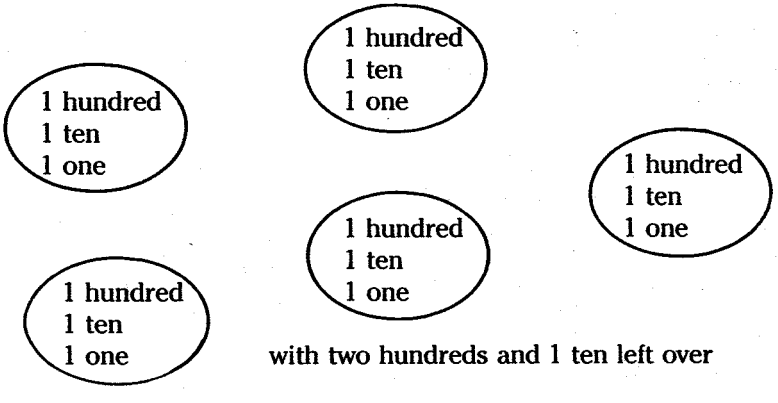
$$5\overline{)765} \tag{17}$$

for example, the division does not come out even if you just consider hundreds. If we were talking about dividing up 765 dollars, for example, in the form of 7 hundred dollar bills, 6 tens, and 5 ones, the "answer" to 7 hundreds divided by 5 would be "one remainder two." If we change the remainder (2) into tens, however, the division does come out evenly because five divides 20 (tens) evenly. When we add a decimal point and zeros to a whole number quotient, we are carrying out this process in terms of smaller units; for example, changing one dollar bills to dimes, or in more abstract terms, changing units to tenths. The concept is the same—that is, taking one kind of unit and transforming it into an equivalent quantity in different size units—whether we are changing hundreds to tens or units to tenths. So learning to deal with the remainder needs to begin with the very first step, even when the division "comes out even" in terms of whole numbers.

The language of classroom discourse in this part of the lesson sequence connected abstract talk about units and groups and parts of units, and talk about what special constraints needed to be considered when dividing up concrete objects like pennies or cookies or people. Students made up their own division stories to go with the symbols that they would then manipulate to find the quotient, giving them some ownership over the meaning of the outcome. A more visually rich sort of representation was used to help students think about how to divide up the remainder when there were no longer enough units to go around, and it incorporated the place-value decompositions that were part of the earlier lessons.

The first problem in this section of the unit was to figure out how to represent the division of 765 into 5 equal groups. One student argued that if you "counted by fives" you would get to 765 eventually, so there would be no leftovers. Others concurred, making the assertion that "anything that ended in a five or a zero would come out evenly in groups of five." These arguments helped to solidify the relationship between multiplication and addition, and division and subtraction as part of the reasoning process and suggested the question of what to do with the "remainder." (A later series of lessons taught to this group of students in the fifth grade would be more focused on the question of how to give meaning to the remainder that is left when all of the whole units have been distributed.)

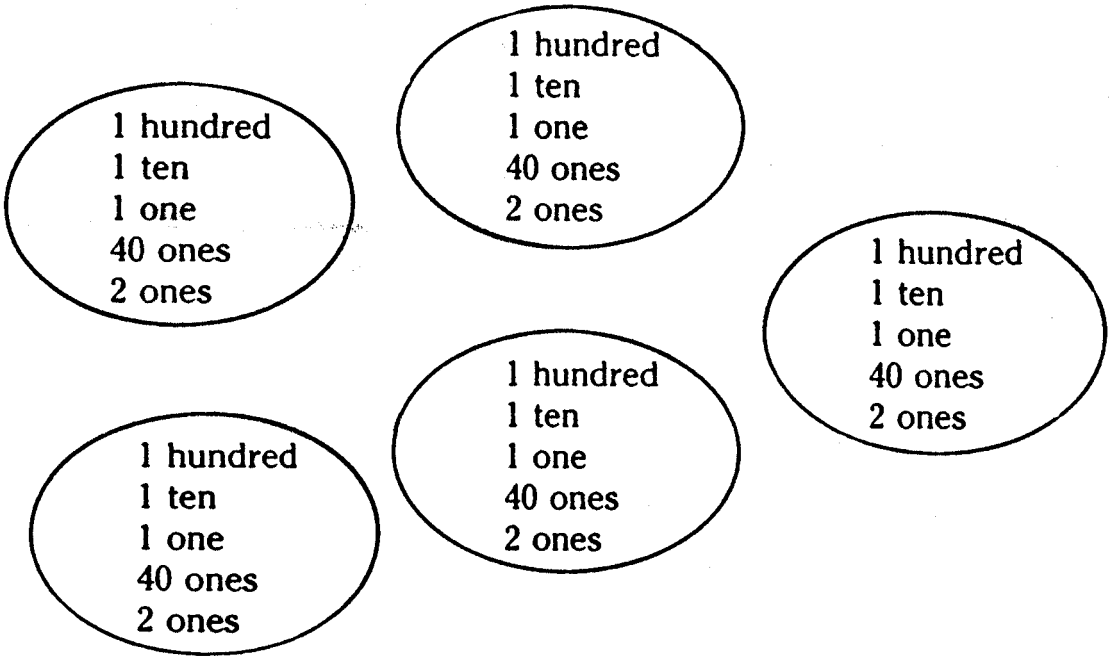
In order to keep the value of each place prominent while at the same time decomposing the number, I suggested that the class, "Think of 765 as 7 hundred dollar bills, 6 ten dollar bills, and 5 one dollar bills." This decomposition deflects students from the procedural first step ("How many times does five go into seven?") which loses the meaning of the numbers being considered and may make it more difficult to reason about the remainder at each step of the decomposition. Associating the digits with amounts of money has the potential to represent the magnitude of the "7" while at the same time breaking it off from the other digits for consideration. It is similar to the approach taken in the fourth-grade book in the *Real Math* series described earlier. The five groups were represented by five circles on the blackboard, and the problem was to get an equal amount of money into each circle. Each of the paper money denominations were distributed, which resulted in:



with two hundreds and 1 ten left over

This representation, unlike the conventional answer to the division problem, retains the idea of five groups; that is, it shows how many are in each group if there are five groups. The two hundreds and the one ten are "remainders" that are now left to be traded so that they can be distributed among the groups.

Once this representation was up on the board, I referred back to an assertion made earlier by a member of the class, saying, "I thought you said it was going to come out even." This challenged students to work with place value to resolve the discrepancy. It gave them a reason for changing hundreds to tens and tens to ones, rather than just doing it as the next step in a formal procedure. The discussion resulted in the following revision to the aforementioned grouping:



because the initial suggestion was to change 2 hundreds to 200 ones and give each group 40. Now the "contents" of each group could be added up, to yield "153" dollars in each.

Now I asked students to think about "What else it could mean to divide 765 into 5 groups," beginning the story-telling activity that would provide a basis for thinking about different ways to dispose of the remainder. Some of their stories included:

765 transformers were sold in five stores. How many did each store sell?

There were 765 reading books in the whole school and each child is supposed to get five books. How many children can get books?

You had 765 people you wanted to invite to five parties. How many would you invite to each party?

The second story is not isomorphic, either to the money story, or to the other two, about stores and parties, even though the procedure for obtaining the numerical answer would be the same. I noted this difference, using a drawing to illustrate, and the contrast became the focus for another day's

lesson. We did not spend much time on the two possible interpretations of division, but students made drawings (e.g., Fig. 4.4) to explore what each might mean.

The purpose of these stories is not to "motivate" students (as stories are reputed to do in the *Addison-Wesley Mathematics* textbook series [Eicholz et al., 1985]) nor to help them see that long division is "useful" for solving real-world problems (as they are intended to do in *Mathematics Unlimited*). My intention in using the stories was to have students construct an expression of the meaning of the operation of division in terms of an activity whose elements they understood, like giving out books or planning parties. (cf. Lampert, 1986a, 1986b.) I did not assume that I was teaching students how to use long division when confronted with such problems in life outside of school; my intention was to teach them to create connections between concrete and familiar situations and the abstractions of mathematical operations and relationships.

Making Connections Between Reasoning About Relationships and Conventional Expressions of Long Division Problems

After discussing and representing the distinction between the two interpretations of the symbols that are used to indicate division (finding the number in each group and finding the number of groups), I turned the unit back toward the place-value decompositions involved in "long" division. The first problem to be tackled was:

$$65\overline{)787} \tag{18}$$

After writing these symbols on the board, I assigned the class the task of writing stories that would fit these symbols. (I deliberately began the lesson with a division of a three-digit number by a two-digit number that would not "come out even.") When all of the students had stories in their notebooks, I chose one for discussion: "Seven hundred and eighty-seven kids were in school and they have to be divided up into different classrooms. How many in each classroom? How many won't be put in classrooms?" The first part of the discussion was an interpretation of the story that was directed toward an estimation of how many students would be in each classroom if they were divided up evenly. (I regularly added conditions to the students' stories in discussions to make explicit that division implied making equal groups. The fact that the students' story does not mention equal groups should not be taken to mean that she did not have equal groups in mind. Within the framework of the conversation, that

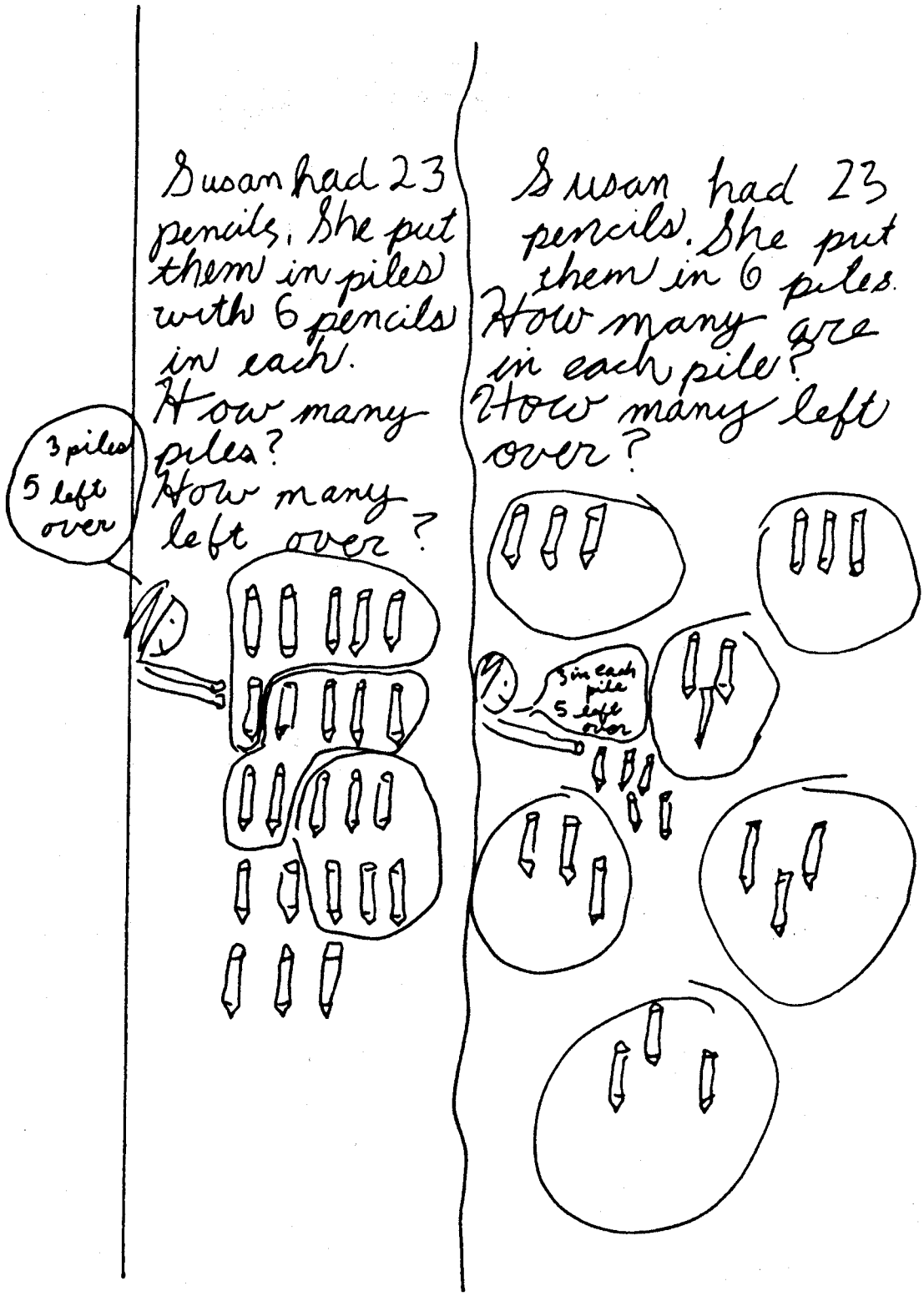


FIG. 4.4. Two interpretations of division.

condition may be assumed, as well as that the students were to be divided into sixty-five classrooms. The story chosen is printed as it was written in the student's notebook.)

To begin the estimation process in a way that would deliberately deflect students away from attempts to use a procedure they did not understand, I asked the class: "Do you think the number of kids in a class is going to be more or less than a hundred? More or less than ten? Twenty?" Following a familiar class routine, the students were expected not only to give an estimate, but to explain why their assertion was mathematically reasonable. The student who answered said that he was sure that it would be more than ten because ten times sixty-five is six hundred and fifty, and that is less than the total number of kids. His reasoning could be expressed in terms of the unequal proportional relationship:

$$\frac{10}{787} < \frac{10}{650} \tag{19}$$

The next student who spoke said he was sure it had to be less than twenty "Because two times 65 is 130, and if you do twenty times 65 you get 1,300, which is way too much." His thinking expressed another unequal proportional relationship:

$$\frac{20}{1300} < \frac{20}{787} \tag{20}$$

The symbolic expression of these proportions was not made an explicit part of the lesson. They are indicated here to illustrate the relationship between the way long division was approached in these lessons and the students' understanding of the concept of multiplicative structures. I asked if another student could give a reason why the number of students would be less than twenty, and he said, "If you doubled what you had for ten, it would be over a thousand." In terms of functions, these students are trying to find X such that $f(X) = 787$. The value of $f(10)$ is too small, and the value of $f(20)$ is too large. The last student to speak was reasoning that $f(20)$ would be double $f(10)$. As in earlier lessons, the students' answers suggest that they have theories (or "theorems in action") about how numbers are related in the operation of division that could be mapped onto the conceptual web that defines the mathematical structure of division. The teacher's work in this circumstance is encouraging students to articulate and use their theories rather than guessing mindlessly or relying on the teacher to give them a procedure.

At this point, one member of the class asked if he could "guess" a number between 10 and 20. I asked him, "How would you know whether your guess was a good guess?" The student answered that he would multi-

ply it by 65, and see if it came out to 787. There followed some discussion of the assertion, made by another class member, that no number multiplied by 65 was going to result in 787, and various students gave reasons for why this was true. One student suggested that it would be a good idea to try 15 × 65. She thought that this multiplication would help figure out the answer because it was "right in between ten and twenty, and so we would know right away whether there were more or less than 15 students." I then wrote on the board:

$$\begin{array}{r} 65 \\ \times 15 \\ \hline 325 \\ 650 \\ \hline 975 \end{array} \tag{21}$$

and a student said, "That's too high. 15 is too high." She was reasoning on the basis of the relationship between 15 and 975 and comparing it with the relationship between some unknown number and 787. At this point, another girl in the class said she knew it was 12, because she "did it" herself. Upon being queried, she explained that what she "did" was multiply 12 × 65 using the conventional algorithm. What was going on in this conversation was the collaborative construction of a mapping table to figure out the proportional relationship between 65 and 787.

The conversation that followed illustrates how students used the story context to help them think about how to interpret the remainder:

Candice: I multiplied 12 times 65 and it's 780.
Teacher: But there were 787 kids all together.
Janet: Seven had to go to another school.
Juan: They went to chapter one.

These students were figuring out what to do with the remainder, while another student entered the discussion to explain his reasoning about the relationship between 787 and 65:

Ralph: If you take away 200 from 975, you get 775.
Teacher: What does that have to do with the problem?
Ralph: Well here's what I did. You keep taking sixty-five away from 975 until you get close to the number of kids.

Then I wrote on the board as he spoke, relating the subtraction to multiplication and the multiplicative structure of this particular division problem:

$$\begin{array}{r} 975 \\ -65 \\ \hline 910 \\ -65 \\ \hline 845 \\ -65 \\ \hline 780 \end{array} \qquad \begin{array}{l} 15 \times 65 \\ \\ 14 \times 65 \\ \\ 13 \times 65 \\ \\ 12 \times 65 \end{array} \tag{22}$$

It is not clear from what he said what this successive subtraction had to do with taking 200 away from 975; he may have reasoned that he would have to take away 65 about four times (because 65 doubled is 130 and 130 doubled is 260) to get near 787. At this point, the student said, "That is as close as you are going to get to 787, and it's under." He was using successive subtractions of groups of 65 to arrive at his answer: 65 was a unit, and every time he took away one such group-unit from the total, he took away a single unit from the multiplier. His subtraction could be written as a proportion, as well:

$$\frac{975}{15} = \frac{910}{14} = \frac{845}{13} = \frac{780}{12} \tag{23}$$

where what he was aiming for was the denominator that would have a numerator close to 787.

Being able to make the sorts of estimates that make it possible to get started on a long division problem is a matter of compressing this kind of "additive" thinking into a multiplicative structure, thinking in terms of groups of groups and their relationship rather than in terms of one group at a time.[13] In this lesson, the class went through a similar discussion with the divisions

$$79\overline{)1584}$$

and

$$89\overline{)2567} \tag{24}$$

The numbers for these problems were proposed by students, within the teacher's constraint that a four-digit number be divided by a two-digit

[13]The approach that is taken to long division in *Mathematics Around Us* (Bolster et al., 1978, pp. 230–238) begins with the process of taking groups the size of the divisor away from the quotient and counting how many you have taken away, then progressing to taking more than one group away at a time, and finally taking away tens or hundreds of groups away. This approach retains the meaning of the operation, but it does not make explicit the proportional relationship entailed in the more efficient process. This point is important to note because *Mathematics Around Us* was the series used in other classes in the school where the author taught at the time the research reported here was conducted.

number. I focused the discussion of these two problems on successively closer estimations of the quotient; students refined their estimates using information derived from multiplying an earlier estimation by the divisor. Particular attention was paid to strategies for estimating that made careful use of the data. The estimates could be refined using proportional reasoning, and this is what I encouraged students to do, not by formalizing the process, but by responding to its informal use by some members of the class. By the end of working through the third problem, the class had expressed the goal of "getting close to the answer with the fewest estimates," and so proportional reasoning became a tool for accomplishing their ends.

Students' Independent Use of Proportional Reasoning to Do Division

In order to assess the extent to which students would independently connect proportional reasoning to doing division, and concurrently, to encourage them to do so, the next few lessons in my fourth-grade class were organized around students working in pairs with a calculator. Each pair of students was given a division of a four-digit number by a two-digit number written on an index card, a sheet on which to record their estimates (Fig. 4.5), and a calculator. The divisions all involved numbers that would not produce remainders because the class had not yet had the opportunity to explore what it might mean to represent a remainder as a decimal number and this background would have been required to make their work with the calculator meaningful. (At this point a teacher could take the journey off into the territory of figuring out what the decimal numbers that appear on the calculator mean, but I decided not to do that with this group at this time.)

In each lesson in this part of the unit, I first conducted a whole-class demonstration to illustrate to students the kind of thinking they might do to pursue these progressive estimations. Before students started working in pairs, I did a problem at the board (I with calculator, students making estimates) and commented on those guesses that were particularly good examples of proportional reasoning strategies. I asked those students who made assertions to explain their thinking so that other students would have examples of how one might go about using number relationships to reason through a division problem. When students worked together in pairs, they were observed to mimic the language I had used to direct one another's thinking about these problems.

My partner's division

98 ⟌2450

First guess: 10	Multiplication: 10 X 98	Answer: 980
Second guess: 60	Multiplication: 60 X 98	Answer: 5880
Next guess: 45	Multiplication: 45 x 96	Answer: 4410
Next guess: 27	Multiplication: 27 x 98	Answer: 2646
Next guess: 23	Multiplication: 23x98	Answer: 2254
25	25 x 98	2450

FIG. 4.5. Estimation game with a partner.

The whole-class discussions began with stories to set a context for making sense of the number manipulations:

Teacher: Before we start working on guesses, I want some stories for:

$$96\overline{)5184} \tag{25}$$

(on blackboard)

Barbara: 5,184 kids in school and 96 classrooms. How many kids in each classroom?

Matthew: 5,184 ships divided into 96 in each group. How many groups?

Sally: 5,184 potato chips in a bag and 96 bags. How many chips in all?

After this contribution, I wrote on the board:

$$\begin{array}{r} 5184 \\ \underline{\times 96} \end{array} \qquad (26)$$

and said: That's the problem she told me."

No other judgment was expressed. The stories continued, and Sally eventually revised hers.

Joshua: 5,184 mice in a laboratory and divide into 96 different cages. How many in a cage?

Carl: 5,184 transformers put in piles of 96. How many piles? How many left over?

Eylie: 5,184 pizzas—put in 96 groups. How many in each group? How many groups?

Sally: I want to change mine. 5,184 potatoes chips put into 96 bags. How many in each bag?

Peter: 5,184 fultrons divided into 96 groups. Put 96 fultrons on each planet. How many fultrons on each planet?

Allison: 5,184 roses had to put them into 96 groups. How many roses in each group?

Teacher: You could use vases to divide them into.

The students who contributed these stories were a diverse group, and for some this was the first occasion on which they had done any "public" thinking about division. Although the language of Peter's story did not represent the division appropriately, I did not call attention to it; I decided to focus this lesson on estimation. The focus of discussion was on what makes a good estimate, and why, and thus proportional reasoning was made part of the public discourse about this operation. The conceptual content was similar to work we had done with smaller numbers, but the calculator enabled us to extend that thinking into the domain of large numbers.

Teacher: I want you to start guessing for the problem $96\overline{)5184}$.

Sally: 51.

Teacher: (To the whole class) Why do you think she picked 51?

Alfred: If you started out with a hundred, 100 × 96 = 9,600—is close to 10,000. Half of 10,000 is 5,000. Need a little over half of one hundred.

Teacher: (Using calculator, writes on the board: 96 × 51 = 4,896.) Sally and Alfred have come up with a good first guess.

Matthew: I would pick 53 because if you add 96 to 4,896 you come close, and then I added another 96.

Teacher: What multiplication problem is 4,992 the answer to? (Writing on the board:)

$$\begin{array}{r} 4896 \\ +\,96 \\ \hline 4992 \end{array} \tag{27}$$

Ana: 96 × 52.

Teacher: Then Matthew added another 96 that's 96 × 53. (Writing on the board:)

$$\begin{array}{r} 4892 \\ +\,96 \\ \hline 5088 \end{array} \tag{28}$$

At this point, a chorus of students asserted with certainty: "It's fifty-four!" and I wrote on the board, as I did the computation on the calculator:

$$96 \times 54 = 5184 \tag{29}$$

There was some discussion of why so many people were sure that fifty-four was going to work, integrating the addition and subtraction of groups that go into creating a mapping table like this:

X	$f(X)$
100	9600
50	4800
51	4896
52	5088
54	5184

with the operations of multiplication and division whose relationship enables the use of the "multiplying calculator" as a tool in developing a strategic approach to estimating the answer.

The teacher's contributions here were articulations (or explanations) that linked what a student said he or she was thinking to the web of conceptual relationships in mathematics that structures the operation of long division and to the conventions of mathematics that characterize school discourse about division and related topics. All three—proportional reasoning, mathematical concepts, and conventional terms and symbols—combine to define what students might come to know from participating in these lessons. The lessons are constructed so that students operating at different levels of skill and knowledge can be challenged to think about the topic at hand at an appropriate level.

After the whole-group discussion, the students worked in pairs, taking turns in the role of estimator and multiplier, with the multiplier using the calculator to tell the estimator the product of his or her estimate and the divisor. As with the other activities in the unit, the emphasis here was on thinking of the numbers as quantities and thinking about the relationship between multiplication and division. By using the calculator to work with the numbers without decomposing them into place-value related parts, students were encouraged to bring the structure that relates division "facts" to multiplication "facts" to bear on large numbers.

An example of the sort of conversation that occurred between pairs of students as they worked with the calculator follows. Nancy and Janice were working on the division $98\overline{)2450}$.

Nancy: Try ten.

Janet: Nine hundred and eighty [i.e., $98 \times 10 = 980$]. Take your time! Think, you know, anything times a number is going to make it hundred if you use ten.

Nancy: Um; sixty.

Janet: (Using calculator) Five thousand six hundred and eighty [i.e., $98 \times 60 = 5{,}680$]. Why don't you try something lower. That's about twice as much.

Nancy: Forty-five.

Janet: Four thousand four hundred and ten [i.e., $98 \times 45 = 4{,}410$]. I think you can try something lower. Try twenty-seven.

Nancy: Okay, twenty-seven.

Janet: Two thousand—a little too high—two thousand six hundred and forty-six [i.e., $98 \times 27 = 2{,}646$].

Nancy: Twenty-three.

Janet: Two thousand two hundred and fifty-four [i.e., $98 \times 32 = 2{,}254$]. Too low.

Nancy: Twenty-five.

Janet: You got it on your sixth guess. That's better than me.

At first, it seemed as if Nancy was not making strategic guesses, although she was not wildly off the mark. Toward the end of working on this problem, Nancy seemed more thoughtful, raising or lowering her guesses in a range that got her closer to the target.

On the first occasion of doing this activity, many students had long lists of somewhat random guesses as to what the quotient might be. After several trials, in which there was considerable discussion between pairs about how to make a good guess, as well as some conversation along these lines between pairs and the teacher, the number of guesses required to get close to the quotient diminished considerably. One incentive that the pairs invented to get one another to make better guesses was a competition for how many problems a given pair could conquer in the course of one class period.

Putting It All Together

In the final lesson of the unit on division, I attempted to bring together the various themes of each of the unit segments: place value, stories, proportional reasoning, estimation, and interpreting the conventional symbols. In this final lesson, as in each lesson in the unit, I tried to portray the "coherence" of the mathematical activities in which we were engaged so that students would be able to make sensible relationships among mathematical ideas in their own thinking. (Cf. Stigler & Baranes, 1988.) The discussion in the last lesson suggests that students were able to make productive connections among ideas in the conceptual web that constitutes the mathematical meaning of long division. There were, at the same time, several instances during the discussion when students acted in ways that did not express attempts to make sense of this procedure—they focused instead on mechanical processes for getting the answer.

The task that I gave students at this point is very similar to tasks that constitute conventional textbook lessons on long division. Part of what I wanted to assess was whether students would take an active role in making sense of problems that were posed in that form. I began the lesson by putting these divisions on the board:

$$6\overline{)49} \qquad 63\overline{)495} \qquad 63\overline{)4956} \qquad 635\overline{)4956} \tag{30}$$

and I directed the students to "Think about how you would estimate the answers." and "Think about what the relationship might be among the answers to these problems." To focus students on exploring the proportional relationship among the problems, I asked: "How many groups of 6 in 49?" The class chorused "Eight" and then I asked "How many groups of 63 in 495?" and raised similar questions about the other divisions. A major portion of the dialogue that ensued is presented here to communicate the diversity, scope, and content of student thinking that went into the discussion.

Teacher: What about this one? (pointing to $63\overline{)495}$.)

Rose: [The groups of 63 in 495 is] Less than 10, not bigger.

Candice: Check 5 because it's in the middle of 1 and 10.

Teacher: What about this one: $63\overline{)4{,}956}$ (pointing to it on the board). Is it less than ten or more than ten?

Janet: Bigger than 10. Because 10×63 is 630.

Teacher: Is it bigger than a hundred?

Students: (in chorus) No.

Teacher: Nancy, what's 100×63?

Nancy: 6,300.

Teacher: So is the answer to this (pointing to $63\overline{)4{,}956}$) as big as 100?

Peter: I think it should be 15.

Teacher: Other ideas on that one? (No response)

Teacher: What about (pointing to it on the board) $635\overline{)4{,}956}$. Let's think about this problem.

Eylie: I think it's smaller than 10 because 6,350 is too big.

Matthew: I think it's seven.

Teacher: Those guesses are kind of different from one another. One is an estimate based on thinking about the number relationships, and the other is an exact guess.

Candice: I want to change 5 to 6 because I worked it out on paper.

Teacher: We're trying to think right now. Not working it out on paper. What else do you think about $63\overline{)495}$?

Janet: 6. Kara said it and I thought it over, and though 6×63. Because I figured out five times 63 is 315 and six times 63 is 378.

Teacher: How much bigger will that be than 315? (No response). How much am I adding on?

Janet: 63. But I was thinking about it different. I was multiplying not adding.

Juan: Try 9.
Teacher: $9 \times 63 = 567$, too high.
Barbara: 8.
Teacher: $8 \times 63 = 504$. Do you think it's closer to 8 or 7?
Eylie: $7\frac{1}{2}$.
Matthew: It's more than that. It's pretty close to 8.

Collectively, these students had worked out the proportional relationship that would get them close to finding out $63\overline{)495}$. They had built on one another's thinking to come up with the progressive proportional estimation:

$$\frac{5}{315} = \frac{6}{378} = \frac{9}{567} = \frac{8}{504} = \frac{7\frac{1}{2}}{487.5} = \frac{\text{"almost eight"}}{495} \tag{31}$$

Except for Janet and Candice, students used proportional reasoning to refine their estimate. Candice used the conventional algorithm (which she had been taught at home) to get the "answer" and Janet did not build directly on an earlier estimate to get a better one. Eylie did not compute $f(7\frac{1}{2})$, but Matthew thought that taking half a group of 65 away from 504 would be taking away too much. The distinction between students who "added on" (i.e., used additive strategies) and those who thought in multiplicative terms is interesting, especially because the students themselves thought it important to make the distinction. They were progressively multiplying one digit after another, but they were using multiplication.

I did not press for a more precise answer than "almost eight" but moved the discussion toward the next division:

$$63\overline{)4956} \tag{32}$$

The first student to give an estimate said "Try forty." He did not seem to derive any information from the discussion on the previous problem. He did have the order of magnitude correct, but he did not seem to be considering the multiplication of 40×63 in making his guess. I gave the results of $40 \times 63 = 2{,}520$ using the calculator, and then the same student (Peter) called out: "Try 85×63." He may have been reasoning proportionally to arrive at the idea that the estimate should be a little more than double forty. I did not inquire about his thinking, however, because several other students were now anxious to contribute an estimate.

Teacher: $85 \times 63 = 5{,}355$. Now there was some good thinking, very close!
Students: (in chorus) Try 83.

Teacher: $83 \times 63 = 5{,}229$. I would like to take estimates from people who can convince me they've been thinking, and have thought it through.

Matthew: I think it should be 78 because it needs to be a little lower than 80, and I don't think 79 is lower enough.

Peter: I still think it will be 77 because I'm narrowing the problem down.

Teacher: $78 \times 63 = 4{,}914$.

Peter: 77.

Teacher: Peter, when I used 78 I got 4,914.

Peter: Oh, then it should be 79.

Teacher: $79 \times 63 = 4{,}977$. Which is better, 78 or 79?

Students: (in chorus) 79.

Teacher: Why?

Candice: If you have 14 and you try 50, if you have 77 it's only about 20 away.

At this point I turned the discussion toward general observations of the relationships among the four division problems with which the lesson began:

$$6\overline{)49} \qquad 63\overline{)495} \qquad 63\overline{)4956} \qquad 635\overline{)4956} \tag{33}$$

Students made observations about the similarities and differences among the answers to these problems, all of which were expressed in approximate, whole-number form. Their observations included talk about place-value relationships, and comments about how such different problems could have such similar answers.

Getting students to the point where they are able to move back and forth between confidently using their own sense-making strategies and working productively on problems of the sort that are posed in school is a step along the way toward their being able to participate in the culture of mathematics. Although finding the "answer" to a division like

$$63\overline{)4956} \tag{34}$$

is an academic rather than a practical task, learning to approach this task as a process of mathematical sense making rather than as a process of following mechanical rules seems to have the potential to move students toward more meaningful and satisfying engagement in mathematical activity both in school and out.

CONCLUSIONS AND IMPLICATIONS

The lessons described here could be counted as successful on two counts. First of all, they resulted in fourth-grade students thinking and talking about the multiplicative structure that underlies division in terms of their own "theorems in action." Second, they demonstrated that the teaching of long division lessons could be coherently organized and at the same time move around flexibly in the web of concepts that justify the mathematical legitimacy of procedures. There is also evidence that a third goal, that of having students tie their understandings of the mathematical structure of division to the conventional symbols and procedures, was accomplished for at least some of the students. In Case's (1988) terms, the design of these lessons worked to enable the teacher to "stay in tune with" students' mental models and the changing nature of their partial understandings; what was described here is teaching that produced the articulation of students "major conceptual errors" and attempted to "head them off at the pass" (cf. Case, 1988, p. 269). In terms used by Hiebert and Behr (1988), the lessons served to "facilitate students' constructions of meaning for written mathematical symbols" (p. 15); their meanings were expressed in the class discussions of relationships among numbers and the representation of these relationships using conventional symbols.

Implications for Teaching Practice

In contemporary classroom practice it is problematic to have students work on the sorts of problems that take them into the web of concepts related to division if this sort of work is simply laid on top of the straight and narrow path through whole number and fraction arithmetic. The conceptual approach is problematic in this context because some students will have mastered the mechanics of the algorithm (having been taught it by another teacher or at home) and some believe they should not yet be "on" division because they have not even mastered their times tables. At all of these levels, students give some meaning to symbolic expressions like $56\overline{)693}$, even if it is only to see this as a representation of something yet to be studied or as a signal to start by figuring out "how many times 56 goes into 69." Rarely do students treat these symbols as a meaningful expression that asks a question about relationships among quantities. In order to teach long division or any other mathematical procedure for understanding, a teacher needs to intervene in the classroom culture that shapes these attitudes. This sometimes means countering resistance from students who

would prefer "just to be told how to get the answers" (Cooney, 1987; Stephens & Romberg, 1985).

Classroom practices shape what teachers and students think it means to know and learn mathematics, and what teachers and students think shapes classroom practices. Given this circle of beliefs and expectations, treating division as a concept about which one can reason and communicate requires the teacher to aggressively challenge both students' expectations about what is going to occur in lessons and the framework they have been using to assess their own progress as learners of mathematics. Getting students to reveal their "theorems in action" in the course of a classroom lesson requires convincing students that their own reasoning processes are relevant to the activity of learning mathematics in school. But even more challenging for the teacher is the implication of treating division as a concept related to other important pieces of mathematics and to students' thinking in the process of designing instruction. If this view of the subject matter and the student is taken seriously, the teacher must be prepared to move around in the conceptual territory of multiplicative structures in response to what he or she can find out about what students already know about these structures. Given what we know about teachers' beliefs about mathematics, and the argument that their classroom practices are intimately related to their beliefs, changing instruction in the direction of attention to students' reasoning processes is not going to be a simple matter.

Implications for Curriculum

Although the lessons described here were constructed by the author, a similar approach to making sense of long division could be undertaken in conjunction with activities in textbooks, carefully chosen. The important issue is how the operation is interpreted, not whether it is or is not taught. For example, the kinds of discussions that are described here could be constructed around the sort of long division lessons that are included in the *Real Math* unit (Willoughby et al., 1987a). The way in which problems are posed in *Real Math* suggests that proportional reasoning, with place-value considerations in mind, is an appropriate way to go about working on long division. Activities that would promote the kind of discourse reported here are actually available in several other textbooks as well, but they are rarely part of the main lessons. For example, *Mathematics Unlimited* includes an "Enrich" activity in which students are introduced to the "Egyptian Method" of dividing by doubling: a procedure that follows the proportional relationships in the numbers and is recorded in a way that makes the

meaning of the procedure explicit (Fennell et al., 1987, p. 173). In *Silver Burdett Mathematics* students are directed in "resource activities" to use patterns to make relationships and find missing quotients (Orfan, Vogeli, Krulik, & Rudnick, 1987), and *Harper & Row Mathematics* includes activities like those described here in "Challenge" and "Super Challenge" sections (Payne et al., 1985).

None of these textbooks suggests discussing the patterns and relationships that underlie reasoning about division, however. And the activities that might lead to students' noticing such ideas and thinking that they are important are not included in the main lessons; often they are suggested only for those students who finish more conventional assignments. They are not generally part of "reteaching" or "extra practice" activities that are suggested for students who have difficulty with the conventional arithmetic procedures. The research reported here suggests that a wider range of students can participate in both doing and discussing such activities and relating them to the more conventional aspects of work on long division.

Implications for a Theory of Teaching and Learning Mathematics for Understanding

Teaching division—and expecting students to understand it as a procedure connected to a set of related mathematical concepts that go substantially beyond counting—requires a serious intervention in the procedure-oriented culture of mathematical knowing in the classroom. It is not a matter of enrichment or supplementary activities. In the currently typical scenario for teaching and learning arithmetic in elementary school, long division is the capstone of whole number procedures. If a student can do long division successfully, it often is taken as an indication that he or she can do all of the other whole number procedures as well and is ready to move on to fractions. This perspective on long division is consonant with the common belief that elementary mathematics is a collection of procedures that need to be learned in order, and that any tampering with that order will be confusing rather than meaningful to learners. If students believe they cannot know division unless their teachers have done lessons on division, or unless they have gotten to it in the textbook, they will be unlikely to bring their understandings of proportional relationships—undeveloped as they may be—to the task of understanding division. If, in contrast, the goal of mathematics curriculum and instruction is to have students learn to become confident in their capacity to reason about mathematical entities, the research reported here suggests that it may be worth

trying to develop instruction interactively with their "theorems in action" rather than following a straight and narrow path through the subject.

What has been portrayed here is not a collection of lessons to be reproduced so that students will predictably learn the meaning of divisions with large numbers. It is a description and analysis of an exerpt from a year's work with a particular class, in which every teaching and learning interaction was organized to give meaning to the activity that was being carried out in the social setting of the classroom. The purpose of the research reported here was to explicate further what might be meant by mathematical pedagogy (Lampert, 1988) in a domain that often is regarded as antithetical to promoting mathematical understanding. Teaching decisions were made with this basic principle in mind: Mathematical knowledge is warranted by establishing the legitimacy of procedures in the domain of number and examining the appropriateness of those procedures for modeling processes in other domains. This principle has several implications for developing a theory of mathematical pedagogy. In the classroom setting, the culture determines the way in which knowledge is regarded and acquired, and the teacher has a leading role in shaping that culture toward students taking responsibility for warranting the procedures they use. Communication is a tool that the teacher can use to shape the classroom culture. The meaning of knowledge is created as students and teacher talk about their common activities and visually represent mathematical concepts. And these concepts form the emerging curriculum of the mathematics class, as the teacher investigates students' ways of thinking about problems and constructs connections among these ideas and important ideas in a mathematical domain. All of these elements of mathematical pedagogy can be represented in lessons on long division as they are in other mathematical arenas if teacher and students are working together to investigate the ideas in their midst.

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