Q-Theory and Real Business Cycle Analytics [1]

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September 2, 2003

[1] I would like to thank Philippe Weil, Matthew Shapiro and Kenneth West for their encouragement in writing this paper and many cohorts of students who gave reactions to the initial, rough versions of this material.

Abstract

A mathematical and graphical treatment of the Q-theory extension of the Basic Real Business Cycle model of Prescott indicates that several key results are robust to both investment adjustment costs and to variation in the shape of the utility function and the production function while other customary results are fragile. It also demonstrates some of the richness of general equilibrium analysis. One key result relevant to recent debates about the empirical effects of technology is that an immediate, permanent improvement in technology unavoidably raises output, investment and the real interest rate, given uncontroversial assumptions such as normality of consumption and leisure and constant returns to scale in production. A permanent increase in government purchases financed by an increase in lump-sum taxes is also shown to unambiguously raise output, investment and the real interest rate.

1 Introduction

Since the eclipse of the purely literary approach to economics, the three primary modes of economic analysis have been mathematical analysis, computer simulation, and statistical analysis of empirical data. Research projects tend to combine these three modes in varying shares. Because Real Business Cycle Theory arose at a time of cheap computing power, computer simulation has been the dominant mode of analysis in studying Real Business Cycle models. Theory in the sense of a thorough mathematical analysis has remained relatively underdeveloped compared to the rich development of a computational understanding of themes and variations on Real Business Cycle models. A diminishing marginal productivity argument suggests the value of further development of the mathematical analysis of Real Business Cycle models. This paper explores what can be gained from pushing further the mathematical analysis of stripped-down Real Business Cycle models by a thorough application of standard tools in the economist's toolkit: control theory, duality, and the even more basic tool of graphical analysis.

One strength of a theoretical analysis is that it allows one to look at general functional forms and a wide range of parameter values to distinguish which results are general and which ones are special to particular functional forms.[1] The approach in this paper will be to start with very general functional forms and then narrow the focus with assumptions on the key functions that have a transparent economic meaning.

Substantively, this paper analyzes the Basic Real Business Cycle Model familiar from Prescott (1986) and the extension of this model that incorporates Hayashi's (1982) Neoclassical interpretation of Tobin's Q-theory. Equivalently, it analyzes the extension of Abel and Blanchard's () general equilibrium Q-theory model that allows for variable labor supply. The style of analysis is inspired by papers such as Cass (), Abel and Blanchard () and Mankiw ()—a style emphasizing graphical analysis centered around the phase diagram.[2]

[1] For example, motivated by the arguments in Basu and Kimball (2002) and the empirical evidence cited there that consumption and labor are unlikely to be additively separable in the utility function, one of the key dimensions of generality I allow for in this paper is nonseparability between consumption and labor.

[2] Because this style of analysis has pedagogical as well as substantive value, this paper is written with several audiences in mind, not least of which is the audience of graduate stu-

Although in relative terms, the amount of theory on Real Business Cycle models pales in comparison to the amount of computational work, in absolute terms, there is a large quantity of theoretical papers on Real Business Cycle models. Some of the more obvious examples are Barro and King (),King, Plosser and Rebelo (1988), Rogerson (), Benhabib, Rogerson and Wright (1991), and Campbell (1994). But each of these papers and others in the literature has other objectives and does not have this focus on the graphical analysis of the Prescott's (1986) Basic Real Business Cycle Models or it Q-theory extension. In general, two factors that may have excessively inhibited the literature from pursuing such a graphical treatment are (1) a preference for formulating business models in discrete time, which makes phase diagrams less natural and (2) a belief that uncertainty makes the use of perfect foresight models inappropriate. It is worth dealing with each of these concerns up front.

The modest language barrier between discrete and continuous time is unfortunate, since with few exceptions continuous and discrete-time models get at the very same economics. As a formal matter, continuous time is particularly convenient when working with phase diagrams and often simplifies formulas, while side-stepping having to specify inconsequential details of timing; discrete time is easier to work with computationally and when using recursive techniques in proofs. But as long as the length of the period in discrete time models is allowed to vary parametrically, discrete-time models are essentially equivalent in their economics to the corresponding continuous-time models.[3] For example, computational power is now sufficient that it is a trivial matter to implement discrete-time business cycle models with time periods of

dents in economics. I hope that more experienced economists will forgive my explanations of certain things that may seem obvious.

[3]When the length of a period is routinely fixed at one quarter or one year, rather than being varied parametrically, certain dangers and temptations arise. For one thing, there are issues like those discussed in Hall (1988) with handling time-averaged data that would be easy to miss when thinking in terms of just one time unit. The issues with time averaging illustrate why the fact that data often comes with a quarterly frequency is not a sufficient reason for fixing a model's time interval unalterably at one quarter. Second, when the period is routinely fixed at a quarter, it is easy to fall into the implicit and often undefended calibration of key parameters by the arbitrary length of the period. It is no accident that models often assume that prices are fixed for three months, that the velocity of money is four times per year, or in an oligopolistic supergame that the length of time a firm can get away with undercutting its rivals is three months.

one hundredth of a year or less to make the gap between discrete-time and continuous-time models negligible. Even a quarter is a short enough length of time that the difference between one quarter and Thus, the continuous-time and discrete-time versions of a business cycle model, each convenient for certain purposes, should both be part of the dialogue about that model.

As for uncertainty, it is true that uncertainty can cause departures from certainty equivalence, but in representative agent macro models, these departures are typically small. This is for the simple reason that macroeconomic annual standard deviations are typically on the order of, say, 3% or .03, which implies a variance of only about .0009 per year to interact with any relevant curvature of the functional forms. The products of small variances with modest curvatures are often reasonable to neglect, as is done routinely when doing log-linear computations such as those implemented by the AIM program. If one is making a certainty-equivalence approximation for computational purposes, a log-linearized perfect foresight model will deliver exactly the same impulse responses. In other words, in representative agent macro models, the certainty equivalence approximation is typically good enough that uncertainty, while a major force *ex post*, is only a minor force *ex ante*. In the early days of phase diagram analysis, some authors were a bit embarrassed at the seeming need to discuss the effects of shocks that were completely unforeseen, but the justification for the analytical procedure in question is much stronger. It is simply the use of the certainty-equivalence approximation *ex ante* with the *ex post* analysis of the effects of the realization of a shock that had the potential to go in either direction. In principal, the impulse responses deduced from a perfect foresight model could be combined with variances and covariance of shocks to get the variances and covariances of macroeconomic variables that Prescott (1986) recommends focusing on to see how well a model is doing, but in recent years, macroeconomists have gradually been coming around to the view that the simulated impulse responses themselves are often more transparently informative of a model's workings than variance-covariance matrices.

After all such programmatic statements, the proof of the pudding is still in the eating. To advertise the menu of what follows from studying the Basic RBC model and the QRBC model, here are some of the most interesting results established and discussed in what follows.

- In both the Basic RBC Model and in the QRBC model, regardless of functional forms, if the utility function has normal consumption and leisure, an immediate, permanent improvement in technology cannot cause *output* or *investment* to fall on impact. Moreover, a phased-in improvement in technology can only cause output or investment to fall if consumption rises.

- In both models, a positive permanent technology shock or separable government purchases shock raises investment and Q, *and* unambiguously raises the real interest rate on impact. Thus, in general equilibrium, interest rate effects on investment are necessarily overwhelmed by changes in the demand for capital services reflected by the rental rate of capital in reaction to permanent technology and fiscal shocks.

- Regardless of the complexity of the driving shock processes, the behavior of the model economy at any point in time can be reduced to a few dimensions: the capital stock is a sufficient statistic for the past, the marginal value of capital is a sufficient statistic for the future, and these plus the *current* values of the exogenous variables are enough to determine the current values of all of the endogenous variables.

Following the advice of Polya (1957) for tackling math problems, the first few sections do a fair bit of pre-processing of the elements of the model, so that the hard core of the problem of characterizing the QRBC model is revealed.

One element of pre-processing that should be done before doing anything else is detrending. In terms of understanding the real world, it makes sense to think in terms of a model with steady-state growth. For this application, think of the steady-state growth as coming from exogenous trend growth in technology and population. The model can then be detrended by dividing quantities through by their trend values and adjusting interest rates, rental rates, utility discount rates and depreciation rates (or the equivalent) appropriately. This transformation is standard, so I do not do it explicitly. Thinking of an ostensibly static model as a detrended version of a model with exogenously-driven trend growth does affect the appropriate calibration of the model because of the adjustments to rates just mentioned, but does not affect the analysis itself—the tools for analyzing a static model are perfectly good for analyzing the departures from trend of a model with exogenously-driven trend growth.[4] The consistency of labor augmenting technological progress with trend growth motivates the interest in labor-augmenting technology shocks below. (Technology has an upward trend, but may improve in fits and starts.) In the absence of trend improvement in the home production technology at exactly the same rate as the market technology, one can also argue that consistency with a model that has steady-state growth should also impose an extra constraint on the utility function, *a la* King, Plosser and Rebelo (), but since this constraint on the utility function will be discussed as an optional extra assumption since it is not central to the analysis.

2 The Social Planner's Problem

The QRBC Model is the solution to the following social planner's problem:

$$V(K_0) = \max_{C,N} \int_0^{\infty} e^{-\rho t} U(C,N)\, dt \tag{1}$$

subject to

$$\dot{K} = KJ\left(\frac{F(K,N,Z) - C - G}{K}\right) \tag{2}$$

and

$$K(0) = K_0.$$

Time zero is the moment when information about the realization of a shock arrives. K is the capital stock. $V(K_0)$ is the optimized value given initial capital stock K_0. C and N are the consumption and the labor hours of the infinitely-lived representative consumer, ρ is the impatience parameter (the utility discount rate), K is the capital stock, Z is the level of labor-augmenting technology and G is exogenously given government purchases that may add to utility in an additively separable way that is not explicitly represented, but does not have any direct interaction with $U(C,N)$ or $F(K,N)$. Many other exogenous government policy, technology and preference shifters could be considered after appropriate modifications of the model

[4] Of course, to analyze the effects of changes in the trend growth rate of technology or population, it would be better to use a model that represents growth explicitly.

(including shocks to home production technology that are observationally equivalent to preference shifters), but it is enough here to concentrate on labor-augmenting technology and government purchases that have benefits that are additively separable from what is happening in the private economy.

The assumptions on the three functions U, F and J are given in the following subsections because they require some discussion.

2.1 Felicity

Felicity (the instantaneous utility function) U is monotonic, with $U_C > 0$ and $U_N < 0$ (consumption is a good, labor is a bad); concave, with $U_{CC} < 0$, $U_{NN} < 0$, and $U_{CC}U_{NN} - [U_{CN}]^2 > 0$; normal in consumption, with

$$\frac{\partial \ln\left(\frac{-U_N}{U_C}\right)}{\partial N} = \frac{U_{NN}}{U_N} - \frac{U_{CN}}{U_C} > 0 \tag{3}$$

and normal in leisure, or equivalently, inferior in labor, with

$$\frac{\partial \ln\left(\frac{-U_N}{U_C}\right)}{\partial C} = \frac{U_{CN}}{U_N} - \frac{U_{CC}}{U_C} > 0. \tag{4}$$

Figure 1 shows how having the slope of the indifference curves increasing in both C and N guarantees that consumption will increase and labor will decrease as one moves to a higher indifference curve to a point with the same slope. This implies that the expenditure expansion path or *Engel curve* slopes down.

2.2 The Production Function

The production function $F(K, N, Z)$ is positive and increasing in each argument, with $F_K > 0$, $F_N > 0$ and $F_Z > 0$. It is concave in K and N, with $F_{KK} < 0$, $F_{NN} < 0$ and $F_{KK}F_{NN} - [F_{KN}]^2 > 0$. The production function has constant returns to scale in K and N: $F(\zeta K, \zeta N, Z) = \zeta F(K, N, Z)$. Also, the formal statement of Z being labor- augmenting technology is that $F(K, \zeta^{-1}N, \zeta Z) = F(K, Z, N)$. Finally, I assume that F is supermodular in Z and N—that is $F_{NZ} > 0$, so that an improvement in technology will raise labor demand. Other than the last condition, of supermodularity between technology and labor, all of these conditions combined are equivalent to

$$F(K, Z, N) = ZNf\left(\frac{K}{ZN}\right), \tag{5}$$

where $f(\Gamma) > 0$, $f'(\Gamma) > 0$ and $f''(\Gamma) < 0$ where $\Gamma = \frac{K}{ZN}$ is the effective capital/labor ratio.

Substituting from (5) into the supermodularity condition $F_{NZ} > 0$ yields

$$\begin{aligned} \frac{\partial^2}{\partial Z \partial N} ZNf\left(\frac{K}{ZN}\right) &= f\left(\frac{K}{ZN}\right) - \left(\frac{K}{ZN}\right) f'\left(\frac{K}{ZN}\right) + \left(\frac{K}{ZN}\right)^2 f''\left(\frac{K}{ZN}\right) \\ &= f(\Gamma) - \Gamma f'(\Gamma) + \Gamma^2 f''(\Gamma) > 0. \end{aligned} \tag{6}$$

The condition (6) is equivalent to the elasticity of substitution between capital and labor being greater than capital's share. To see this, I anticipate a bit by identifying $f'(\Gamma)$ with the (real) rental rate of capital R and $Z[f(\Gamma) - \Gamma f'(\Gamma)]$ with the (real) wage W. Then if σ is the elasticity of substitution between capital and labor,

$$\begin{aligned} \frac{1}{\sigma} &= \frac{\partial \ln\left(\frac{W}{R}\right)}{\partial \ln\left(\frac{K}{ZN}\right)} \\ &= \frac{\partial[\ln(Z) + \ln(f(\Gamma) - \Gamma f'(\Gamma)) - \ln(f'(\Gamma))]}{\partial \ln(\Gamma)} \\ &= \frac{-\Gamma^2 f''(\Gamma)}{f(\Gamma) - \Gamma f'(\Gamma)} - \frac{\Gamma f''(\Gamma)}{f'(\Gamma)} \\ &= \frac{-\Gamma f''(\Gamma) f(\Gamma)}{f'(\Gamma)[f(\Gamma) - \Gamma f'(\Gamma)} \end{aligned} \tag{7}$$

For comparison, capital's share α is the elasticity of gross output with respect to capital:

$$\alpha = \frac{\partial \ln(ZNf\left(\frac{K}{ZN}\right)}{\partial \ln K} = \frac{\partial \ln f(\Gamma)}{\ln \Gamma} = \frac{\Gamma f'(\Gamma)}{f(\Gamma)}. \tag{8}$$

Thus, the elasticity of capital/labor substitution is greater than capital's share iff

$$\frac{f'(\Gamma)[f(\Gamma)-\Gamma f'(\Gamma)]}{-\Gamma f''(\Gamma)f(\Gamma)} > \frac{\Gamma f'(\Gamma)}{f(\Gamma)}. \tag{9}$$

Multiplying both sides by the positive magnitude $\frac{-\Gamma f''(\Gamma)f(\Gamma)}{f'(\Gamma)}$, (9) is equivalent to

$$f(\Gamma)-\Gamma f'(\Gamma) > -\Gamma^2 f''(\Gamma)$$

or $f(\Gamma)-\Gamma f'(\Gamma)+\Gamma^2 f''(\Gamma) > 0$. The condition that the elasticity of capital/labor substitution be greater than capital's share is automatically guaranteed with Cobb-Douglas technology, and is satisfied by any technology that is not too close to being Leontieff.

2.3 The Capital Accumulation Function

The capital accumulation function J satisfies $J' > 0$ and $J'' \leq 0$. The case $J'' = 0$ corresponds to the Basic RBC model, while $J'' < 0$ corresponds to the QRBC model proper.

To aid in the discussion of J, label gross output Y,

$$Y = F(K,Z,N) = ZNf\left(\frac{K}{ZN}\right) \tag{10}$$

and label gross investment expenditure I:

$$I = Y - C - G = ZNf\left(\frac{K}{ZN}\right) - C - G. \tag{11}$$

Then (2) becomes

$$\dot{K} = KJ(I/K). \tag{12}$$

Note that I is implicitly investment expenditure *inclusive* of adjustment costs, as in Hayashi (). Substantively, this is equivalent to Abel and Blanchard's () alternative convention of bestowing a letter on investment expenditure *exclusive* of adjustment costs, but investment expenditure inclusive of adjustment costs plays a key role in the material balance condition $Y = C + I + G$, so the Hayashi convention is quite convenient. Finally, it is good to have a label for the gross investment rate:

$$X = I/K = \frac{F(K,N,Z) - C - G}{K} \tag{13}$$

I assume there exists a value δ for which $J(\delta) = 0$ ($J' > 0$ implies that this value is unique). This value δ plays the role of the depreciation rate even when the capital accumulation function is nonlinear since $\dot{K} = 0$ when $I = \delta K$.

One can normalize so that $J'(\delta) = 1$. To show that this is only a normalization, suppose that $J'(\delta) = D \neq 1$. Then one can define a new variable $\bar{K}$, new function $\bar{J}$ and new crossing point $\bar{\delta}$ by

$$\bar{K} = K/D,$$

$$\bar{J}(X) = J(X/D),$$

and

$$\bar{\delta} = D\delta.$$

Then

$$\dot{\bar{K}} = \bar{K}\bar{J}(I/\bar{K}),$$

$\bar{J}(\bar{\delta}) = 0$ and

$$\bar{J}'(\bar{\delta}) = \frac{J'(\delta)}{D} = 1.$$

The optimization problem with $\bar{K}$ in place of K is then identical in form to the original problem, except that $\bar{J}'(\bar{\delta}) = 1$. Suppressing the bars in the notation yields the same result as if $J'(\delta) = 1$ were stipulated from the beginning.

The normalization $J'(\delta) = 1$ is convenient because the first- order Taylor expansion of $KJ(I/K)$ around any point (K^*, I^*) where $I^* = \delta K^*$ is

$$\begin{aligned} KJ(I/K) &\approx K^*J(\delta) + J'(\delta)[I - I^*] + [J(\delta) - \delta J'(\delta)](K - K^*) \\ &= I - I^* - \delta(K - K^*) \\ &= I - \delta K. \end{aligned} \tag{14}$$

Among other things, the fact that the first-order Taylor approximation for J is identical to the usual case without investment adjustment costs means that the units used to measure K match as closely as possible the usual empirical procedure of measuring the capital stock by cumulating the amount of resources spent on producing capital and depreciating those expenditures at rate δ.

2.4 The Canonical Equations

Using (5), and Λ for the marginal value of capital, the current value Hamiltonian for the social planner's problem (1) is

$$H = U(C,N) + \Lambda K J\left(\frac{ZNf\left(\frac{K}{ZN}\right) - C - G}{K}\right).$$

The first order conditions for optimal consumption and labor are $H_C = 0$ and $H_N = 0$, or

$$U_C(C,N) = \Lambda J'\left(\frac{ZNf\left(\frac{K}{ZN}\right) - C - G}{K}\right) \tag{15}$$

and

$$-U_N(C,N) = \Lambda J'\left(\frac{ZNf\left(\frac{K}{ZN}\right) - C - G}{K}\right) Z\left[f\left(\frac{K}{ZN}\right) - \left(\frac{K}{ZN}\right) f'\left(\frac{K}{ZN}\right)\right]. \tag{16}$$

The Euler equation, divided through by Λ, is

$$\begin{aligned}\frac{\dot{\Lambda}}{\Lambda} &= \rho - \frac{H_K}{\Lambda} \qquad (17)\\ &= \rho - J\left(\frac{ZNf\left(\frac{K}{ZN}\right) - C - G}{K}\right)\\ &\quad -J'\left(\frac{ZNf\left(\frac{K}{ZN}\right) - C - G}{K}\right)\left[f'\left(\frac{K}{ZN}\right) - \left(\frac{ZNf\left(\frac{K}{ZN}\right) - C - G}{K}\right)\right].\end{aligned}$$

Following the traditional simplification, the transversality condition can be given as

$$\lim_{t\to\infty} e^{-\rho t}\Lambda(t)K(t) = 0. \tag{18}$$

The only role of the transversality condition in the analysis is to help justify the approach of focusing on paths in the phase diagram that lead eventually to the steady state.

3 Decentralizing the Social Planner's Problem

Since there are no distortions, the solution to the social planner's problem is equivalent to competitive equilibrium. This can be seen directly. Viewing the model through the lens of the decentralized competitive equilibrium brings important insights that help to interpret the general equilibrium solution to the social planner's problem.

The key actors for the decentralized economy are the representative household, the representative production firm and the representative capital-owning and leasing firm. There is also a government budget constraint:

$$B_0 + \int_0^\infty e^{-\int_0^t \Re(\tau)\,d\tau} G\,dt = \int_0^\infty e^{-\int_0^t \Re(\tau)\,d\tau} T\,dt \tag{19}$$

where B_0 is the initial government debt, $\Re$ is the instantaneous real interest rate, and T is the instantaneous flow of lump-sum taxes.

In terms of quantities that do not appear directly in the statement of the social planner's problem, output Y and gross investment expenditure I are defined above in a natural way. To justify identification of various magnitudes with prices, we need to show that those magnitudes fit into the household and firm problems in the appropriate way.

3.1 The Production Firm

The representative production firm is the easiest to analyze. It rents capital and hires labor on the spot market, so its optimization problem has no intertemporal aspect. At each point in time the production firm solves the unconstrained maximization

$$\max ZNf\left(\frac{K}{ZN}\right) - WN - RK, \tag{20}$$

where W is the (real) wage and R is the (real) rental rate. The first-order conditions are

$$W = Z\left[f\left(\frac{K}{ZN}\right) - \left(\frac{K}{ZN}\right)f'\left(\frac{K}{ZN}\right)\right] \tag{21}$$

and

$$R = f'\left(\frac{K}{ZN}\right). \tag{22}$$

Constant returns to scale implies that paying the factors exhausts the production firms revenue, so there are no profits to account for.

3.2 The Household

The representative household faces a familiar problem:

$$\max_{C,N} \int_0^\infty e^{-\rho t} U(C,N)\, dt \tag{23}$$

subject to

$$\int_0^\infty e^{-\int_0^t \Re(\tau)\, d\tau} C\, dt = A_0 + \int_0^\infty e^{-\int_0^t \Re(\tau)\, d\tau} [WN - T]\, dt \tag{24}$$

or equivalently

$$\dot{A} = \Re A + WN - C - T, \tag{25}$$

$$A(0) = A_0$$

and

$$\lim_{t\to\infty} e^{-\int_0^t \Re(\tau)\, d\tau} A(t) = 0,$$

where A represents the household's assets. The capital-owning and leasing firm is owned by the household, but the present value of its dividends is capitalized into the value of its stock and included in A. To distinguish the marginal value of wealth from the marginal value of capital in the social planner's problem, let Θ be the marginal value of wealth for the household. The current-value Hamiltonian for the household is then

$$H = U(C, N) + \Theta[\Re A + WN - C - T].$$

The first order conditions are

$$U_C(C, N) = \Theta \tag{26}$$

and

$$-U_N(C, N) = W\Theta. \tag{27}$$

Given the first-order condition (26), I will often refer to Θ as the marginal utility of consumption, even though its more fundamental meaning is the marginal value of wealth.

The household's Euler equation is

$$\frac{\dot{\Theta}}{\Theta} = \rho - \Re. \tag{28}$$

Integrating Equation (28) yields

$$\ln \Theta(t) = \ln \Theta(\infty) + \int_t^\infty [\Re(\tau) - \rho]\, d\tau. \tag{29}$$

where

$$\Theta(\infty) = \lim_{\tau \to \infty} \Theta(\tau).$$

Equation (29) decomposes the log marginal utility of consumption into the sum of a very long-term real interest rate, $\int_0^\infty [\Re(\tau) - \rho]\, d\tau$, and a marginal utility indicator of the household's long-run wealth position, $\ln(\Theta(\infty))$.

3.3 The Leasing Firm

The capital-owning and leasing firm faces the least familiar problem:

$$\max_I \int_0^\infty e^{-\int_0^t \Re(\tau)\, d\tau}[RK - I]\, dt \tag{30}$$

subject to

$$\dot{K} = KJ(I/K) \tag{31}$$

and

$$K(0) = K_0.$$

Denote the costate variable by Q. The current value Hamiltonian is then

$$H = RK - I + QKJ(I/K).$$

In the social planner's problem and in the household's problem, the objective is in utils and both the marginal value of capital Λ, the marginal value of wealth Θ are measured in utils per real dollar, and discounting is at the rate ρ. In the leasing firm's problem, the objective is in real dollars, Q is a pure number, and discounting is at the rate r.

The first order condition is

$$QJ'(I/K) = 1. \tag{32}$$

Since J is concave, J' is decreasing and (32) makes the investment rate I/K an increasing function of Q. Moreover, Q is a sufficient statistic for everything affecting the investment rate I/K, justifying its identification with Tobin's Q.

The Euler equation is

$$\dot{Q} = Q[\Re + (I/K)J'(I/K) - J(I/K)] - R. \tag{33}$$

Using the (32) and (31), this can be rewritten as

$$\Re = \frac{RK - I}{QK} + \frac{\dot{Q}}{Q} + \frac{\dot{K}}{K} \tag{34}$$

Hayashi shows that the value of the firm (in real dollars) is equal to QK. Thus, equation (34) can be stated in words as "The required rate of return must equal the cash-flow to value ratio plus the rate of growth of the firm's value."

The leasing firm's Euler equation (34), together with the leasing firm's transversality condition

$$\lim_{t\to\infty} e^{-\int_0^t \Re(\tau)\,d\tau} Q(t)K(t) = 0, \tag{35}$$

can be integrated to get the asset equation

$$QK = \int_0^\infty e^{-\int_0^t \Re(\tau)\,d\tau}[R(t)K(t) - I(t)]dt. \tag{36}$$

In words, the value of the leasing firm is equal to the present discounted value of its rental income minus its investment.

3.4 Equivalence Between the Social Planner's Problem and Competitive Equilibrium

It is time to do an inventory of key variables. One can think of the state and costate variables K and Λ, the exogenous variables Z and G, the value of the objective function U, and the endogenous control variables C and N as the primitives of the social planner's problem.

Other variables can be defined in terms of the primitives of the social planner's problem. Output Y is defined by (10), gross investment I is defined by (11) and the gross investment rate X by (13), the real wage W is defined by (88), the real rental rate R is defined by (22) and the marginal utility of consumption Θ is defined by (26). Tobin's Q can be defined by

$$Q = \frac{1}{J'(I/K)} \tag{37}$$

Finally, using the household's Euler equation (28), the real interest rate $\Re$ can be defined by

$$\Re = \rho - \frac{\dot{\Theta}}{\Theta} = \rho - \frac{U_{CC}(C,N)\dot{C} + U_{CN}\dot{N}}{U_C(C,N)}. \tag{38}$$

In the QRBC model, the real interest rate is different in character from the other key variables since its definition requires the derivative of primitives of the QRBC model.

Substituting in the definitions of Θ and Q in the first order condition for consumption in the social planner's problem (15) establishes the relationship between these two variables and Λ

$$\Theta = \frac{\Lambda}{Q}. \tag{39}$$

Equivalently,

$$Q = \frac{\Lambda}{\Theta}. \tag{40}$$

In words, Q is equal to the ratio of the marginal value of capital to the marginal utility of consumption. Taking the time derivative of the log of both sides,

$$\frac{\dot{Q}}{Q} = \frac{\dot{\Lambda}}{\Lambda} - \frac{\dot{\Theta}}{\Theta}. \tag{41}$$

The first order condition for labor in the social planner's problem (16) is equivalent to a combination of (39), (27) and (88). The capital accumulation equation for the social planner's problem (2) is equivalent to a combination of the capital accumulation equation for the leasing firm (31) combined with (10) and (11). The Euler equation for the social planner's problem (17) is equivalent to a combination of (41), (28), (2), (11), (37) and (22).

Given all of the equations discussed already, the balance sheet relationships are redundant, but still interesting. Because of factor exhaustion, which can be seen from (88) and (22), $WN + RK = Y$, so

$$C + I + G = WN + RK,$$

or after rearranging,

$$C + T - WN = (T - G) + (RK - I). \tag{42}$$

By (24), the present value of $C + T - WN$ is A; by (19) the present value of $T - G$ is B; by (36) the present value of $RK - I$ is QK. Therefore, taking the present values of both sides of (42) reveals that

$$A = B + QK. \tag{43}$$

The model exhibits Ricardian equivalence. For a given path of government purchases, a higher level of government debt B matched by a higher present-value of lump-sum taxes has no effect on most of the other variables of the model. Thus, in trying to understand general equilibrium it does not make much sense to focus on total household wealth $A = B + QK$, but only on Q and K. To put things another way, B and A have no counterpart in the social planner's problem.

4 The Steady State

In the steady state, $\dot{K} = \dot{\Lambda} = \dot{\Theta} = \dot{Q} = 0$. Given the equation $J(\delta) = 0$ that defines δ, the steady-state version of Equation (12) (equivalent to Equation (2)) is

$$X^* = \frac{I^*}{K^*} = \delta. \tag{44}$$

Equation (44), together with $J'(\delta) = 1$, implies that the steady-state version of Equation (32) is

$$Q^* = 1. \tag{45}$$

All of these facts combined imply that the steady state version of (17) is

$$R^* = \Re^* + \delta. \tag{46}$$

Finally, the steady-state version of Equation (28) is

$$\Re^* = \rho. \tag{47}$$

Because (24) tells the dynamics of A, which is so peripheral to the model that $\dot{A} = 0$ is not necessary for the model to be in all important respects in steady state,[5] and (31) is equivalent to Equation (12), this exhausts the new facts implied by being at a steady state.

Given the steady-state equations (44), (45), (46) and (47), one can solve recursively for all of the key variables in terms of steady-state labor hours N^*. In order, the steady-state equations are

$$R^* = \rho + \delta, \tag{48}$$

[5] The kinds of tax reschedulings contemplated in a discussion of Ricardian equivalence change the path of A and B but have few other effects.

$$\Gamma^* = \frac{K^*}{ZN^*} = f'^{-1}(\rho + \delta), \tag{49}$$

$$W^* = Z[f(\Gamma^*) - (\rho + \delta)\Gamma^*], \tag{50}$$

$$K^* = ZN^*\Gamma^*, \tag{51}$$

$$I^* = ZN^*\delta\Gamma^*, \tag{52}$$

$$Y^* = ZN^*f(\Gamma^*), \tag{53}$$

$$C^* = Y^* - I^* - G = ZN^*[f(\Gamma^*) - \delta\Gamma^*] - G, \tag{54}$$

$$U^* = U(C^*, N^*), \tag{55}$$

and

$$\Lambda^* = \Theta^* = U_C(C^*, N^*). \tag{56}$$

The key to analyzing the steady state is that by Equation (49), the effective capital/labor ratio Γ^* is a constant unless ρ or δ changes.[6]

Figure 2 shows how to determine steady-state labor N^* graphically in N-C space. Let me call Equation (54) the *Long- Run Material Balance* condition: ($\mathrm{MB}^{\mathrm{LR}}$). From Equation (54), the Long-Run Material Balance line has an intercept at $-G$ and a constant slope of $Z[f(\Gamma^*) - \delta\Gamma^*]$. The Long-Run Labor Market Equilibrium curve (LME^{LR}) is the household's Engel curve at the steady-state wage W^*. Figure 2 also illustrates the effect of a permanent increase in government purchases G (financed by an increase in lump-sum taxes). Table 1 shows the directions of effects of the permanent increase in G on each variable.

[6] In more complex models, changes in capital taxation or in the steady-state growth rate can also change the effective capital/labor ratio.

Table 1: Steady-State Effects

	G	Z
K^*	$+$	$+$
Λ^*	$+$	$-$
Θ^*	$+$	$-$
Q^*, X^*	0	0
I^*	$+$	$+$
N^*	$+$	?
W^*	0	$+$
Γ^*	0	0
R^*	0	0
Y^*	$+$	$+$
C^*	$-$	$+$
U^*	$-$	$+$
$\Re^*$	0	0

In Table 1, Q^* and $X^* = \frac{I^*}{K^*}$ share a row because Equation (37) guarantees that X and Q always move in the same direction (except when there are no adjustment costs, so that Q is always equal to 1). The equations above make it clear why the steady state values of some variables do not change after a permanent increase in G. The effect of G on N^* and C^* are directly visible in Figure 2. Of the other variables whose steady-state value changes after a permanent increase in G, the increase in K^*, Y^* and I^* are driven by the increase in N^*, combined with a constant steady-state capital/labor ratio. The reduction in C^* and increase in N^* reduce U^* for obvious reasons. Since Q^* stays at 1, $\Theta^* = \Lambda^*$ in the steady state, but Figure 2 does not make it obvious why Λ^* increases after a permanent increase in G. This result falls out of the graphical analysis below of the full adjustment path after a permanent increase in G.

What about the long-run effects of technology? Equation (54) makes it clear that permanent improvement in technology causes the Long- Run Material Balance line to swing up around its intercept at $-G$. Also, unlike an increase in G, a permanent improvement in labor-augmenting technology Z raises the steady-state real wage W^*. This increase in the steady-state real wage W^* shifts the Long-Run Labor Market Equilibrium curve upward. As shown above in Figure 1, the indifference curves determine an Engel curve as the locus of points at which the indifference curves have a slope equal to a particular real wage. Normality of consumption and leisure (inferiority of labor) implies that the Engel curve in N-C space will be downward sloping. Figure 3 shows how an increase in the real wage shifts the Engel curve upwards and to the right since on each indifference curve a point with a slope equal to the higher real wage is picked out.[7] Figure 4 shows the effects of the upward shift in both MBC^{LR} and LME^{LR}. Steady state consumption C^* unambiguously increases, but it is unclear what happens to N^*. In analyzing the comparative statics for other variables, the following equations come in handy here. They are all easily derived from steady-state above.

$$ZN^* = \frac{C^* + G}{f(\Gamma^*) - \delta\Gamma^*} \tag{57}$$

$$K^* = \frac{\Gamma^*}{f(\Gamma^*) - \delta\Gamma^*}[C^* + G] \tag{58}$$

$$I^* = \frac{\Gamma^*}{f(\Gamma^*) - \delta\Gamma^*}[C^* + G] \tag{59}$$

$$Y^* = \frac{f(\Gamma^*)}{f(\Gamma^*) - \delta\Gamma^*}[C^* + G]. \tag{60}$$

Thus, an increase in C^* after a permanent increase in Z and therefore in $C^* + G$ is an indicator of a higher total amount of effect labor. In other words, the increase in C shows that while steady-state labor N^* may fall, it cannot fall so much that ZN^* falls. The increase in effective labor ZN^* guarantees in turn that the capital stock, investment and output must all increase in the new steady state.

Intuitively, one would expect the improvement in technology to increase U^* because it expands the opportunity set available to the social planner. But while expanding the opportunity set must raise $\int_0^\infty e^{-\rho t} U dt$, one could imagine that coming about from higher U along the transition path rather

[7]The argument that on each indifference curve an increase in the real wage shifts the point upward and to the right holds even if normality of consumption or normality of leisure fails. However, inferiority of consumption would imply an upward-sloping Engel curve over some region, while inferiority of leisure would imply a forward-bending Engel curve over some region. Either failure of normality would complicate both the figure and the inferences that could be drawn from it.

than from a higher steady-state U^*. To show that U^* does increase as one would expect, decompose the change in N^* and C^* into the movement out along the original Engel curve, which clearly increases U^*, and the movement along the new Material Balance line. The slope of the Long-Run Material Balance line, $Z[f(\Gamma^*) - \delta\Gamma^*]$ is greater than the slope $W^* = Z[f(\Gamma^*) - (\rho + \delta)\Gamma^*]$ of the indifference curves along the corresponding Engel curve. The slope of the *new*, steeper, Material Balance line is higher than the slope of all the indifference curves it intersects in the segment between the two Engel curves. Therefore, the movement out along the new Material Balance line also increases U^*.

The effect of a permanent improvement in Z on $\Lambda^* = \Theta^*$ is, again, most easily seen from the graphical analysis below of the full adjustment path.

5 Fashioning Tools for the Analysis of Fluctuations

In this section I detail a number of tools that will help with the general equilibrium analysis. First, since the utility function can be nonseparable between consumption and labor, one cannot depend on familiar results from the additively separable case. It is important to characterize the comparative statics of the household behavioral functions. The general analysis of tradeoffs plays a key role in that characterization. Second, the factor price possibility frontier provides a graphical representation of the relationship between the wage and the rental rate implied the equations for the production firm. Third, at a higher level of integration, supply and demand in the labor market is a key tool. Finally, in contrast to the labor supply and demand diagram, which takes the marginal utility of consumption Θ as fixed, for some purposes it is useful to take the investment rate X as fixed instead. The Contemporaneous Preferences and Technology Diagram is a key tool for analyze the fixed-X equilibrium.

5.1 Characterizing the Household Behavioral Functions

Duality theory applied to the household's Hamiltonian is the royal road to characterizing the household behavioral functions. At least four aspects of the household's behavior are of interest: consumption C, labor hours N, felicity U, and the household's primary saving S, defined by

$$S = WN - C - T. \tag{61}$$

Define the maximized Hamiltonian $\hat{H}(\Theta, W, T)$ for the household by

$$\hat{H}(\Theta, W, T) = \max_{C,N} U(C, N) + \Theta[WN - C - T] \tag{62}$$

Also define the *money-metric indicator* M by

$$M = \Upsilon U(C, N) + WN - C - T.$$

where Υ is the value of a util, and the maximized money- metric indicator $\hat{M}$ by

$$\hat{M}(\Upsilon, W, T) = \max_{C,N} \Upsilon U(C, N) + WN - C - T. \tag{63}$$

Clearly, when $\Upsilon = \frac{1}{\Theta} > 0$, the values of C and N that maximize the money-metric indicator M are the same as the values of C and N that maximize the Hamiltonian H. Clearly, these maximizing values of C and N do not depend on the lump-sum tax T. Let

$$\begin{aligned}(\hat{C}(\Theta, W), \hat{N}(\Theta, W)) &= \arg\max_{C,N} U(C, N) + \Theta[WN - C - T] \\ &= \arg\max_{C,N} \Theta^{-1} U(C, N) + WN - C - T. \end{aligned} \tag{64}$$

Also, define

$$\hat{U}(\Theta, W) = U(\hat{C}(\Theta, W), \hat{N}(\Theta, W)) \tag{65}$$

and

$$\hat{S}(\Theta, W, T) = W\hat{N}(\Theta, W) - \hat{C}(\Theta, W) - T \tag{66}$$

By the envelope theorem,

$$\hat{H}_\Theta(\Theta, W, T) = \hat{S}(\Theta, W, T), \tag{67}$$

$$\hat{H}_W(\Theta, W, T) = \Theta\hat{N}(\Theta, W), \tag{68}$$

and

$$\hat{H}_T(\Theta, W, T) = -\Theta, \tag{69}$$

$$\hat{M}_\Upsilon(\Upsilon, W, T) = \hat{U}(\Upsilon^{-1}, W), \tag{70}$$

$$\hat{M}_W(\Upsilon, W, T) = \hat{N}(\Upsilon^{-1}, W), \tag{71}$$

and

$$\hat{M}_T(\Upsilon, W, T) = 0. \tag{72}$$

Thus, the first derivatives of $\hat{H}$ and $\hat{M}$ represent the main behavioral outcomes for the household. The second derivatives of $\hat{H}$ and $\hat{M}$ convey a great deal of information about the determinants of these behavioral outcomes.

Two results allow one to characterize the comparative statics of C, N, U and S. The first is Young's Theorem that the order of differentiation does not matter for mixed cross-partial derivatives.[8] The second is the following lemma.

Lemma 1 *Let $\mathcal{F}$ be the real-valued function of the three vectors ξ, β and ζ. If $\mathcal{F}(\xi, \beta, \zeta)$ is linear or convex in β, then*

$$\hat{\mathcal{F}}(\beta, \zeta) = \max_{\xi} \mathcal{F}(\xi, \beta, \zeta)$$

is linear or convex in β. Furthermore, $\hat{\mathcal{F}}$ is strictly convex in β if any change in β forces a change in the optimizing value of ξ, or if $\mathcal{F}$ is strictly convex in β. Conversely, $\mathcal{F}(\xi, \beta, \zeta)$ is linear in β and a change in β never requires a change in the optimizing value of ξ, then $\hat{\mathcal{F}}$ is linear in β.

[8] The limitations on Young's Theorem that are sometimes emphasized in Calculus courses are purely technical and have no economic content. In particular, the discrete cross-partial of a function $\phi(x, y)$ is $\phi(x+\Delta x, y+\Delta y) - \phi(x+\Delta x, y) - \phi(x, y+\Delta y) + \phi(x, y)$, which is symmetric in x and y. The discrete cross-partial carries virtually all the economic meaning (as has been demonstrated in the literature on monotone comparative statics and supermodularity), and raises no technical issues.

Proof: *Given any values for* β_0 *and* ζ*, let* $\hat{\xi}(\beta_0,\zeta)$ *maximize* $\mathcal{F}(\xi,\beta_0,\zeta)$*. Then for any* β_1

$$\begin{aligned}\hat{\mathcal{F}}(\beta_1,\zeta) &= \max_{\xi} \mathcal{F}(\xi,\beta_1,\zeta)\\ &\geq \mathcal{F}(\hat{\xi}(\beta_0,\zeta),\beta_1,\zeta)\\ &\geq \mathcal{F}(\hat{\xi}(\beta_0,\zeta),\beta_0,\zeta)+\mathcal{F}_\beta(\hat{\xi}(\beta_0,\zeta),\beta_0,\zeta)[\beta_1-\beta_0]\\ &= \hat{\mathcal{F}}(\beta_0,\zeta)+\hat{\mathcal{F}}_\beta(\beta_0,\zeta)[\beta_1-\beta_0]. \end{aligned} \tag{73}$$

Overall, (73) shows that $\hat{\mathcal{F}}$ is always above its tangent planes in the dimensions represented by β, which guarantees that $\hat{\mathcal{F}}$ is convex in β. The equalities at the beginning and end are the definition of $\hat{\mathcal{F}}$. The first inequality follows from the nature of maximization. It is strict if $\hat{\xi}(\beta_0,\zeta)$ does not maximize $\mathcal{F}(\xi,\beta_1,\zeta)$. The second inequality is a consequence of $\mathcal{F}(\xi,\beta,\zeta)$ being linear or convex in β. It is strict if $\mathcal{F}$ is strictly convex in β. If a change in β never requires a change in the optimizing value of ξ and $\mathcal{F}(\xi,\beta,\zeta)$ is linear in β, every line of (73) holds with equality.

Figure 5 illustrates the basic idea behind Lemma (1). It shows a case where $\mathcal{F}(\xi,\beta,\zeta)$ is linear in β and ξ can is chosen from one of three values. $\hat{\mathcal{F}}$ is the convex upper envelope of the three lines.

5.1.1 Applying Lemma 1

Though it is unnecessary for the economic substance, for heuristic purposes, the applications of Lemma (1) below assume that $\hat{\mathcal{F}}$ is twice-differentiable. Typically, such convenient differentiability would require that ξ be chosen from a continuum of values, unlike the case in Figure 5.

Applying Lemma 1 to the main diagonals of the Hessians of $\hat{H}$ and $\hat{M}$ yields

$$\hat{H}_{\Theta\Theta}(\Theta,W,T)=\hat{S}_\Theta(\Theta,W,T)\geq 0, \tag{74}$$

$$\hat{H}_{WW}(\Theta,W,T)=\Theta\hat{N}_W(\Theta,W,T)\geq 0, \tag{75}$$

$$\hat{M}_{\Upsilon,\Upsilon} = -\Upsilon^{-2}\hat{U}_{\Theta}(\Upsilon^{-1}, W) \geq 0, \tag{76}$$

and

$$\hat{M}_{WW} = N_W(\Upsilon^{-1}, W) \geq 0. \tag{77}$$

Looking at $\hat{H}_{TT}$ or M_{TT} only yields $0 \geq 0$. The meaning of (74) is clear. Both (75) and (77) say that labor supply is increasing in the real wage: $\hat{N}_W(\Theta, W) \geq 0$. There is no possibility of a backward-bending labor supply curve this is a Frisch labor supply curve; holding the marginal utility of wealth Θ constant blocks most of the power of income effects.

Inequality (76) implies that

$$U_{\Theta}(\Theta, W) \leq 0. \tag{78}$$

The intuition for (74) and (78) is made clearer by a simple tradeoff diagram —in this case the felicity-saving possibility frontier shown in Figure 6.

As can be seen from (62) and (61), at any moment the household is maximizing $U + \Theta S$ by choosing the point on the felicity-saving possibility frontier with a slope of $-\Theta$. An increase in the marginal value of wealth Θ makes saving relatively more important compared to the current flow of utility and causes the household to choose a point further down to the right, with higher primary saving S and lower felicity U. Though it is less natural, one can draw a similar tradeoff diagram to give additional intuition for the result $\hat{N}_W(\Theta, W) \geq 0$. The money-metric indicator can be written $[\Upsilon U(C, N) - C - T] + WN$. Putting N on the horizontal axis and $\Upsilon U(C, N) - C - T$ on the vertical axis, the possibility frontier in Figure 7 picks out points that maximize $\Upsilon U(C, N) - C - T$, money-metric felicity net of consumption and taxes, for each value of N. Other values of C yield points inside the possibility frontier. The money-metric indicator is maximized where the possibility frontier in Figure 7 has slope equal to $-W$. A higher real wage W causes the household to choose a point further down to the right, with higher N and lower net felicity $\Upsilon U(C, N) - C - T$.

5.1.2 Applying Young's Theorem

Applying Young's theorem to $\hat{H}_{\Theta T}$ only confirms that $\hat{S}_T(\Theta, W, T) = -1$, while if I had not already argued that $\hat{C}$, $\hat{N}$ and $\hat{U}$ do not depend on

T, applying Young's theorem $\hat{H}_{TW}$, $\hat{M}_{T\Upsilon}$, $\hat{M}_{TW}$ would only confirm that $\hat{N}_T(\Theta, W, T) = 0$ and $U_T(\Theta, W, T) = 0$. Note that

$$C = WN - S - T$$

so that $\hat{C}_T$ would be

$$\hat{C}_T(\Theta, W, T) = WN_T(\Theta, W, T) - S_T(\Theta, W, T) - 1 = 0,$$

confirming that, given Θ and W, consumption is not a function of the current flow of lump sum taxes. This is a reflection of Ricardian equivalence.

Applying Young's theorem to $\hat{H}_{\Theta W}$ yields

$$\hat{S}_W(\Theta, W, T) = \hat{N}(\Theta, W) + \Theta \hat{N}_\Theta(\Theta, W) \tag{79}$$

Substituting into (79) the identity $\hat{S}(\Theta, W, T) = W\hat{N}(\Theta, W) - \hat{C}(\Theta, W) - T$ yields

$$\hat{N}(\Theta, W) + W\hat{N}_W(\Theta, W) - \hat{C}_W(\Theta, W) = \hat{N}(\Theta, W) + \Theta \hat{N}_\Theta(\Theta, W)$$

or equivalently,

$$W\hat{N}_W(\Theta, W) - \hat{C}_W(\Theta, W) = \Theta \hat{N}_\Theta(\Theta, W). \tag{80}$$

Applying Young's theorem to $\hat{M}_{\Upsilon W}$ yields

$$\hat{U}_W(\Upsilon^{-1}, W) = -\Upsilon^{-2}\hat{N}_\Theta(\Upsilon^{-1}, W),$$

implying

$$\hat{U}_W(\Theta, W) = -\Theta^2 \hat{N}_\Theta(\Theta, W). \tag{81}$$

Since $\hat{U}_\Theta(\Theta, W) \geq 0$, normality of leisure, or equivalently, inferiority of labor, implies that $\hat{N}_\Theta(\Theta, W) > 0$. (Normality and inferiority have to do with what happens as one moves to a higher indifference curve holding the real wage W constant.) By (80) and (81), normality of leisure and inferiority of labor imply

$$W\hat{N}_W(\Theta, W) - \hat{C}_W(\Theta, W) > 0 \tag{82}$$

and

$$\hat{U}_W(\Theta, W) < 0. \tag{83}$$

Inequality (82)—or equivalently both sides of (79) being positive—can be interpreted as saying that an increase in the real wage will affect saving by more than the direct effect of the wage on household income for constant N. In other words, an increase in the real wage will raise quantitative saving effort in terms of N and C. This increase in quantitative saving effort is associated with a fall in the current flow of utility indicated by (83). Note that, mechanically, holding Θ constant blocks most income effects, through which one might otherwise expect an increase in the real wage to raises the household's felicity. If nothing else changes, a temporary increase in the real wage will indeed make the household better off, but only by raising *future* felicity. At the moment, the household will be working too hard to enjoy themselves.

Graphically, thinking about the shift in the felicity-saving possibility frontier due to an increase in the real wage W provides the intuition for (83) and (82). For given values of C and N, an increase in the real wage raises primary saving by NdW. Thinking in terms of small changes, the envelope theorem guarantees that in Figure 8, for given U, the value of S on the felicity-saving possibility frontier shifts right by $\hat{N}(\Theta, W)dW$. Inferiority of labor implies that N is highest at points low down to the right on the felicity-saving possibility frontier, where U is low. Therefore, an increase in W shifts the felicity-saving possibility frontier further to the right at lower values of U. As Figure 8 illustrates, this makes the new felicity-saving possibility frontier flatter than the old one at the corresponding point directly to the right. Thus, the point on the new felicity-saving possibility frontier that has the unchanged slope $-\Theta$ must be lower down on the new possibility frontier: the increase in W, holding Θ fixed reduces the optimal value of U, confirming (83). This movement down along the felicity-saving possibility frontier guarantees in turn that primary saving will increase by more than the rightward shift of the possibility frontier $\hat{N}(\Theta, W)dW$—which is the essence of (82).

5.1.3 Summarizing the Characterization of Household Behavioral Functions

At this point we can sign (at least in the weak sense) all but one of the derivatives of $\hat{C}$, $\hat{N}$, $\hat{U}$ and $\hat{S}$:

$$\underset{-\quad ?}{\hat{C}(\Theta, W)}; \qquad \underset{+\quad +}{\hat{N}(\Theta, W)}; \qquad \underset{-\quad -}{\hat{U}(\Theta, W)}; \qquad \underset{+\quad +\quad -}{\hat{S}(\Theta, W, T)} \tag{84}$$

The negative effect of Θ on consumption is due to normality of consumption. The effect of W on consumption remains to be discussed.

The positive effect of Θ on labor is due to normality of leisure, or equivalently, inferiority of labor. The normality of leisure or inferiority of labor also implies the negative effect of W on felicity U through Young's theorem.

The positive effect of W on labor, the negative effect of Θ on felicity U and the the positive effect of Θ on saving all reflect the monotone comparative statics principle that the optimal quantity of something goes up when its weight in the objective function increases—and conversely that the optimal quantity of something goes down when its weight in the objective function declines. This principle is a special case of the principles surrounding supermodularity and submodularity, to be discussed below.

Finally, the negative effect of lump-sum taxes T on saving reflects the fact that consumption and labor are unaffected by lump-sum taxes, given Θ and W—a fact closely connected to Ricardian equivalence.

5.1.4 The Effects of Nonseparability Between Consumption and Labor

What determines the sign of the effect of the real wage on consumption? Looking at the problem of maximizing the household's Hamiltonian,

$$\max_{C,N} U(C, N) + \Theta[WN - C - T]$$

the real wage does not interact directly with consumption. However, labor potentially interacts with consumption in $U(C, N)$, and so the real wage can affect C through its effect on N. Because an increase in W, holding Θ fixed unambiguously raises N, the sign of the effect of W on C is determined by the sign of the cross-partial U_{CN}. If $U_{CN} > 0$, then the increase in N induced by higher W raises the optimal level of C. If $U_{CN} < 0$, then the increase in N induced by higher W lowers the optimal level of C. Finally, in the familiar (but not necessarily realistic) additively- separable case, $U_{CN} = 0$, the increase in N induced by higher W has no effect on the optimal level of C. All of these statements follow from the basic principles of supermodularity. $U_{CN} > 0$ makes the Hamiltonian supermodular in consumption and labor; $U_{CN} < 0$ makes the Hamiltonian submodular in consumption and labor; and $U_{CN} = 0$ makes the Hamiltonian modular in consumption and labor. Because of the concavity and differentiability assumptions I have made, the situation is simpler than the usual situation to which supermodularity principles are applied. In this case, one can focus on the first-order condition for consumption

$$U_C(C, N) = \Theta.$$

Figure 9 shows the case in which $U_{CN} > 0$. The increase in N induced by a higher real wage shifts $U_C(C, N)$ upwards for every value of C. Therefore, the point at which $U_C(C, N)$ crosses the marginal value of wealth line, horizontal at Θ, is pushed further to the right. If $U_{CN} < 0$ the marginal utility of consumption curve is pushed down by the increase in N and the intersection with the horizontal marginal value of wealth line would be shifted to the left. Additive separability would make N irrelevant to the figure.

Basu and Kimball (2002) argue that U_{CN} should be positive because this preserves equality of income and substitution effects on labor supply when the elasticity of intertemporal substitution for consumption is less than one. In particular, King, Plosser and Rebelo (1988) show that to keep hours per worker and the real interest rate constant under conditions of steady-state growth, $U(C, N)$ must be of the form

$$U(C, N) = \frac{C^{1-(1/s)}}{1-(1/s)}\Omega(N), \tag{85}$$

where $s \neq 1$ is the elasticity of intertemporal substitution, or

$$U(C, N) = \ln(C) - v(N) \tag{86}$$

when $s = 1$. Although Equation (86) is convenient, empirical evidence points to a value of s significantly less than one.[9] A variety of empirical evidence also points directly to $U_{CN} > 0$. Basu and Kimball (2002) look at the aggregate evidence directly and discuss the literature on the micro evidence.

5.2 The Factor-Price Possibility Frontier

As illustrated in Figure 10, Equations 88 and 22 define the Factor-Price Possibility Frontier parametrically with

$$(W, R) = (Z[f(\Gamma) - \Gamma f'(\Gamma)], f'(\Gamma)), \tag{87}$$

where $\Gamma = \frac{K}{ZN}$ is the parameter. The labor- augmenting technology Z is the only thing that shifts the factor price possibility frontier. Straightforward calculation shows that the slope of the Factor Price Possibility frontier is equal to $-\frac{1}{Z\Gamma} = -\frac{N}{K}$, which is closer to zero when R/W is higher. Therefore, the factor price possibility frontier is always convex as shown, becoming linear in the case of Leontieff technology.

The Factor Price Possibility Frontier serves as a graphic reminder that the real wage and the real rental rate must always move in opposite directions unless technology changes. This has to do with both constant returns to scale and the constancy of the markup ratio $\frac{P}{MC} \equiv 1$.

5.3 Labor Supply and Demand

Figure 11 shows labor supply and demand. $N = \hat{N}(\Theta, W)$ is the labor supply equation. The labor supply curve is upward sloping because $\hat{N}(\Theta, W)$ is monotonically increasing in W.[10] An increase in Θ shifts the labor supply curve to the right because of the normality of leisure. The marginal utility of consumption Θ is a summary statistic for everything that shifts the labor supply curve, since it is the only other variable that appears in $\hat{N}(\Theta, W)$.

In other words, given Θ, nothing else shifts the labor supply curve. Equation (88) is the labor demand equation. Letting $W^d(N, K, Z)$ be the wage determined by labor demand,

$$W^d(N, K, Z) = Z\left[f\left(\frac{K}{ZN}\right) - \left(\frac{K}{ZN}\right) f'\left(\frac{K}{ZN}\right)\right]. \tag{88}$$

[9]See for example Hall (1988) and Barsky, Juster, Kimball and Shapiro (1997).
[10]See (75) or (77).

The original assumption that the production function is supermodular in labor and technology implies

$$F_{KZ} = W^d_Z(N, K, Z) \geq 0. \tag{89}$$

The other two derivatives are

$$W^d_N(N, K, Z) = \left(\frac{K^2}{ZN^3}\right) f''\left(\frac{K}{ZN}\right) < 0. \tag{90}$$

$$W^d_K(N, K, Z) = -\left(\frac{K}{ZN^2}\right) f''\left(\frac{K}{ZN}\right) > 0. \tag{91}$$

Equation (90) implies that the labor demand curve is downward sloping. Equations (91) and (90) imply that both K and Z shift the labor demand curve upward and outward. Given K and Z, no other variable has any effect on the labor demand curve.

For fluctuations, as for the steady-state, given labor hours N, many other variables can be determined recursively. Thus, the interaction between labor supply and demand that determines N nonrecursively is a central part of the model's mechanism. For use in higher levels of integration in the model, define $\mathcal{W}(\Theta, K, Z)$ as the equilibrium real wage and $\mathcal{N}(\Theta, K, Z)$ equilibrium quantity of labor determined in the labor market.

5.4 The Contemporaneous Preferences and Technology Diagram

The Contemporaneous Preferences and Technology Diagram is illustrated in Figure 12. The Contemporaneous Preferences and Technology Diagram takes the investment rate X as fixed. Given X, the Contemporaneous Material Balance (CMB) curve is

$$C = F(K, N, Z) - XK - G = ZNf\left(\frac{K}{ZN}\right) - XK - G. \tag{92}$$

The Contemporaneous Material Balance curve is upward sloping and concave in N-C space since $F_N > 0$ and $F_{NN} < 0$.

Since any point on the Contemporaneous Material Balance curve allows the same level of government purchases and investment $I = XK$ for the future, all points are equally good *as far as the future is concerned.* Therefore, given K and X, the social planner should maximize current felicity U. In other words, a characteristic of the overall solution to the social planner's problem must be that, whatever the optimal value of K and X at any moment, along with the exogenous values of Z and G, the social planner will put the economy at the highest level of felicity available along the Contemporaneous Material Balance curve. Of course, the highest level of U is achieved at the point at which the Contemporaneous Material Balance curve is tangent to an indifference curve at that point.

Because of the tangency condition, in the Contemporaneous Preferences and Technology Diagram, both the level and the slope of the curves matter. The comparative statics of the level and slope of the Contemporaneous Material Balance curve can be seen from Equation (92). An increase in either X or in G shifts the Contemporaneous Material Balance curve downward in a parallel shift that does not change the slope at a given level of G. Because of the assumption that $F_{ZN} > 0$, an increase in Z raises both the level and slope of the Contemporaneous Material Balance curve at a given level of N. An increase in the capital stock raises the level of the Contemporaneous Material Balance curve at a given level of N as long as

$$F_K - X = R - X > 0,$$

and always raises the slope of the Contemporaneous Material Balance curve at a given N.

Since I am abstracting from preference shocks that would change the shape of felicity $U(C, N)$ the indifference curves do not shift. Indeed, the indifference curves are invariant to shocks to impatience ρ as well.

On the household's side, the real wage in the tangency condition is determined by a function $\hat{W}(C, N)$:

$$\begin{aligned} W &= \frac{-U_N(C,N)}{U_C(C,N)} \\ &= \underset{+\ \ +}{\hat{W}(C, N)}. \end{aligned} \tag{93}$$

Equations (3) and (4) imply that $\hat{W}(C, N)$ is increasing in both arguments. In particular, moving up along the Contemporaneous Material Balance curve the needed wage on the household side, $\hat{W}(C, N)$, increases, while the wage at which the firm demands labor $F_N(K,N,Z) = W^d(K,N,Z)$ decreases with the increase in N. Thus, if a particular point on the Contemporaneous Material Balance curve has $F_N(K,N,Z) > \hat{W}(C,N)$, then the tangency point must be further up along the Contemporaneous Material Balance curve. Conversely, if a particular point on the Contemporaneous Material Balance curve has $F_N(K,N,Z) < \hat{W}(C,N)$, then the tangency point must be lower down along the Contemporaneous Material Balance curve.

The direction of movement of labor N, consumption C, the real wage W and felicity U are readily apparent in the Contemporaneous Preferences and Technology Diagram. It is also not too hard to use the behavior of W and N to determine what is happening to the rental rate R, the effective capital/labor ratio $\Gamma = \frac{K}{ZN}$ and output Y. With X fixed, the behavior of $\dot{\Lambda}$ will turn out to depend primarily on the rental rate R. Less obviously, the Contemporaneous Preferences and Technology Diagram indicates a fair bit about the movement of the marginal utility of consumption Θ. First, there is the direct dependence of $U_C(C,N)$ on C and N according to $U_{CC} < 0$ and U_{CN}, which can be of either sign as discussed above. Second, the equation

$$U = \hat{U}(\Theta, W) \tag{94}$$

can be inverted in its first argument to yield the function $\hat{\Theta}(U,W)$.

$$\hat{\Theta}(\underset{-}{U}, \underset{-}{W}). \tag{95}$$

Holding W fixed, the negative effect of Θ on U in $\hat{U}(\Theta,W)$ implies a negative effect of U on Θ in the inverted function $\hat{\Theta}(U,W)$. When the real wage W increases, the direct effect on U in Equation (94) is negative. The value of U in Equation (94) can be held fixed by reducing Θ as W increases. This reduction of Θ along with W in order to hold U fixed in Equation (94) corresponds to a negative effect of W on $\hat{\Theta}$ in Equation (95) when holding U fixed.

There is another way to see why normality of leisure, which is behind the negative effect of W on $\hat{U}(\Theta,W)$, implies that W has a negative effect on $\hat{\Theta}(U,W)$. As shown in Figure 1, normality of leisure corresponds to the positive effect of C on $\hat{W}(C,N)$. If the point on an indifference curve directly above another point has a higher slope, then the vertical distance between indifference curves must increase as one follows both indifference curves to the right. With the gap in felicity U fixed by a focus on these two indifference curves, the increasing vertical distance implies that in comparing these pairs of points, $\frac{\Delta C}{\Delta U}$ is increasing as we follow both indifference curves up to the right. Then it only requires focusing on very close indifference curves to see that

$$\Theta = \frac{\partial U}{\partial C} = \lim_{\Delta U \to 0} \left(\frac{\Delta C}{\Delta U} \right)^{-1}$$

is decreasing as we follow an indifference curve around to the right, where W is higher.

6 Contemporaneous General Equilibrium

The QRBC model, and any similarly complex dynamic general equilibrium model has at least four levels of integration:

1. Household and Firm Optimization
2. Market Equilibrium
3. Contemporaneous General Equilibrium
4. Dynamic General Equilibrium.

Household and firm optimization are clear concepts. Labor market equilibrium is a good example of what I mean by the market equilibrium level of integration.

Contemporaneous general equilibrium is the least familiar concept. Contemporaneous general equilibrium is defined as the solution to the model when the current values of the endogenous state variables, costate variables and exogenous variables are taken as given. It is a very useful concept because the structure of contemporaneous general equilibrium is invariant to the time- series properties of the exogenous variables. The structure of contemporaneous general equilibrium makes the endogenous state variable(s) a sufficient statistic for the past and the costate variable(s) a sufficient statistic for the future, so that these plus the *current* values of the exogenous variables are enough to determine contemporaneous general equilibrium. In the

QRBC, this means that the current values of K, Λ, Z and G are enough to determine all of the other key variables except the real interest rate. The real interest rate $\Re$ is not determined in contemporaneous general equilibrium; in the QRBC model, it is by nature a dynamic general equilibrium object.

Even in an exact solution to a dynamic stochastic general equilibrium model, without linearization or a certainty- equivalence approximation, the structure of contemporaneous general equilibrium is invariant to the stochastic processes of the exogenous variables.

Dynamic general equilibrium is the overall outcome of the model. In the QRBC model, there is only one endogenous state variable, so the dynamics of K and Λ in dynamic general equilibrium can be illustrated by a two-dimensional phase diagram. In a linearized or log-linearized model, using a certainty-equivalence approximation, the essential structure of dynamic general equilibrium is given by the impulse responses of all the key variables to each relevant type of shock.

To understand the model, it is useful to think of each point on the phase diagram not just as a (K, Λ) pair that determines a dynamic arrow $(\dot{K}, \dot{\Lambda})$, but as a particular contemporaneous general equilibrium that determines many other variables of interest at the same time as it determines $\dot{K}$ and $\dot{\Lambda}$. Contemporaneous general equilibrium is the key to determining the mapping from (K, Λ) to $(\dot{K}, \dot{\Lambda})$ that governs the phase diagram dynamics.

The structure of contemporaneous general equilibrium means that the impulse responses of K and Λ that come out of the phase diagram, combined with the impulse responses of Z and G, generate impulse responses for all of the other variables determined as functions of K, Λ, Z and G in contemporaneous general equilibrium. In a more intricate way, these impulse responses also allow one to determine the impulse response of the real interest rate $\Re$.

In the Ramsey-Cass-Koopmans model that is often used to teach the use of phase diagrams with dynamic general equilibrium models, contemporaneous general equilibrium is trivial, but contemporaneous general equilibrium is not trivial in the QRBC model. Because the concept of contemporaneous general equilibrium is relatively unfamiliar, its behavior may not always seem intuitive. In particular, holding the marginal value of capital Λ fixed can seem as unnatural in its own way as holding price fixed to think about shifts in the supply and demand curves often seems to students in "Principles of Economics," who often have more intuition for market equilibrium (at least in simple cases) than for the analytical constructs of supply and demand that economists use to think about market equilibrium.

In the QRBC model, the analysis of contemporaneous general equilibrium is closely linked to the analysis of equilibrium between investment demand and saving supply, since it is straightforward to determine the values of all the other variables in contemporaneous general equilibrium once the investment rate and the marginal utility of consumption are determined by investment demand and saving supply. But before moving on to investment demand and saving supply, there is some housekeeping to do.

6.1 Taking logarithms of the dynamic variables

Equations (12), (17), (28) and (33), suggest that it is convenient to express the dynamic equations of the QRBC model and its partial equilibrium in terms of logarithmic time derivatives: $\frac{\dot{K}}{K}$, $\frac{\dot{\Lambda}}{\Lambda}$, $\frac{\dot{\Theta}}{\Theta}$ and $\frac{\dot{Q}}{Q}$. Using small letters for logarithms, these equations can be rewritten

$$\dot{k} = J(X), \tag{96}$$

$$\dot{\lambda} = \rho - J(X) - J'(X)[R - X], \tag{97}$$

$$\dot{\theta} = \rho - \Re, \tag{98}$$

and

$$\dot{q} = \Re - J(X) - J'(X)[R - X]. \tag{99}$$

Note that Equation (97) has the same form as Equation (99), except that the real interest rate is replaced by the utility discount rate. Thus, Equation (97) also admits of a required-rate-of-return interpretation, but with utils as the numeraire instead of dollars:

$$\rho = \frac{RK - I}{QK} + \dot{k} + \dot{\lambda}. \tag{100}$$

I will resist the temptation to fully log-linearize the model here, since that would take things in a different direction, toward quantitative analytics. But even in studying the qualitative analytics in a way that allows for the nonlinearities in contemporaneous general equilibrium and at least the perfect- foresight version of dynamic general equilibrium,[11] it is convenient to analyze the dynamics in terms of the logarithms k, λ, θ and q. Of course, every comparative static statement about Θ in the discussion of household behavioral functions applies equally well to θ and every comparative static statement about K in the discussion of labor demand applies equally well to k.[12]

6.2 The Investment Demand Curve

It works well to analyze investment demand and saving supply in X-θ space. On the horizontal axis, this means thinking about investment and saving in relation to the preexisting size of the capital stock. It makes sense to put (log) marginal utility θ on the vertical axis because θ is central to household behavior and therefore to saving supply.

Defining the function $\hat{q}(X)$ by

$$\hat{q}(X) = -\ln(J'(X)), \tag{101}$$

Equation (39) can be rewritten as

$$\theta = \lambda - \hat{q}(X). \tag{102}$$

Equation (102) is the investment demand (II) curve. As illustrated in Figure 13, the investment demand curve intersects the vertical line $X = \delta$ at the level λ. The investment demand curve is downward sloping since $\hat{q}(X)$ is increasing in X. An increase in λ causes a parallel upward shift in the investment demand curve. In X-θ space the (log) marginal value of capital λ is a sufficient statistic for everything that shifts the investment demand curve.[13]

[11] This can be useful: Kimball (1993) shows how to use the nonlinearities in a perfect foresight model to derive a perturbation-theory approximation to dynamic stochastic general equilibrium.

[12] However, the derivations of many of these comparative statics results were made easier by the use of K and Θ rather than k and θ.

[13] In I-θ space, an increase in k would shift the investment demand curve out and up.

6.3 Saving Supply

Saving supply can be thought of either as a function of Θ, K, Z and G, through the lens of *Fixed-Θ equilibrium*, or as a function of X, K, Z and G, through the lens of *Fixed-X equilibrium*. Both Fixed-Θ equilibrium and Fixed-X equilibrium are important for analyzing the comparative statics of Contemporaneous General Equilibrium, which always lie between the comparative statics for the Fixed- Θ and the comparative statics for the Fixed-X equilibria. The Fixed-Θ equilibrium corresponds to the Contemporaneous General Equilibrium when $J(X) = X - \delta$, so that there are no investment adjustment costs. Conversely, very large investment adjustment costs make Contemporaneous General Equilibrium behave much like Fixed-X equilibrium. Intermediate degrees of adjustment costs make Contemporaneous General Equilibrium something in between.

For clarity, let me say that in naming "Fixed-Θ" or "Fixed-X" equilibrium, I am using "fixed" in the sense of "stipulated," "specified" or "known," rather than in the sense of "unchanging." Fixed-$Theta$ equilibrium, for example, is what one gets by treating the marginal utility of consumption Θ *as if* it were exogenous, while Fixed-X equilibrium is what one gets by treating the investment rate X as if it were exogenous. In both cases, K is treated as if it were exogenous, which is natural for the analysis of a single point in time. Implicitly, if either Θ or X is treated as if it were exogenous, Λ must be treated as endogenous. Thus, on the phase diagram, Fixed-Θ Equilibrium picks out an iso-Θ locus. The effect of a change in K on Fixed-Θ equilibrium shows up as a movement along the iso-Θ locus, while the effects of a change in Z or G show up as a shift of the iso-Θ locus. Varying Θ picks out a different iso-Θ locus. Similarly, Fixed-X Equilibrium picks out an iso-X locus. Some of the importance of Fixed-X equilibrium comes from that fact that iso-X locus with $X = \delta$ is the $\dot{k} = 0$ locus.

6.3.1 Fixed-Marginal-Utility-of-Consumption Equilibrium

Gross national saving is $Y - C - G$. This is not the same as the household's primary saving S. The relationship is

$$Y - C - G = S + RK + T - G.$$

Remember that primary saving S does not include interest income or debt payments. Define the function $\mathcal{X}(\Theta, K, Z, G)$ by

$$\mathcal{X}(\Theta, K, Z, G) = \frac{F(K, \mathcal{N}(\Theta, K, Z), Z) - \hat{C}(\Theta, \mathcal{W}(\Theta, K, Z)) - G}{K}, \quad (103)$$

where $\mathcal{N}$ and $\mathcal{W}$ give the labor market equilibrium values of labor and the real wage. Then the saving supply curve, which focuses on gross national saving relative to the capital stock, is given by

$$X = \frac{Y - C - G}{K} = \mathcal{X}(\Theta, K, Z, G) = \mathcal{X}(e^{\theta}, e^{k}, Z, G).$$

It is clear from Equation (103) that saving supply is intimately bound up with labor supply and demand. The four variables Θ, K, Z and G that determine saving supply are also enough to determine the values of $N = \mathcal{N}(\Theta, K, Z)$, $W = \mathcal{W}(\Theta, K, Z)$,

$$\Gamma = \frac{K}{Z\mathcal{N}(\Theta, K, Z)}, \quad (104)$$

$$R = f'\left(\frac{K}{Z\mathcal{N}(\Theta, K, Z)}\right), \quad (105)$$

$$C = \hat{C}(\Theta, \mathcal{W}(\Theta, K, Z)), \quad (106)$$

$$U = \hat{U}(\Theta, \mathcal{W}(\Theta, K, Z)), \quad (107)$$

$$Y = (K, \mathcal{N}(\Theta, K, Z), Z), \quad (108)$$

$$I = K\mathcal{X}(\Theta, K, Z, G), \quad (109)$$

$$\dot{k} = \frac{\dot{K}}{K} = J(\mathcal{X}(\Theta, K, Z, G)) \quad (110)$$

$$\begin{aligned}
\dot{\lambda} &= \frac{\dot{\Lambda}}{\Lambda} \\
&= \rho - J(\mathcal{X}(\Theta, K, Z, G)) \\
&\quad - J'(\mathcal{X}(\Theta, K, Z, G))[f'\left(\frac{K}{Z\mathcal{N}(\Theta, K, Z)}\right) - \mathcal{X}(\Theta, K, Z, G)] \quad (111)
\end{aligned}$$

The comparative statics for all of these variables in labor market equilibrium are given in Table 2.[14] In addition to the assumptions in Section 2, the effect of K on X, and the effect of G on $\dot{\lambda}$ depend on being close enough to the steady state.

Table 2: Fixed-Marginal-Utility-of-Consumption Comparative Statics

	Θ	K	Z	G
λ	+	+	+	−
N	+	+	+	0
W	−	+	+	0
Γ	−	+	−	0
R	+	−	+	0
Y	+	+	+	0
C	−	?	?	0
U	−	−	−	0
I	+	+	+	−
X, q, $\dot{k}$	+	+	+	−
$\dot{\lambda}$	?	+	+	−

Let me spell out the reasons for each of these signs, beginning with the effects of an increase in marginal utility Θ for fixed K, Z and G.

As mentioned above, treating Θ as exogenous means that λ will have to vary. By Equation (102),

$$\lambda = \theta + \hat{q}(X). \quad (112)$$

[14]Given the additional assumption $U_{CN} > 0$, which I favor, the effects of K and Z are unambiguously positive.

Therefore, holding θ constant, λ must move in the same direction as q and X. The last three signs in the row for λ are copied from those in the row for X. The logic for the direction of effects on X in Fixed-Θ equilibrium are given below. When Θ increases, holding K, Z and G constant, q and X turn out to increase as well, so the required value of λ goes up. This accounts for the upper-left-hand sign in Table 2.

$\Theta\uparrow$: As shown in Figure 14, an increase in marginal utility Θ (or equivalently, an increase in log marginal utility θ) shifts the labor supply curve out with no effect on labor demand, raising labor N and lowering the real wage W. The increased N lowers the effective capital labor ratio Γ; the relative shortage of capital raises the rental rate R.

Because felicity may not be additively separable in consumption and labor, it is not as obvious why consumption falls with an increase in Θ as one might think. Figure 15 shows why consumption falls in response to an increase in Θ, regardless of the sign of U_{CN}. The lower real wage W shifts the relevant Engel curve downward. Combined with the rightward shift in N, this guarantees that consumption must fall. Note that the normality of consumption and leisure are important for this argument, since an upward-sloping or forward-bending Engel curve would undercut the argument.

Felicity U falls because labor goes up and consumption goes down. Output increases because labor increases, with K and Z fixed. Since output increases, while consumption falls, $I = Y - C - G$ goes up, as does $X = I/K$, since K is fixed. The increase in the investment rate X raises the growth rate of the capital stock $\dot{k}$ in a mechanical way and must reflect an increase in Tobin's q.

The effect of an increase in Θ on $\dot{\lambda}$ is ambiguous because there are two opposing effects. Taking a total differential of both sides of Equation (97),

$$d(\dot{\lambda}) = -J''(X)[R - X]dX - J'(X)dR. \tag{113}$$

Using Equation (100) to aid in interpretation, the first term the reduction in the rate of return when Q increases along with X. A higher util capital gains rate $\dot{\lambda}$ can help meet the required rate of return. The direct costs and benefits of X cancel out because of the leasing-firm's first-order condition, Equation (32). The last term, $-J'(X)dR$, reflects the ability of a higher rental rate to meet the required rate of return as an alternative to a higher util capital gains rate $\dot{\lambda}$.

In response to an increase in Θ, the rental rate R increases, so the second term $-J'(X)dR$ will be negative. As for the first term, since $X^* = \delta$, while $R^* = \rho + \delta$, anywhere reasonably close to the steady-state $R - X$ will be positive, so $-J''(X)[R - X]$ will be positive. Since the increase in Θ raises X, the entire first term $-J''(X)[R - X]dX$ is positive. Thus, the effect of an increase in Θ on $\dot{\lambda}$ is ambiguous. If investment adjustment costs are small, then $-J''(X)$ will be small and the $\dot{\lambda}$ will fall when Θ increases. If investment adjustment costs are large, then $-J''(X)$ will be large and $\dot{\lambda}$ may fall when Θ increases. This cannot be ruled out, since neither $\mathcal{X}(\Theta, K, Z, G)$, nor the determination of R in Equation (105) are affected by the shape of the accumulation function J.

$K \uparrow$: As shown in Figure 16, an increase in capital K (or equivalently, an increase in the log of the capital stock k) shifts labor demand out with no effect on labor supply. The outward shift in labor demand increases both labor N and and the real wage W. The increase in both productive factors K and N raises output Y.

As shown in Figure 17, the Factor Price Possibility Frontier implies that the increase in the real wage W must correspond to a decrease in the rental rate R. Since $R = f'(\Gamma)$, this decreasing in the rental rate has to come from an increase in the capital to effective labor ratio $\Gamma = \frac{K}{ZN}$.

With Θ fixed, felicity U falls because because the increase in K raises the real wage W, and $\hat{U}_W(\Theta, W) \leq 0$. (See Figure 8.)

As for consumption C, as shown in Figure 9, in my preferred case $U_{CN} > 0$ (supermodularity between consumption and labor), $\hat{C}_W(\Theta, W)$, so the increase in the real wage induced by an increase in the capital stock reduces consumption. However, the claim that $U_{CN} > 0$ is more controversial than the assumptions I made in Section 2, so the entry in Table 2 shows a question mark to account for the possibility that $U_{CN} < 0$, which would reverse the sign of $\hat{C}_W$ and cause C to fall when K increases.

The strong possibility that consumption rises when K increases, holding Θ fixed makes it more difficult to sign the change in I and $X = \frac{I}{K}$. Totally differentiating expressions for I and X yields

$$
\begin{aligned}
dI &= d(Y - C - G) \\
&= F_K(K,N,Z)dK + F_Z(K,N,Z)dZ - dG \\
&\qquad +[F_N(K,N,Z)\hat{N}_\Theta(\Theta,W) - \hat{C}_\Theta(\Theta,W)]d\Theta \\
&\qquad +[F_N(K,N,Z)\hat{N}_W(\Theta,W) - \hat{C}_W(\Theta,W)]dW \\
&= RdK + F_Z(K,N,Z)dZ - dG \\
&\qquad +[W\hat{N}_\Theta(\Theta,W) - \hat{C}_\Theta(\Theta,W)]d\Theta \\
&\qquad +[W\hat{N}_W(\Theta,W) - \hat{C}_W(\Theta,W)]dW
\end{aligned}
$$

and

$$
\begin{aligned}
dX &= d\frac{Y - C - G}{K} \\
&= [R - X]\frac{dK}{K} + \frac{F_Z(K,N,Z)dZ - dG}{K} \\
&\qquad +[W\hat{N}_\Theta(\Theta,W) - \hat{C}_\Theta(\Theta,W)]\frac{d\Theta}{K} \\
&\qquad +[W\hat{N}_W(\Theta,W) - \hat{C}_W(\Theta,W)]\frac{dW}{K}.
\end{aligned}
$$

In this column of Table 2, Θ, Z and G are held constant. Inequality (82) guarantees that $W\hat{N}_W - \hat{C}_W \geq 0$, so the increase in the real wage induced by a higher capital stock brings forth more saving effort from the household. This, combined with the direct increase in output represented by the term $[R - X]\frac{dK}{K}$ guarantees a higher level of investment. For the investment rate X the story is similar, but slightly more complex. Since $R^* - X^* = \rho$, the factor $R - X$ is positive for some range around the steady state. Of course, q and $\dot{k}$ increase along with X.

$Z \uparrow$**:** The effects of an increase in Z are very similar to the effects of an increase in K. Because $F_{KZ}(K,N,Z) > 0$, the labor demand curve shifts out, so labor N and the real wage W increase. Output increases both because of the direct effect Z and the induced increase in N. Felicity U falls because of the increase in the real wage. Equations (114) and (114) indicate that investment I and the investment rate X rise, but without the quibbling about being close enough to the steady state. q and $\dot{k}$ rise along with X.

As with an increase in K, the change in consumption C is driven by the change in the real wage. The direction of the effect on consumption is given by the sign of $C_W(\Theta, W)$, which in turn is the same as the sign of U_{CN}. In my preferred case, $U_{CN} > 0$, the improvement in technology causes consumption to rise for fixed Θ.

The one big difference between an increase in K and an increase in Z is that the increase in Z shifts the factor price possibility frontier out, as shown in Figure 18. The rental rate R increases as well as the real wage W because with both Z and N increasing, $\Gamma = \frac{K}{ZN}$ must fall and so $f'(\Gamma)$ must rise.

$G \uparrow$: Holding Θ, K and Z fixed, an increase in government purchases has no effect on either labor supply or labor demand. Thus, nothing happens to N or W. Since K, N and Z are unchanged, $\Gamma = \frac{K}{ZN}$, R and Y are all unchanged.

With both marginal utility Θ and the real wage W unchanged, $C = \hat{C}(\Theta, W)$ and $U = \hat{U}(\Theta, W)$ are unchanged.

With output and consumption unchanged, $I = Y - C - G$ and $X = \frac{I}{K}$ both fall, accompanied by q and $\dot{k}$.

Finally, with R unchanged and X lower, Equation (113) implies that $\dot{\lambda}$ decreases as long as $R - X > 0$. Again, since $R^* - X^* = \rho$, $R - X > 0$ in a substantial region around the steady-state.

What is happening is that holding marginal utility Θ constant zeroes out wealth effects and interest-rate effects on household behavior, leaving only the direct effect of government purchases on X because of economy-wide material balance, and the consequences of a lower investment rate for q and the rate of return.

6.3.2 Fixed-Θ Equilibrium and the Comparative Statics of the Saving Supply Curve

The behavior of the SS curve reflects the effects of Θ, K, Z, and G on X, which can be summarized as follows:

$$\underset{}{\mathcal{X}}\,(\underset{+}{\Theta}\,,\underset{+}{K}\,,\underset{+}{Z}\,,\underset{-}{G}) \tag{114}$$

Of course, the sign of the effect of θ is the same as sign of the effect of Θ, and the sign of the effect of k is the same as the sign of the effect of K.

Figure 19 shows the Saving Supply (SS) Curve. The positive effect of θ on X guarantees that the Saving Supply Curve slopes up. Going up along the SS curve, in addition to the increase in the three monotonically related variables X, q and $\dot{k}$ shown by the increase in X on the graph, it is useful to think of N, R, Y and I going up as one goes up the curve, while W, Γ and U are lower at higher points on the SS curve.

An increase in the (log) capital stock k shifts the SS curve to the right. At the point on the new SS curve directly to the right of a point on the old curve, not only X, q and $\dot{k}$, but also N, W, Γ, Y, I and $\dot{\lambda}$ (and probably C) are higher, while R and U are lower.

An improvement in technology Z also shifts the SS curve to the right. At the point on the new SS curve directly to the right of a point on the old curve, not only X, q and $\dot{k}$, but also N, W, R, Y, I and $\dot{\lambda}$ (and probably C) are higher, while Γ and U are lower.

An increase in government purchases G shifts the SS curve to the left. At the point on the new SS curve directly to the right of the point on the old curve, $\dot{\lambda}$ is lower in addition to X, q and $\dot{k}$ being lower.

6.3.3 Fixed-Investment-Rate Equilibrium

Where Fixed-Θ equilibrium picks out the point on the new SS curve directly to the right of the relevant point on the old SS curve when K, Z or G change, Fixed-X equilibrium picks out the point on the new SS curve directly below or above the relevant point on the old SS curve.

Table 3 details the comparative statics for Fixed-X equilibrium. The row for q and $\dot{k}$ is easy since both are monotonically increasing functions of X by itself. Also, Equation (112), $\lambda = \theta + \hat{q}(X)$, implies that holding X constant, λ must move in the same direction as θ. Since an increase in X holding K, Z and G constant turns out to increase θ as well, such an increase in X raises the required value of λ.

I will discuss the rest of the comparative statics results in Table 3 column by column.

Table 3: Fixed-Investment-Rate Comparative Statics

	X	K	Z	G
λ	$+$	$-$	$-$	$+$
N	$+$	?	?	$+$
W	$-$	$+$	$+$	$-$
Γ	$-$	$+$	$-$	$-$
R	$+$	$-$	$+$	$+$
Y	$+$	$+$	$+$	$+$
C	$-$	$+$	$+$	$-$
U	$-$	$+$	$+$	$-$
I	$+$	$+$	0	0
θ	$+$	$-$	$-$	$+$
q, $\dot{k}$	$+$	0	0	0
$\dot{\lambda}$	?	$+$	$-$	$-$

The Contemporaneous Preferences and Technology Diagram is the key to analyzing Fixed-X equilibrium.

X ↑: The effects of increasing X in Fixed-X equilibrium must be qualitatively the same as the effects of increasing Θ in Fixed-Θ equilibrium, since both cases correspond to movements up along the SS curve, holding K, Z and G fixed. Still, it is useful to see how these results show up in the Contemporaneous Preferences and Technology Diagram as a way of building more intuition for the use of the diagram.

As shown in Figure 20 an increase in X causes a parallel downward shift in the Contemporaneous Material Balance curve. Because the indifference curve slope $\hat{W}(C,N)$ is increasing in C, at the point directly below the original tangency, the Contemporaneous Material Balance curve slope $W^d(K,N,Z)$ is greater than the slope $\hat{W}(C,N)$ of the indifference curve through that point (not shown), implying that the new tangency is further up the new Contemporaneous Material Balance curve, at a higher level of N. The slope W at the new tangency will be lower than at the original tangency. With K and Z unchanged, $\Gamma = \frac{K}{ZN}$ will be lower and R higher than before. The increase in N also guarantees that Y increases. The fall in $\frac{K}{ZN}$ raises the rental rate R, in accordance with the factor price possibility frontier, and with X fixed, Equation (97) implies that $\dot{\lambda}$ moves opposite to the rental rate R.

$I = XK$ rises because K is unchanged. Felicity U at the new tangency falls because the opportunity set is worse. Since both U and W are lower at the new tangency, $\Theta = \hat{\Theta}(U, W)$ will be higher.

Finally, to see what happens to C in going to the new tangency, consider the point on the new Contemporaneous Material Balance curve directly to the right of the original tangency, where C is the same, but N is higher. If W_0 is the slope at the original tangency, $\hat{W}(C, N) > W_0 > W^d(K, N, Z)$ at this point, since the indifference curve slop is increasing in N, while the slope of the new Contemporaneous Material Balance curve at a higher value of N is lower than the slope of the original Contemporaneous Material Balance curve at the original value of N. Therefore, the new tangency is lower down on the new Contemporaneous Material Balance curve than the point that has the original level of consumption C.

$G \uparrow$: Since in increase in G also causes a parallel downward shift in the Contemporaneous Material Balance curve, the comparative statics effects of G conditional on X are identical to the effects of an increase in X, except for the obvious lack of effect of G on q, $\dot{k}$ and I, given X. Note that the confrontation of higher government purchases with a Contemporaneous Material Balance curve that does not allow G to simply crowd out investment as in the Fixed-Θ comparative statics yields more intuitive effects of government purchases.

$K \uparrow$: As shown in Figure 21, an increase in K shifts the Contemporaneous Material Balance curve up (as long as $R - X > 0$) and increases its slope at a given level of labor N because $F_{KN} > 0$. Also, the concavity of the Contemporaneous Material Balance curve guarantees that it has a higher slope at the point on the new curve with the original level of consumption C.

Because the slope of indifference curves is increasing in both N and C, the point on the new Contemporaneous Material Balance curve directly to the left of the old tangency point has a flatter indifference curve and a steeper Contemporaneous Material Balance curve than the original tangency. Moving up along the Contemporaneous Material Balance curve unambiguously steepens the slope of the relevant indifference curve at each point, while flattening the slope of the Contemporaneous Material Balance curve. Therefore, the new tangency will be at a point with higher consumption C than the old tangency. Equation (92) then implies that Y must be higher at the new tangency, since $Y = C + I + G$ and both C and $I = XK$ are higher.

Because the new Contemporaneous Material Balance curve is concave and is steeper than the original Contemporaneous Material Balance curve for given N, the point on the new Contemporaneous Material Balance curve on the same Engel curve as the original tangency must have $W^d(K, N, Z) > \hat{W}(C, N) = W_0$. Therefore, the new tangency must have a higher real wage W than the original tangency. According to the Factor-Price Possibility Frontier, the higher value of W must correspond to a higher $\Gamma = \frac{K}{ZN}$ and a lower R. Conditional on X, the lower value of R must correspond to a higher value of $\dot{\lambda}$. Since U and W are both higher, $\Theta = \hat{\Theta}(U, W)$ must be lower at the new tangency. Felicity U is clearly higher at the new tangency, since the contemporaneous opportunity set has improved.

The change in labor N in moving to the new tangency is ambiguous, since at the point directly above the old tangency, both the Contemporaneous Material Balance curve and the relevant indifference curve are steeper than before. If the Contemporaneous Material Balance curve is tilted up more at that point, then the new tangency will be at a higher N. If the relevant indifference curve is tilted up more at that point, then the new tangency will be at a lower N. A quasilinear utility function of the form $U(C, N) = C - v(N)$ will have indifference curves directly above with exactly the same slope; so it is clear that for some utility functions, the new tangency can be at a higher N. Similarly, a quasilinear utility function of the form $u(C) - N$, with a strongly concave $u(C)$ will make the indifference curves directly above have a much higher slope; so it is clear that for some utility functions, the new tangency can be at a lower N.

$Z \uparrow$: On the Contemporaneous Preferences and Technology Diagram an increase in Z looks quite similar to an increase in K. The Contemporaneous Material Balance curve shifts up and becomes steeper at a given N, but without needing to assume that $R - X > 0$ for the upward shift. Unlike when K increases, I does not increase, but C increases since the point to the left original tangency has a demand wage greater than the supply wage. $Y = C + I + G$ increases because C increases. W increases because the demand wage is greater than the supply wage at the intersection of the original Engel curve and the new Contemporaneous Material Balance curve. U increases since the opportunity set is greater. Θ is lower because both U and W are higher. The movement in labor is ambiguous, as it was for an increase in K.

Although the change in N is ambiguous, since output is an increasing function of K and ZN, and output Y has increased, ZN must be higher. With K unchanged, the higher value of ZN pushes the *effective* capital/labor ratio $\Gamma = \frac{K}{ZN}$ down and the rental rate R up. Given X, the increase in R pushes $\dot{\lambda}$ down. These three effects are opposite to the effects of an increase in K.

6.4 II-SS and Contemporaneous General Equilibrium

Equilibrium between investment demand and saving supply is the fulcrum of Contemporaneous General Equilibrium. The major new complication in Contemporaneous General Equilibrium is that the investment rate X and the marginal utility of consumption Θ are endogenous.[15]

Table 4 gives the signs of the effects of λ, k, Z and G on the key variables. The complications due to the endogeneity of (log) marginal utility θ and the investment rate X in Contemporaneous General Equilibrium are detailed below. As before, some of the signs noted depend on $R - X > 0$, which will hold in a substantial region around the steady state.

[15] While the capital stock k never jumps and the marginal value of capital λ can jump only when new information arrives, even a fully foreseen change in Z or G can cause θ to jump by shifting the SS curve. Since the real interest rate is

$$\Re = \rho - \dot{\theta},$$

the (*ex ante*) real interest rate would be infinite at that point (that is, there would be conversion ratio different from 1 for real funds before and after the point at which Z or G changed in a foreseen way). This kind of a spike in the real interest rate is a reflection of the fact that $\Re$ is not itself determined in Contemporaneous General Equilibrium, but has to do with the rate of change of Contemporaneous General Equilibrium variable θ.

Table 4: Contemporaneous General Equilibrium Comparative Statics

	λ	k	Z	G
N	+	?	?	+
W	−	+	+	−
Γ	−	+	−	−
R	+	−	+	+
Y	+	+	+	+
C	−	?	?	−
U	−	?	?	−
θ	+	−	−	+
I	+	+	+	−
x, q, $\dot{k}$	+	+	+	−
$\dot{\lambda}$	?	+	?	−

6.4.1 $\lambda \uparrow$: Shifting Out the II Curve

As shown in Figure 22, an increase in the marginal value of capital λ shifts the II curve out, moving the equilibrium to a higher point on the SS curve. Marginal utility θ, the investment rate X, Tobin's q and $\dot{k}$ are all higher. Since K, Z and G are being held fixed, in terms of Table 2, only the increase in Θ is operative. Therefore the signs in the column for the effects of λ in Table 4 are identical to those in the column for Θ in Table 2. However, an increase in λ causes a less-than-one-for-one increase in θ.

6.4.2 Shifts of the SS curve

Figure 23 shows an outward shift of the SS curve. This outward shift in the SS curve could come from an increase in K, and increase in Z, or a decrease in G. Depending on the slope of the II curve, the new intersection could be anywhere on the striped segment of the new SS curve. In other words, for all of the shifts of the SS curve—by K, Z, or G, Contemporaneous General Equilibrium effects will be between the effects when X is fixed and the effects when θ is fixed. This is obvious when linearizing or log- linearizing. Lemma 2 gives the same result without using linearization.

Lemma 2 *If a variable moves in an unambiguous direction when moving up along the SS curve, and if a shift in the SS curve by K, Z or G has a consistent effect on the variable when comparing between Fixed-Θ equilibrium and in Fixed-X equilibrium, then the effect of the shift in the SS curve on the variable in Contemporaneous General Equilibrium is the same as that consistent effect. If the effect in one Fixed equilibrium is zero, then the effect of the shift in the SS curve will be in the same direction as the effect in the other Fixed equilibrium.*

Proof: *The move to the new intersection of II and SS can be decomposed into either the move to the new Fixed-Θ equilibrium plus a movement along the SS curve, or decomposed into a move to the new Fixed-X equilibrium plus a movement along the SS curve in the opposite direction. If the variable moves the same direction in both Fixed Θ equilibrium and Fixed-X equilibrium, the effect of the move along the SS curve must be the same as initial shift to a new SS curve in one of these two cases. If the effect on the variable is zero in either Fixed-Θ or Fixed-X equilibrium, this yields an unambiguous prediction for the Contemporaneous General Equilibrium effect, which must be consistent with the direction for the other Fixed equilibrium, since the other Fixed equilibrium is further along the SS curve in the same direction beyond the new intersection of II and SS.*

The one Contemporaneous General Equilibrium variable Lemma 2 does not cover is $\dot{\lambda}$, where the effect of a movement along the SS curve is ambiguous.

The converse to Lemma 2—which I will state less formally—is also useful: unless the movement of a variable depends on the degree of investment adjustment costs even in the Fixed equilibria, if the direction a variable moves in Fixed- Θ equilibrium is opposite to the direction it moves in Fixed-X equilibrium, it must be genuinely ambiguous in Contemporaneous General Equilibrium. The reason is that by varying the degree of the investment adjustment cost, the slope of the II curve can be varied to make Contemporaneous General Equilibrium effects arbitrarily close to Fixed-Θ effects or Fixed-X effects. Also, for the same reason, unless the movement of a variable depends on the degree of investment adjustment costs even in the Fixed equilibria, the structure of Contemporaneous General Equilibrium cannot eliminate a genuine ambiguity present in one of the Fixed equilibria.

The only variable in which the degree of investment adjustment costs matters for the direction of an effect in Fixed-Θ or Fixed-X equilibrium is $\dot{\lambda}$. Thus, Lemma 2 and its converse provide the entries in Table 4 for every row but the row for $\dot{\lambda}$.

For $\dot{\lambda}$, in the case of an increase in K or G, we can use Equation (113) to say that when the investment rate X and the rental rate R move in opposite directions, $\dot{\lambda}$ must move in the same direction as X. The effect of Z on $\dot{\lambda}$ is genuinely ambiguous, since a very small investment adjustment cost will make $\dot{\lambda}$ go opposite to R and therefore fall when Z increases in Contemporaneous General Equilibrium. To show that $\dot{\lambda}$ can go up in Contemporaneous General Equilibrium with an increase in Z, fix a particular strength of the investment adjustment costs, thereby determining a given relationship between the movement of X and Θ when the shift in the SS curve moves the intersection of II and SS along the II curve. Then making the production function f more and more linear will make any movement in $R = f'\left(\frac{K}{ZN}\right)$ very small, so that the positive effect of X on $\dot{\lambda}$ will dominate.

Finally, note that the additional assumption $U_{CN} > 0$ could eliminate the ambiguity of the effect of k and Z on consumption in Contemporaneous General Equilibrium: both would then have a positive effect, since they would have a positive effect in both Fixed-Θ equilibrium and Fixed-X equilibrium. The effect of k and Z on consumption in Contemporaneous General Equilibrium is also unambiguously positive in the additively separable case $U_{CN} = 0$.

6.5 An Important Special Case: The Basic Real Business Cycle Model

Let me call the model of Prescott (1986), with a more general felicity function U and a more general production function f, the Basic Real Business Cycle Model. Then the Basic Real Business Cycle Model is the limiting special case of the QRBC Model when there is no investment adjustment costs: $J(X) = X - \delta$. Since with $J'(X) \equiv 1$,

$$\Theta = \Lambda J'(X) = \Lambda,$$

the II curve $\theta = \lambda + \ln J'(X)$ is flat, and Contemporaneous General Equilibrium for the Basic Real Business Cycle Model is identical to Fixed-Θ equilibrium. Moreover, the absence of investment adjustment costs simplifies the determination of $\dot{\lambda}$, since with $J'(X) \equiv 1$, since Equation (97)reduces to

$$\dot{\lambda} = \rho + \delta - R.$$

Since $\Theta = \Lambda$,

$$\rho - \Re = \dot{\theta} = \dot{\lambda} = \rho + \delta - R,$$

implying

$$r = R - \delta$$

in the absence of investment adjustment costs.

The additional structure on $\dot{\lambda}$ insures that $\dot{\lambda}$ always moves opposite to the rental rate in the Basic Real Business Cycle Model. Thus, the one change to Table 2 as a description of the Contemporaneous General Equilibrium comparative statics for the Basic Real Business Cycle model is that in the absence of investment adjustment costs, an increase in $\Theta = \Lambda$ unambiguously lowers $\dot{\lambda}$.

6.6 Immediate Applications of Contemporaneous General Equilibrium

In addition to its role as a building block for determining dynamic general equilibrium, Contemporaneous General Equilibrium also serves as a useful approximation for the effects of shocks to Z or G that are so short-lived that they have relatively little effect on either k or λ. In other words, the last two columns of Table 4 show the effects of very short-lived shocks to Z and G.

Another immediate application of contemporaneous general equilibrium is that it shows the effects of anticipated, but sudden movements in Z or G. The log marginal value of capital λ can only jump when there is new information. Thus, when sudden movements in Z or G are foreseen, λ cannot jump at that moment. The capital stock K never jumps. As a result, the changes at that moment of sudden, anticipated change are driven by the changes in Z or G holding λ and K fixed.

Thus, treating λ as if it were exogenous is very useful in cases where λ will not move very much. It is also useful in situations where the direction

λ jumps or evolves is known. What about cases such as a permanent improvement in technology, in which λ can jump either up or down on impact, depending on parameters? Then Contemporaneous General Equilibrium is not as immediately helpful. For this particular case, knowledge about the behavior X on impact, together with the structure of Fixed-X equilibrium, can help in the analysis. More generally, any time one has knowledge of the behavior of a variable, structural analysis treating that variable as if it were exogenous can sometimes be useful.

7 Dynamic General Equilibrium

My ultimate objective is to characterize the impulse responses to technology and government purchase shocks. These impulse responses come out of *Dynamic General Equilibrium.*

7.1 The Phase Diagram

The Contemporaneous General Equilibrium comparative statics for $\dot{k}$ and $\dot{\lambda}$ indicate the dynamics that show up directly on the phase diagram. The dynamics of k and λ are central to dynamic general equilibrium because k can never jump, while λ can jump only when new information arrives. In the analysis of the impulse response to a single bundle of new information arriving at "time zero" this means that the impulse response for λ can only jump at time zero, at the moment when the new information arrives. Beyond that moment, both intertemporal variables k and λ act as slowly moving anchors to the behavior of the model, helping to determine the rest of the key variables through the machinery of Contemporaneous General Equilibrium.

The positive effects of both k and λ on $\dot{k}$ in Table 4 indicate that the $\dot{k} = 0$ locus is downward sloping, with $\dot{k} > 0$ above the locus and $\dot{k} < 0$ below.

The positive effect of k on $\dot{\lambda}$ indicates that $\dot{\lambda} > 0$ to the right of the $\dot{\lambda} = 0$ locus, while $\dot{\lambda} < 0$ to the left. Because of the ambiguous effect of λ on $\dot{\lambda}$, whether the $\dot{\lambda} = 0$ locus is upward-sloping, downward-sloping, or vertical depends on parameters.[16] With no investment adjustment costs, the Basic

[16] Totally inelastic labor supply and no investment adjustment costs yields the Ramsey-Cass-Koopmans Model which has a vertical $\dot{\lambda} = 0$ locus.

Real Business Cycle Model has a negative effect of λ on $\dot{\lambda}$ and therefore an upward-sloping $\dot{\lambda} = 0$ locus.

If the $\dot{\lambda} = 0$ locus were not only downward-sloping, but tilted back so much that it was flatter than the $\dot{k} = 0$ locus, it would make the dynamics totally unstable. Fortunately, the positive effect of K on $\dot{\lambda}$ in the Fixed-X equilibrium guarantees that the $\dot{\lambda} = 0$ locus is downward sloping, it must be more steeply downward sloping than the $\dot{k} = 0$ locus where the two loci intersect. The key to seeing this is that $\dot{k} = 0$ implies $X = \delta$, so X *is* fixed as one moves along the $\dot{k} = 0$ locus. Increasing k while letting λ fall to stay on the $\dot{k} = 0$ equilibrium makes $\dot{\lambda}$ rise in accordance with Fixed-X equilibrium. Therefore the points on the $\dot{k} = 0$ locus to the right of the intersection must be in the *region* to the right of the $\dot{\lambda} = 0$ locus, which can only happen if the $\dot{\lambda} = 0$ locus is more steeply downward-sloping, vertical or upward-sloping.

Figure 24 shows the three possible cases for the phase diagram. In each case, there is a downward-sloping saddle path that shows the dynamics if Z and G are constant. Since Λ is the marginal value of capital, $\Lambda = V_K(K)$, and the saddle-path gives the relevant value of Λ for the optimal dynamic program, the downward-sloping saddle path is a reflection of the concavity of the value function, $V_{KK}(K) < 0$. The concavity of the value function in turn arises from the fact that the social planner's problem is a concave problem in all of its components.

7.2 Behavior Along the Saddle Path

Table 5 indicates how key variables evolve as the economy moves along the saddle path. Since λ is fully endogenous in Dynamic General Equilibrium, it makes sense to think of the behavior along the saddle path as governed by the value of the (log) capital stock k inherited from the past at any given moment. An increase in k corresponds to moving down along the saddle path to a higher capital stock, as the economy will do if the initial capital stock is below its steady-state value. If instead the initial capital stock is higher than the steady-state value, the economy will move up the saddle path on the other side of the steady state, and all the signs in Table 5 get multiplied by the negative movement of the capital stock, as the capital stock gradually falls back to its steady-state value.

Table 5: Moving Along the Saddle Path

	k
λ	$-$
X, q, $\dot{k}$	$-$
$\dot{\lambda}$	$+$
N	?
W	$+$
Γ	$+$
R	$-$
Y	?
C	$+$
U	$+$
θ	$-$
I	?

Besides the marginal value of capital λ falling with k, the structure of convergence to the steady-state along the saddle path guarantees that if the capital stock is below the steady-state value so that $\dot{k} > 0$, the growth rate of the capital stock must slow down toward to end so that the economy can converge toward $\dot{k} = 0$ in the Steady State. It may be possible for $\dot{k}$ to be nonmonotonic at an earlier stage, but close enough to the steady state, $\dot{k}$ must monotonically fall toward zero as k increases. The entry for $\dot{k}$ in Table 5 reflects this monotonicity that necessarily holds in the last stage of convergence and would appear in any linearization of the model around the steady state. By a similar line of reasoning, with $\dot{\lambda} < 0$ as k increases, convergence to $\dot{\lambda} = 0$ at the steady state requires the *growth rate* of the marginal value of capital $\dot{\lambda}$ to increase from a negative value towards zero in the final stage of convergence to the steady state from a capital stock below the steady-state value.

The direction of movement of the real wage W, the effective capital/labor ratio, Γ, the rental rate R and the marginal utility of consumption θ are clear from the Contemporaneous General Equilibrium Comparative Statics since the increase in k and the fall in λ both push each of these variables the same direction. The fall in θ can be seen in Figure 25. The increase in k shifts out the SS curve, while the fall in λ shifts the II curve down, leading to an unambiguous fall in θ. The fall in θ reduces labor supply, while the increase in k raises labor demand, guaranteeing an increase in the real wage W, accompanied by an increase in the effective capital/labor ratio Γ and a decrease in the rental rate R in accordance with the Factor Price Possibility Frontier.

The signs for consumption C and felicity U are guaranteed by the structure of Fixed-X equilibrium, in view of the gradual decline in the investment rate X in the last stage of convergence as the economy nears a higher steady-state level of capital. Intuitively, as capital accumulates and the effort toward accumulating yet more capital abates, the economy has more resources for current felicity. The increase in real wage as the economy goes down the saddle path then guarantees that the improvement in felicity will come more from increased consumption and less from increased leisure than it would along an Engel curve.

What about the real interest rate $\Re$? Unlike the other variables, the real interest is not determined by k, λ, Z and G at a given moment in Contemporaneous General Equilibrium. But in Dynamic General Equilibrium, the behavior of $\Re$ can be seen from

$$\Re = \rho - \dot{\theta}.$$

Since θ declines as the capital stock increases along the Saddle Path,

$$\Re > \rho$$

along the left-hand arm of the Saddle Path where $\dot{k} > 0$ while $\Re < \rho$ on the right-hand arm of the Saddle Path where $\dot{k} < 0$. In either case, the real interest $\Re$ must eventually go back towards a value of ρ at the Steady State.

7.3 The Dynamic General Equilibrium Response to a Permanent Increase in Government Purchases

There are two ways to analyze how the two isoclines $\dot{k} = 0$ and $\dot{\lambda} = 0$ shift when there is a permanent increase in additively separable government purchases, financed by an increase in lump-sum taxes. One way is to look at the effects of G in Contemporaneous General Equilibrium of on $\dot{k}$ and $\dot{\lambda}$. The negative effect G on $\dot{k}$ in Contemporaneous General Equilibrium means it would require an increase in either k or λ to cancel out the effect of the increase in G on $\dot{k}$. Therefore, an increase in G shifts the $\dot{k} = 0$ locus outward and upward. The negative effect of G on $\dot{\lambda}$ in Contemporaneous General Equilibrium means that, holding λ fixed, it would require an increase in k to cancel out the effect of the increase in G on $\dot{\lambda}$. Therefore the $\dot{\lambda} = 0$ locus shifts to the right. The $\dot{\lambda} = 0$ locus shifts right with an increase in G regardless of the slope of the $\dot{\lambda} = 0$ locus, since the variations in the slope of the $\dot{\lambda} = 0$ locus come from the ambiguity in the effect of λ on $\dot{\lambda}$, which I sidestepped by considering how to get back to $\dot{\lambda} = 0$ while holding λ fixed. The other way to analyze how the two isoclines shift is to realize that the intersection of the $\dot{k} = 0$ locus and the $\dot{\lambda} = 0$ locus must shift in accordance with the steady-state effects on k and λ indicated by Table 1: the intersection must be at a higher level of both k and λ. Figure 26 shows the two main cases for the dynamic path on the phase diagram. The case of a vertical $\dot{\lambda} = 0$ locus would have an appearance between the appearance of these two cases. In both cases, the (log) marginal value of capital λ must jump up on impact and then fall gradually as the economy moves down the saddle path to a new, higher level of capital. The reason the slope of the $\dot{\lambda} = 0$ locus makes so little difference to the qualitative picture is: (1) the direction in which the new steady state shifts is totally unaffected by adjustment costs and (2) the saddle path is always downward sloping.

What about the other key variables? The QRBC model has only one endogenous state variable, k. Therefore the response to immediate permanent shocks is composed of an impact effect, a saddle path effect as the capital stock adjusts to the new value, and a steady-state effect that is the sum of the impact effect and the saddle-path effect. Table 6 shows for key variables the impact effect, saddle path effect, and steady-state effect of an immediate, permanent increase in additively separable government purchases, financed by an increase in lump-sum taxes. Given the increase in the steady-state capital stock generated by a permanent increase in G, the saddle-path effects for each variable are those given in Table 5. The steady-state effects for are the same as those given in Table 1.

Table 6: Dynamic General Equilibrium Effects of
An Immediate, Permanent Increase in Government Purchases

	Impact	Saddle-Path	Steady-State
k	0	+	+
λ	+	−	+
X, q, $\dot{k}$	+	−	0
$\dot{\lambda}$	−	+	0
N	+	?	+
W	−	+	0
Γ	−	+	0
R	+	−	0
Y	+	?	+
C	−	+	−
U	−	+	−
θ	+	−	+
I	+	?	+
$\Re$	+	−	0

The impact effects for N, W, Γ, R, Y, C, U and θ can be signed by the fact that the jump up in λ pushes these variables in the same direction as the Contemporaneous General Equilibrium effect of the increase in G. Intuitively, the key is the increase in the marginal utility of consumption θ that can be seen in the II-SS diagram as investment demand increases with the jump up in λ, while saving supply is reduced by the increase in G. This increase in θ then combines with the zero effect in Fixed-Θ equilibrium of G on N, W, Γ, R, Y, C and U.

The positive impact effect of the increase in G on $\dot{k}$, and therefore on X, q and $I = XK$, stems from the necessity apparent in the phase diagram for the capital stock to grow to reach its higher steady-state level. The negative impact effect on $\dot{\lambda}$ is also apparent from the phase diagram. $\dot{\lambda} = 0$ in the initial steady state, than after λ jumps up, $\dot{\lambda} < 0$ until the new steady-state is reached, although $\dot{\lambda}$ is less and less negative as time goes on.

Finally, as discussed above, because $\dot{\theta} < 0$ along the saddle path, as indicated by Figure 25 and by the negative sign in the saddle-path column for θ, the real interest rate $\Re$ must jump up from its initial steady-state rate of ρ to something higher, then decline along the saddle path as the economy approaches the new Steady State and return to a value of ρ at the new Steady State.

7.4 The Dynamic General Equilibrium Response to a Permanent Improvement in Technology

An increase in Z raises $\dot{k}$ in Contemporaneous General Equilibrium. This increase can be cancelled out by an reduction in either k or λ. Therefore, an increase in Z causes the $\dot{k} = 0$ locus to shift down. The ambiguous effect of Z on $\dot{\lambda}$ in Fixed-X equilibrium indicates that the $\dot{\lambda} = 0$ locus can shift either right or left. In accordance with Table 1, the new steady state must be at a higher capital stock k and a lower marginal value of capital λ. As shown in Figure 27, the key ambiguity is that the marginal value of capital can jump either up or down on impact, depending on the slope of the saddle path in comparison with the vector direction of the shift from the old Steady State to the New Steady State. (As in Figure 26, the qualitative possibilities are not much altered when the $\dot{\lambda} = 0$ locus if vertical or downward sloping, since the shift of the new steady state is unaffected by adjustment costs.) This ambiguity in which way λ jumps makes it impossible to sign the impact effects of an immediate, permanent increase in Z simply by adding or subtracting columns of Table 4.

Fortunately, the structure of Dynamic General Equilibrium guarantees that since the new steady-state value of capital is higher, $\dot{k} > 0$ and $\dot{\lambda} < 0$ along the saddle path, which in turn implies that the investment rate $\dot{k}$, q and X must jump up on impact.[17] Since both Z and X jump up on impact, while G does nothing and K cannot jump, the structure of Fixed-X equilibrium guarantees that R, Y and I increase on impact, while $\Gamma = \frac{K}{ZN}$ jumps down. The impact effects on λ, N, W, C U and θ are ambiguous. Although the initial jump in θ on impact is ambiguous, by the logic of Figure 25, it must be that $\dot{\theta} < 0$ thereafter as the economy moves down the new saddle path to the new Steady State at a higher level of capital. Since $\Re = \rho - \dot{\theta}$, this implies that $\Re > \rho$ along the saddle path. Therefore, $\Re$ must jump up on impact from its original value of ρ.

By particular numerical examples of log-linearized models, I have verified

[17] If the shock to Z hits an economy that begins out of steady-state, this statement relies on linearization.

that the impact effect on λ, N, W, C, U and θ are all genuinely ambiguous, but the details I leave to a future companion paper.[18][19]

labor N goes up or down as the economy goes down the saddle path as capital's share α is higher or lower than the elasticity of intertemporal substitution s. As for output Y, in the limiting case where $U(C, N) = \ln(C) - N$, output will go up or down with evolution down the saddle path depending on whether capital's share α is greater or less than 1/3.

[18] In brief, suppose one focuses on the Basic Real Business Cycle model, which is the limit of the QRBC model as the investment adjustment cost becomes very small (so that $\theta = \lambda$), with King-Plosser-Rebelo preferences of the form

$$U(C, N) = \frac{C^{1-s^{-1}}}{1 - s^{-1}} \phi(N)$$

and impatience ρ very close to zero (for convenience, one can look at the limiting equations as $\rho \to 0$). Then and an elasticity of intertemporal substitution s significantly greater than one, coupled with a low elasticity of substitution between capital and labor (low enough that it barely obeys the restriction $\sigma > \alpha$ that is equivalent to $F_{NZ} > 0$), makes the impact effects on λ, N and θ positive while the impact effects on W, C and U are negative. On the other hand, if the production function is Cobb-Douglas, with capital's share equal to 1/3, while the elasticity of intertemporal substitution s is less than 1/3, the impact effects on λ, N and θ are negative, while the impact effect on W, C and U are positive. The value of the labor supply elasticity determined by the shape of $\phi(N)$ is not crucial for any of these statements, though there are some restrictions on $\phi(N)$ needed to guarantee concavity of $U(C, N)$ and the normality of consumption and leisure.

[19] I have also verified by numerical examples closely related to those of the previous footnote that the directions of evolution of N, Y and I along the saddle path are ambiguous. With $\rho \approx 0$ in the Basic RBC model with King-Plosser-Rebelo preferences and a Cobb-Douglas production function that has capital's share $\alpha = 1/3$, moving down the saddle path will cause labor, output and investment to go down for a high enough elasticity of intertemporal substitution s (and in the case of output, a high enough labor supply elasticity); while a low enough value of s will ensure that N, Y and I go up as the economy moves down the saddle path.

Table 7: Dynamic General Equilibrium Effects of An Immediate, Permanent Improvement in Technology

	Impact	Saddle-Path	Steady-State
k	0	+	+
λ	?	−	−
X, q, $\dot{k}$	+	−	0
$\dot{\lambda}$	−	+	0
N	?	?	?
W	?	+	0
Γ	−	+	0
R	+	−	0
Y	+	?	+
C	?	+	+
U	?	+	+
θ	?	−	−
I	+	?	+
$\Re$	+	−	0

8 The Implications of Empirical Evidence on the Short-Run Effects of Technology Shocks in the Light of Theoretical Results for the QRBC Model

Basu, Fernald and Kimball (2003) estimate the empirical response of many of the key variables here to a technology shock. As mentioned above, the technology shocks they estimate look permanent and are not Granger caused in any obvious way by other variables. This tends to confirm that modeling technology shocks as immediate, permanent changes in technology is reasonable. However, Basu, Fernald and Kimball (2003) find a significant decline in investment in the first year in response to an improvement in technology. The generality of the model above allows me to say that this cannot be generated by the QRBC model, regardless of parameter values, within the basic assumptions made in this paper. In the wake of Gali (1999), much of the attention garnered by the evidence for contractionary technological improvements has focused on the estimated negative response of employment to an improvement in technology found by both Gali (1999) and Basu, Fernald and Kimball (2003), by very different methods. But the decline in investment found by Basu, Fernald and Kimball (2003) is more decisive in rejecting a real business cycle model explanation of the effects of technology shocks than the decline in employment, since it a straightforward matter to find parameter values for which the QRBC model, or other closely related models imply a negative response of employment to a technological improvement. By contrast, it is quite difficult (though presumably not impossible, given enough cleverness) to generate a decline in investment in response to a technological improvement as the sensible response of a social planner who ultimately wants to get to a *higher* capital stock. A *decline in output* Y or the real interest rate would also be relatively decisive in rejecting a model like the QRBC model. Although these results are not statistically significant, Basu, Fernald and Kimball do find a decline in output in the first year of a technological improvement, and a long-lived decline in the real interest rate after a technological improvement begins.

What about the possibility that, despite point estimates to the contrary by Basu, Fernald and Kimball (2003), a technology shock involves a phase-in period in which true total-factor productivity will be even higher tomorrow than it is today as the new technology is more fully implemented? It is easy to see from the phase diagram that an anticipated future improvement in technology, which has an immediate effect only through the marginal value of capital λ can cause a contraction in employment, output and investment if λ jumps down enough in response to the news. The relevant case is associated with a wealth effect that is large in comparison to the interest rate effect, as can be seen most easily by looking again at Equation (29), which with $\theta = \ln(\Theta)$, is

$$\theta(t) = \theta(\infty) + \int_t^{\infty} [\Re(\tau) - \rho] d\tau.$$

If technology improves now and will improve more in the future, it is like a combination of an immediate permanent technology shock of the size of the immediate improvement in technology with an anticipated future technology shock that goes further.

In analyzing a phase-in period for technology as an explanation for a decline in investment caused by a technological improvement within the framework of the QRBC model, the robustness of Contemporaneous General Equilbrium structures to the dynamic path of variables is useful. The perspective of Fixed-X equilibrium as described in Table 3 indicates that if the investment rate falls on impact in response to a technological improvement, consumption and the real wage must go up. (Just subtract the X column from the Z column.) Here the robustness to the details about the future path of technology comes from the fact that, given the current values of K, Z and G, the current investment rate X embodies the key information about the future.

Alternatively, take the observation that employment falls in response to a technological improvement. In the QRBC model, labor demand depends only on the current values of K and Z. An increase in Z shifts labor demand out. Given slow movement in K so that we can neglect its effects, if N falls at the same time labor demand shifts out, the real wage W must increase in the QRBC model.

The increase in the real wage W coupled with a decrease in N then implies an increase in consumption. Along a given Engel curve, a decrease in N is associated with an increase in C. A higher real wage pushes the representative household above the original Engel curve, in the direction of higher C.

Thus, in the QRBC model, an increase in *current* Z associated with *either* a fall in the investment rate I/K *or* a fall in employment N, must be associated with an increase in consumption and the real wage, *regardless* of what technology is going to do beyond that moment in time. This result is based primarily on the assumptions that consumption and leisure are normal and that technology and labor are supermodular in the production function: $F_{NZ} > 0$.

Is the entire exercise here self-destructive? Is it an analysis of the QRBC model only to show that it cannot be true? My answer is no. Models build on one another. In particular, a Neoclassical sticky-price model needs a solid real business model as its backbone, describing the nature of the full employment equilibrium that the economy tends toward as prices adjust. Kimball (1995) implicitly uses a variant of the QRBC model in just this fashion.

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Figure 1: Normality of Consumption and Leisure

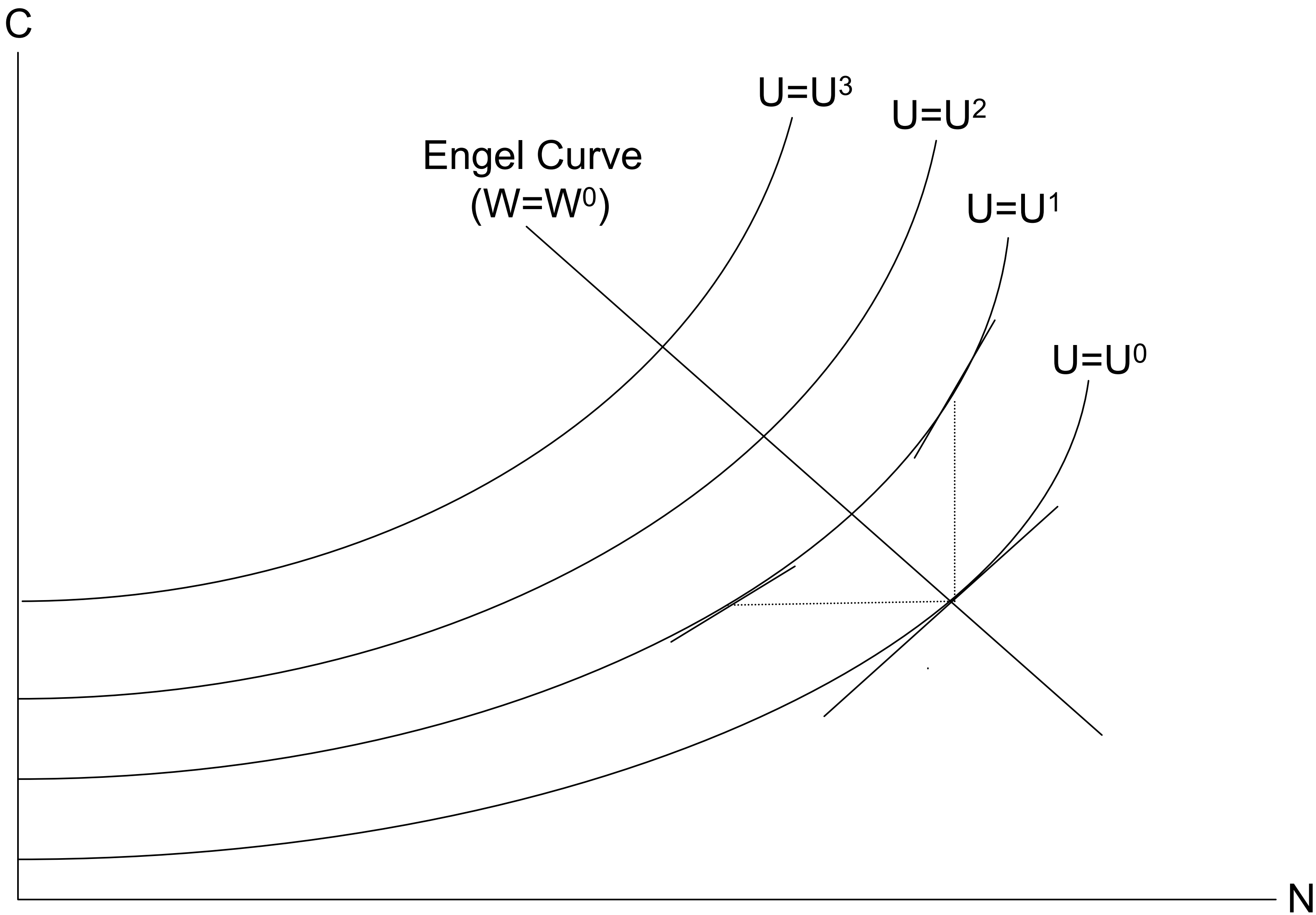

Figure 2: Long-Run Preferences and Technology Diagram (The Effect of an Increase in G)

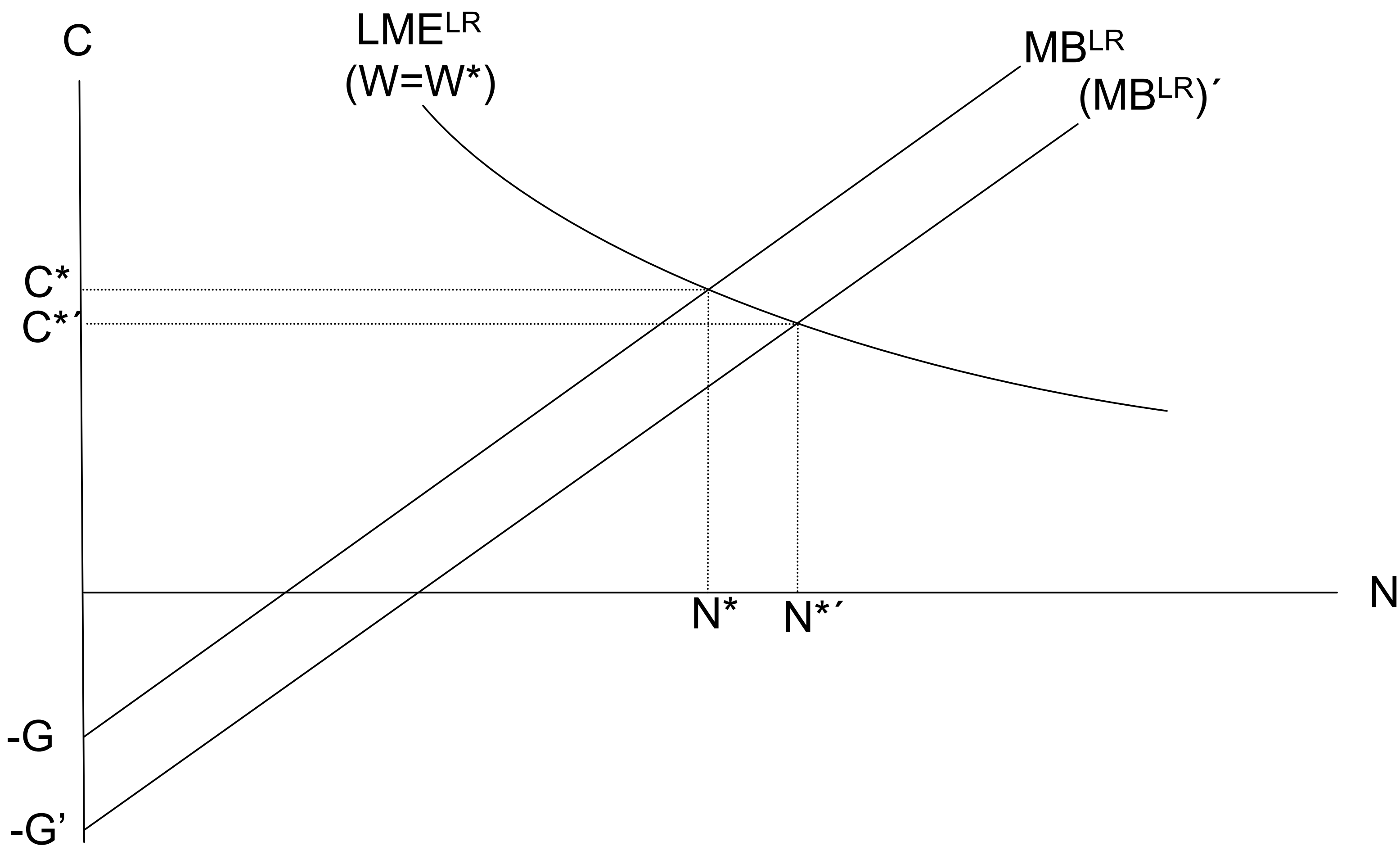

Figure 3: A Higher W* Raises LME^{LR} (the Engel Curve for W*)

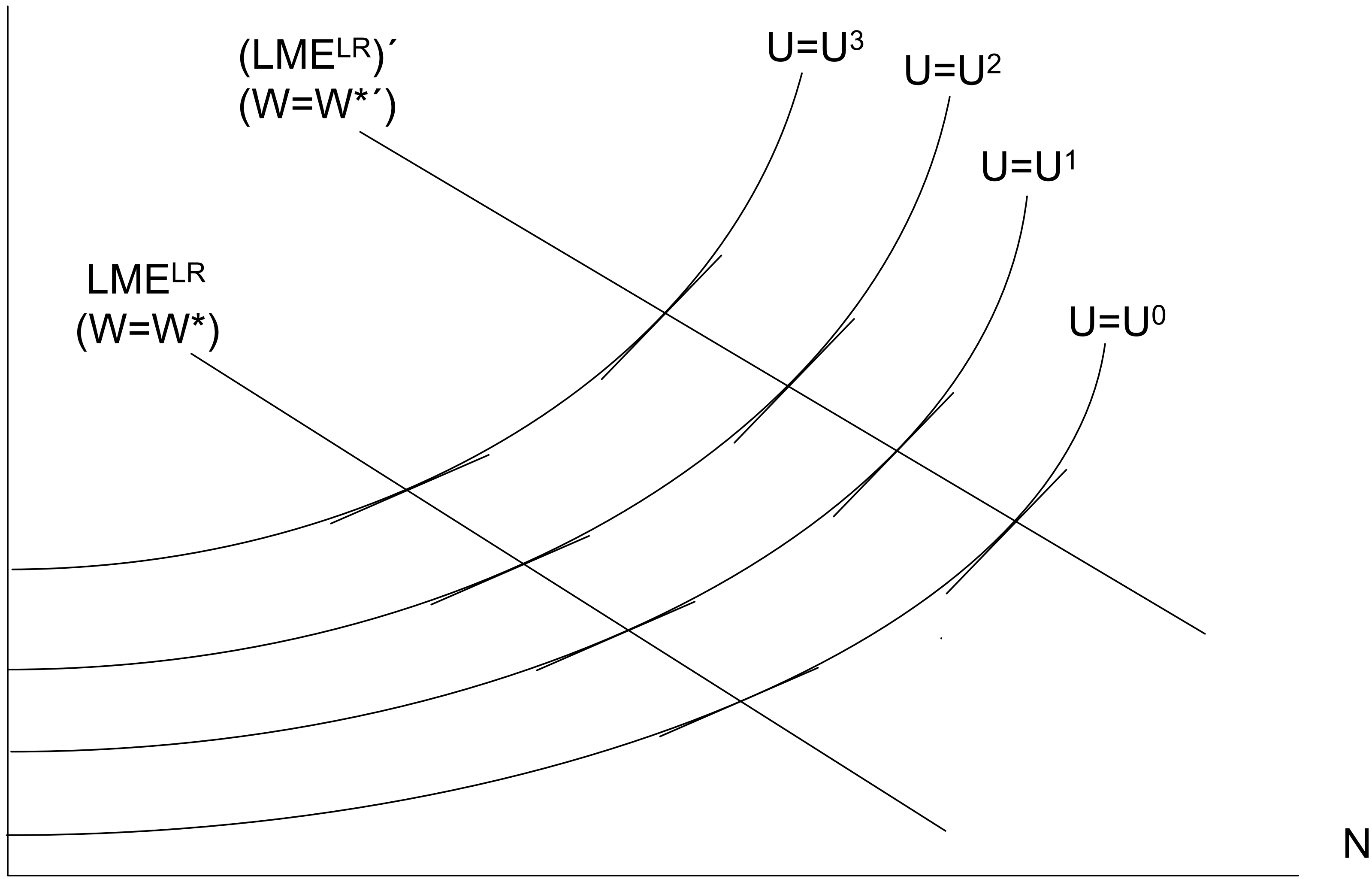

Figure 4: Effect of an Increase in Z On Long-Run Equilibrium

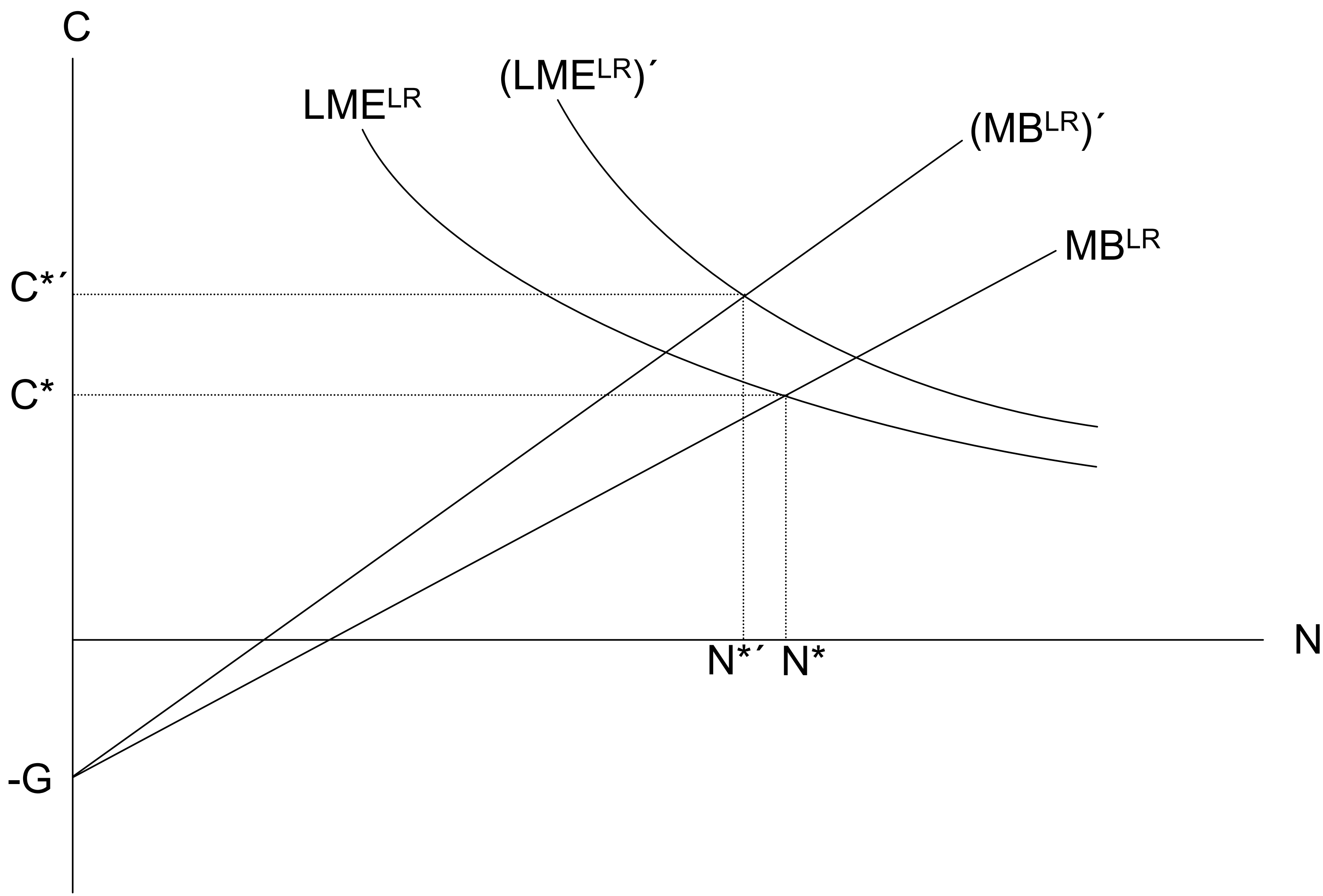

Figure 5: Illustration of Lemma 1

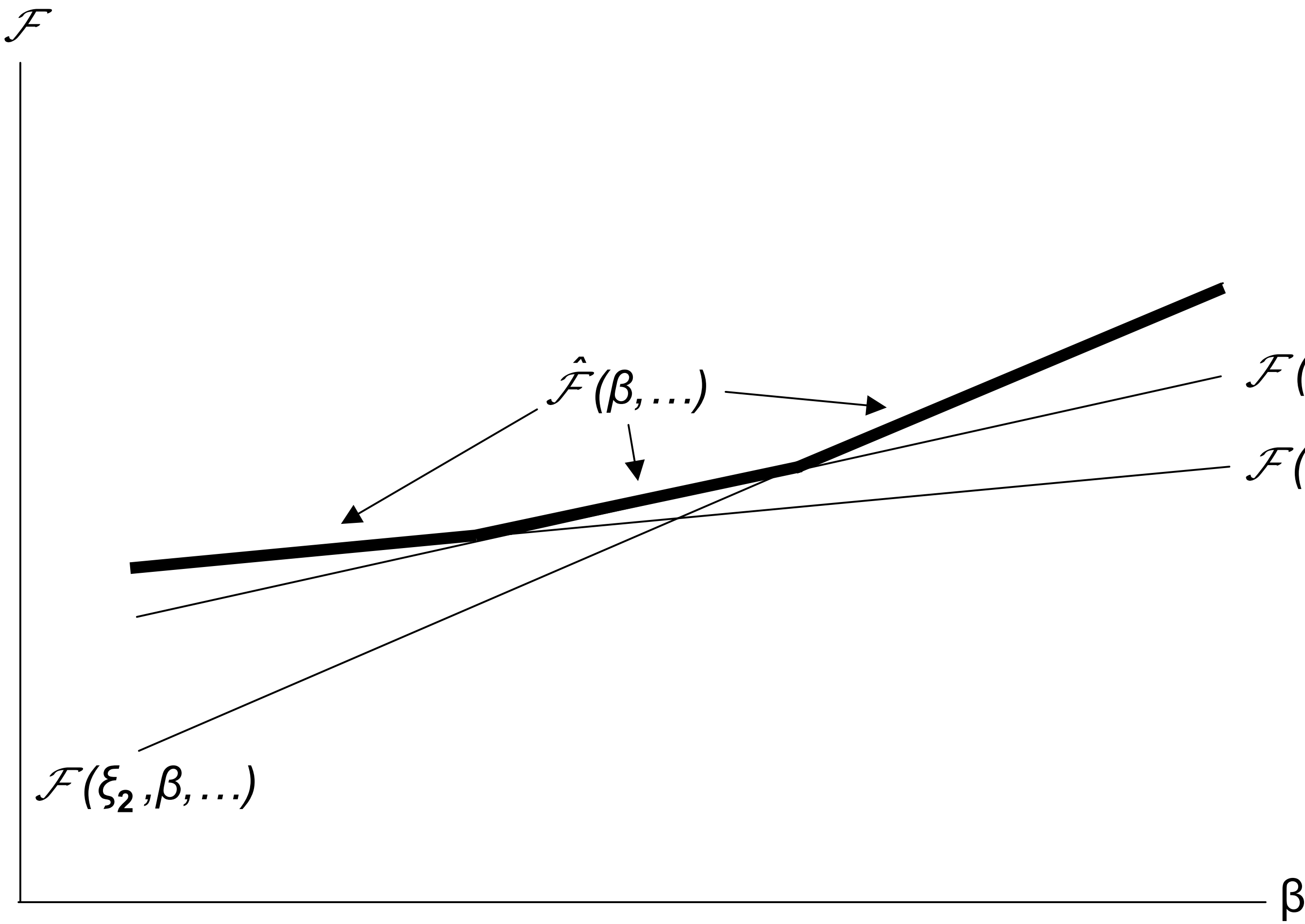

Figure 6:
Felicity-Saving Possibility Frontier

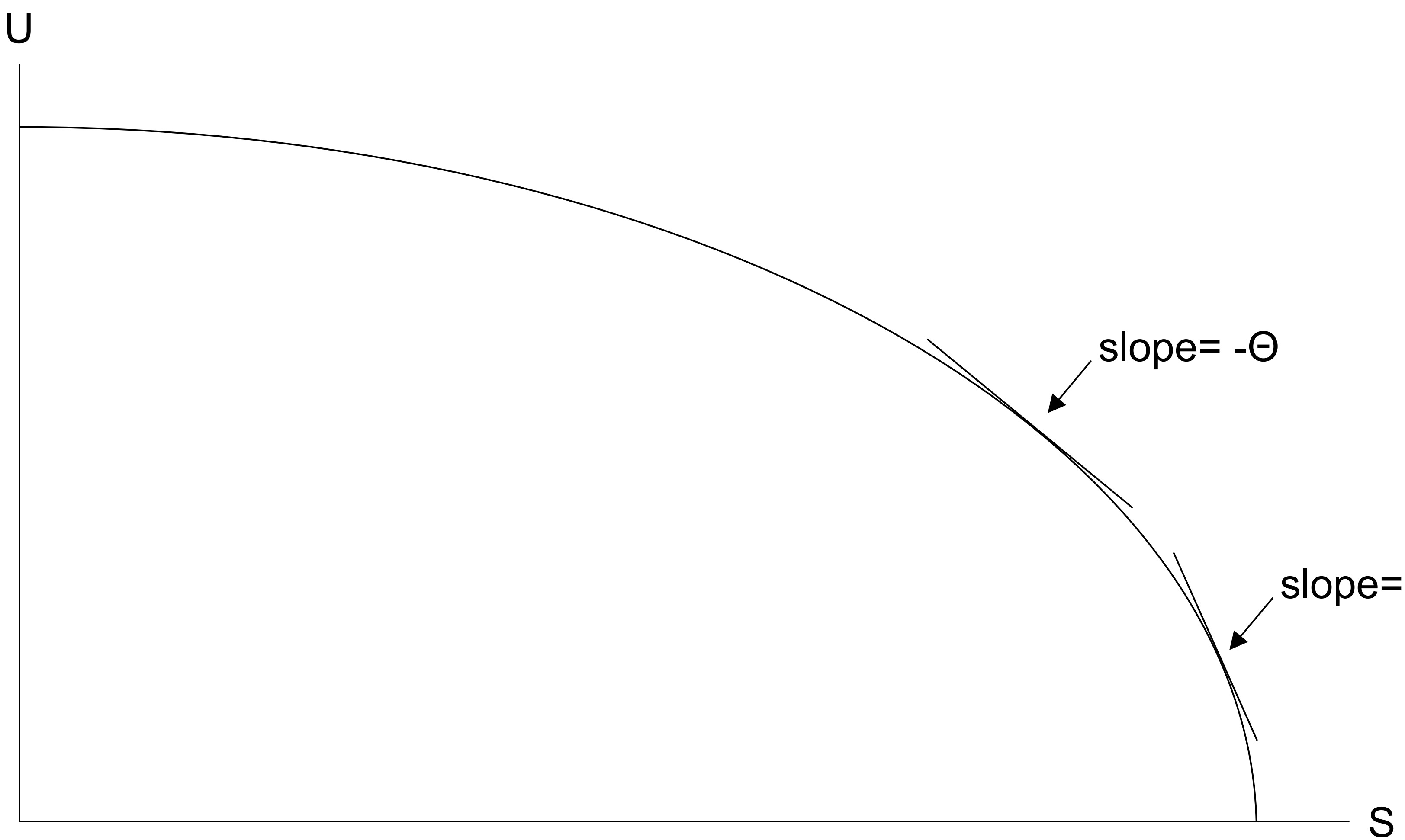

Figure 7: Money-Metric Net Felicity-Labor Possibility Frontier

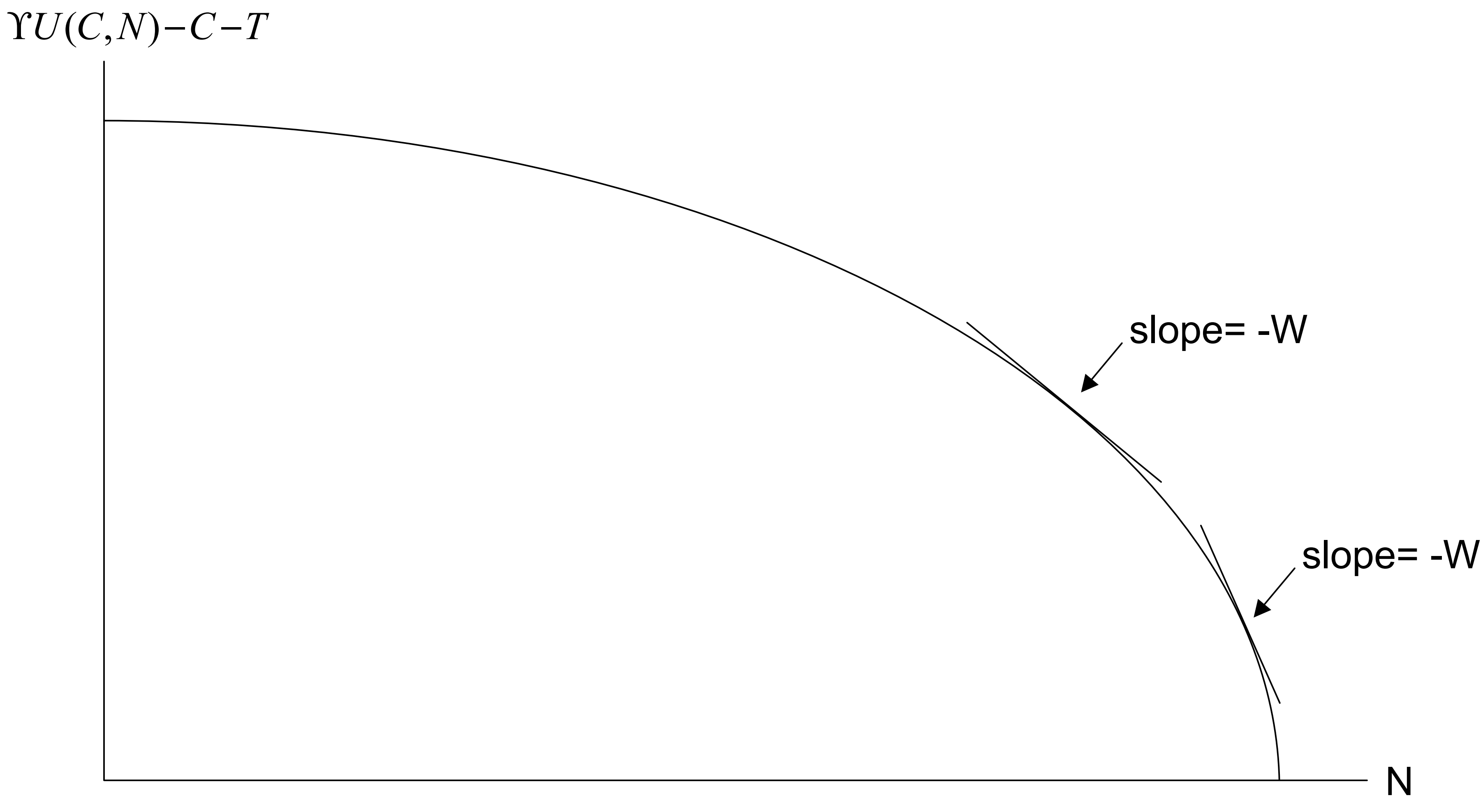

Figure 8: Effect of a Higher Wage on the U-S Possibility Frontier

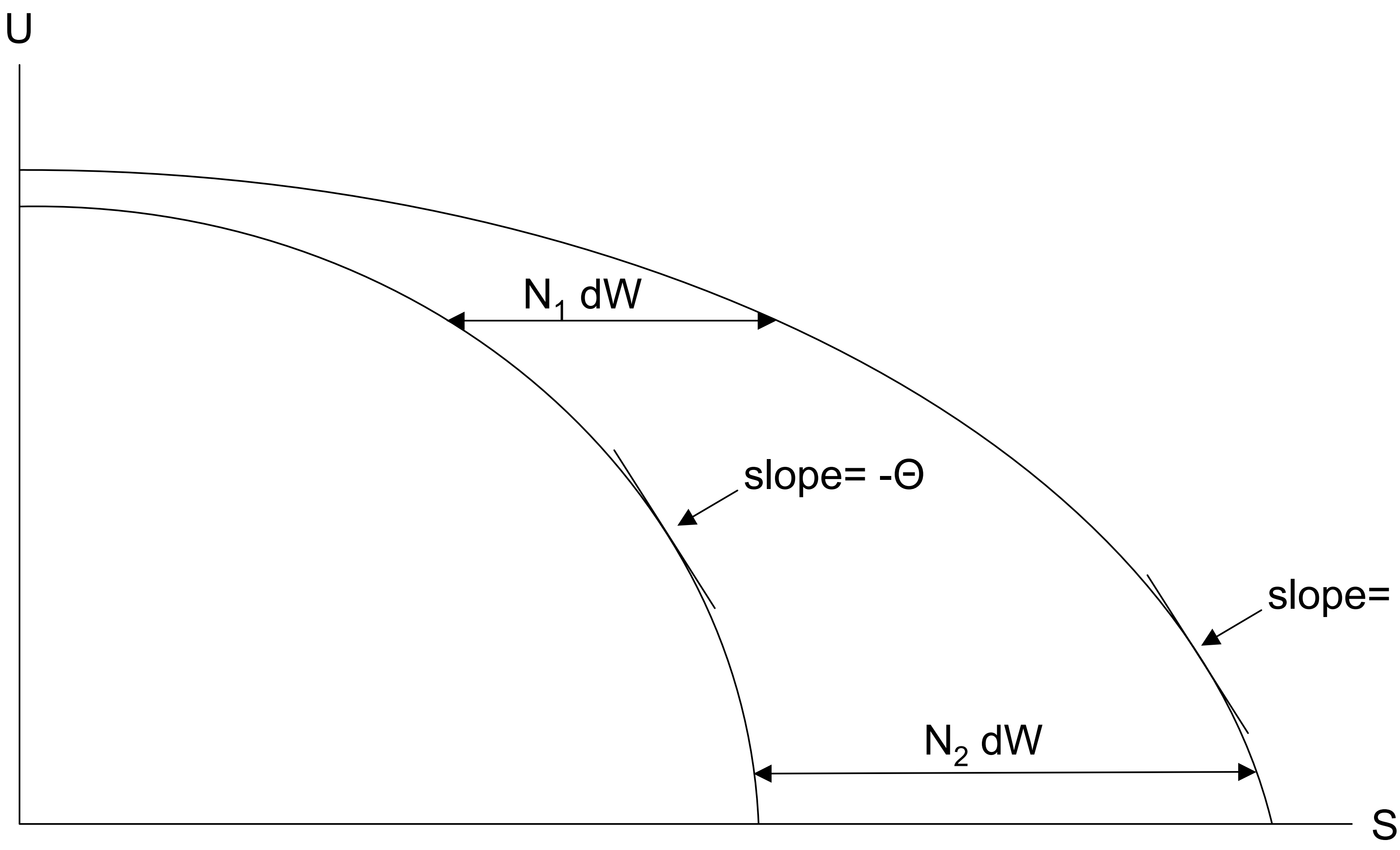

Figure 9: The Effect of Higher N on Optimal C when $U_{CN}>0$

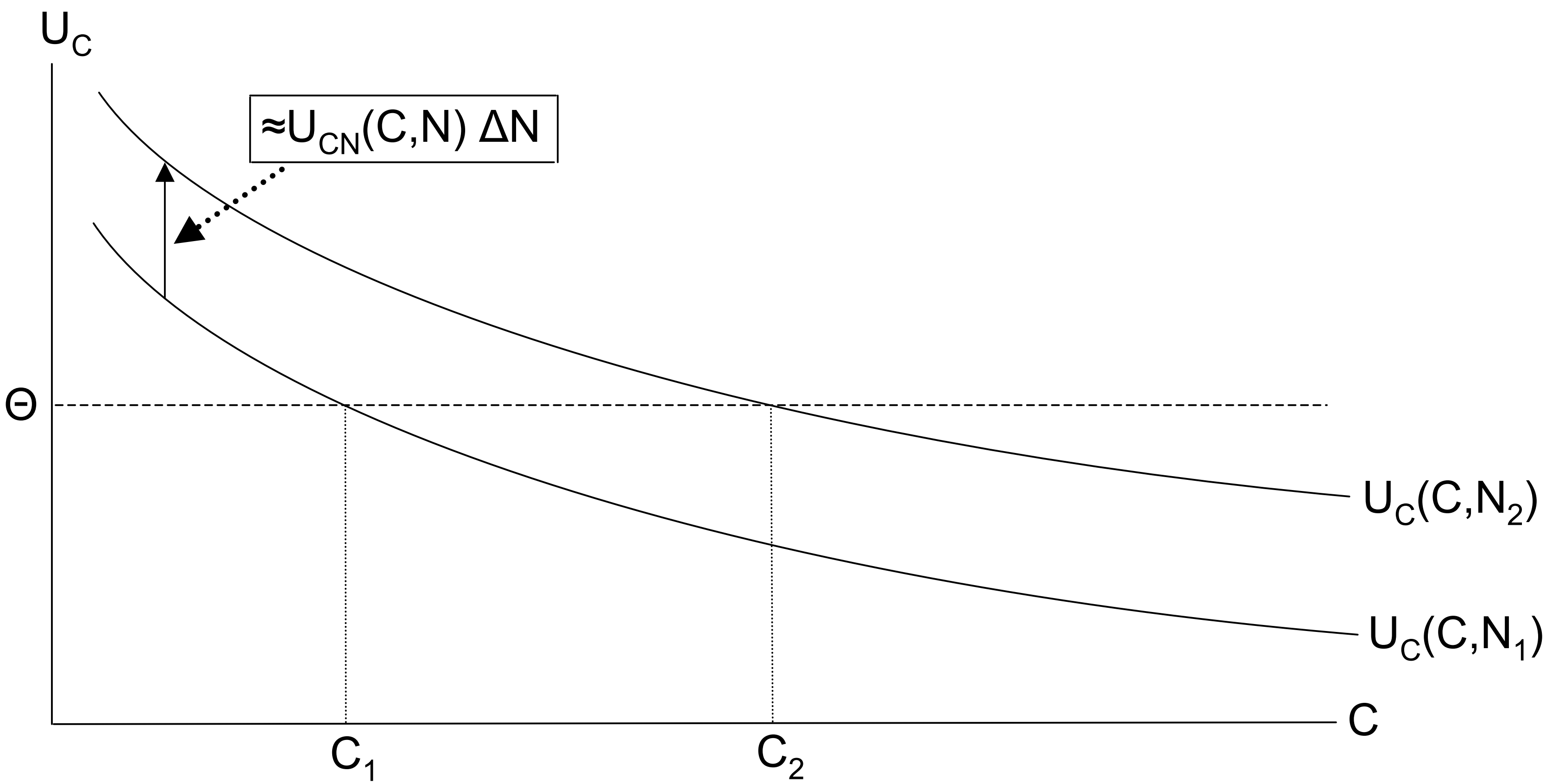

Figure 10: The Factor Price Possibility Frontier

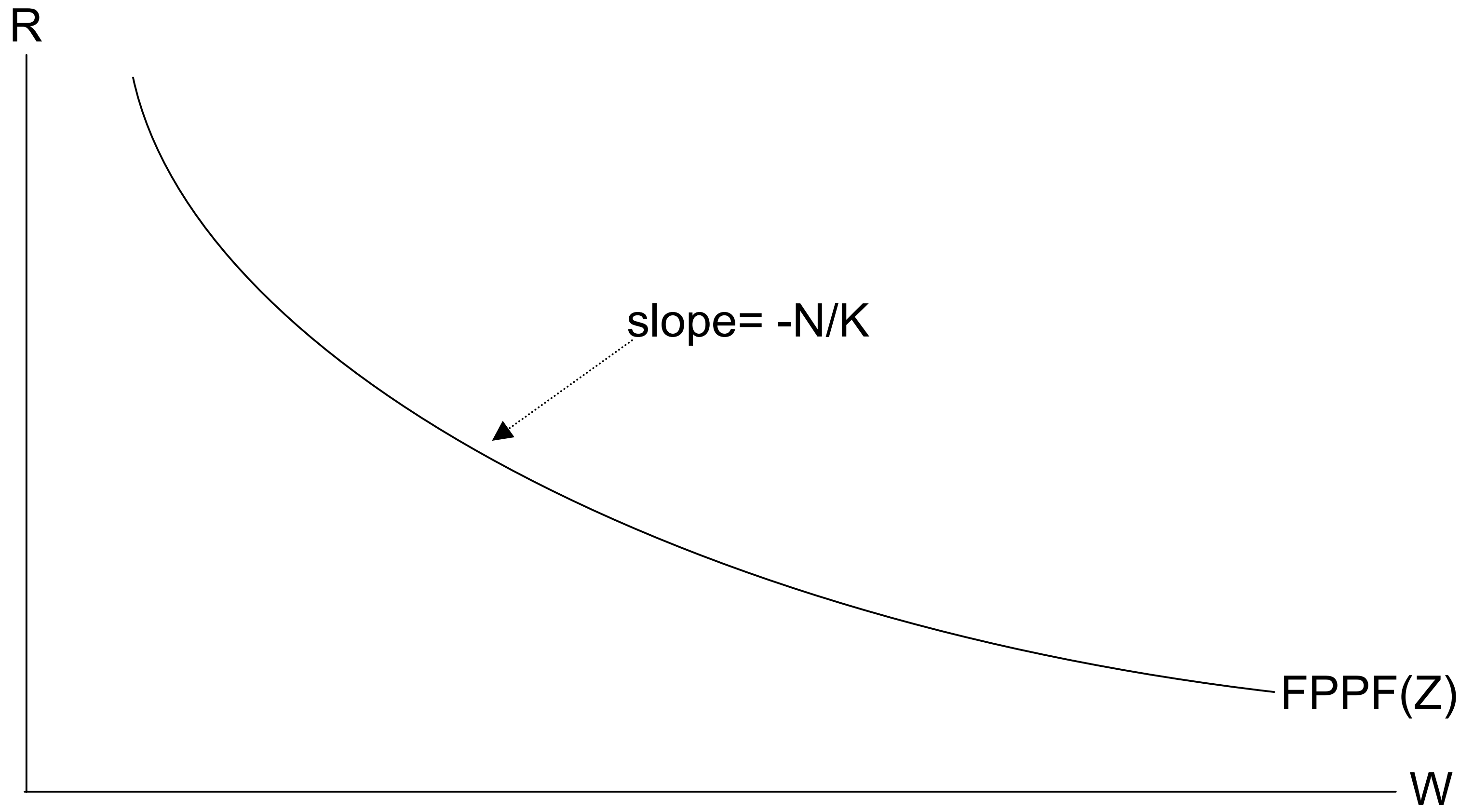

Figure 11: Labor Supply and Demand

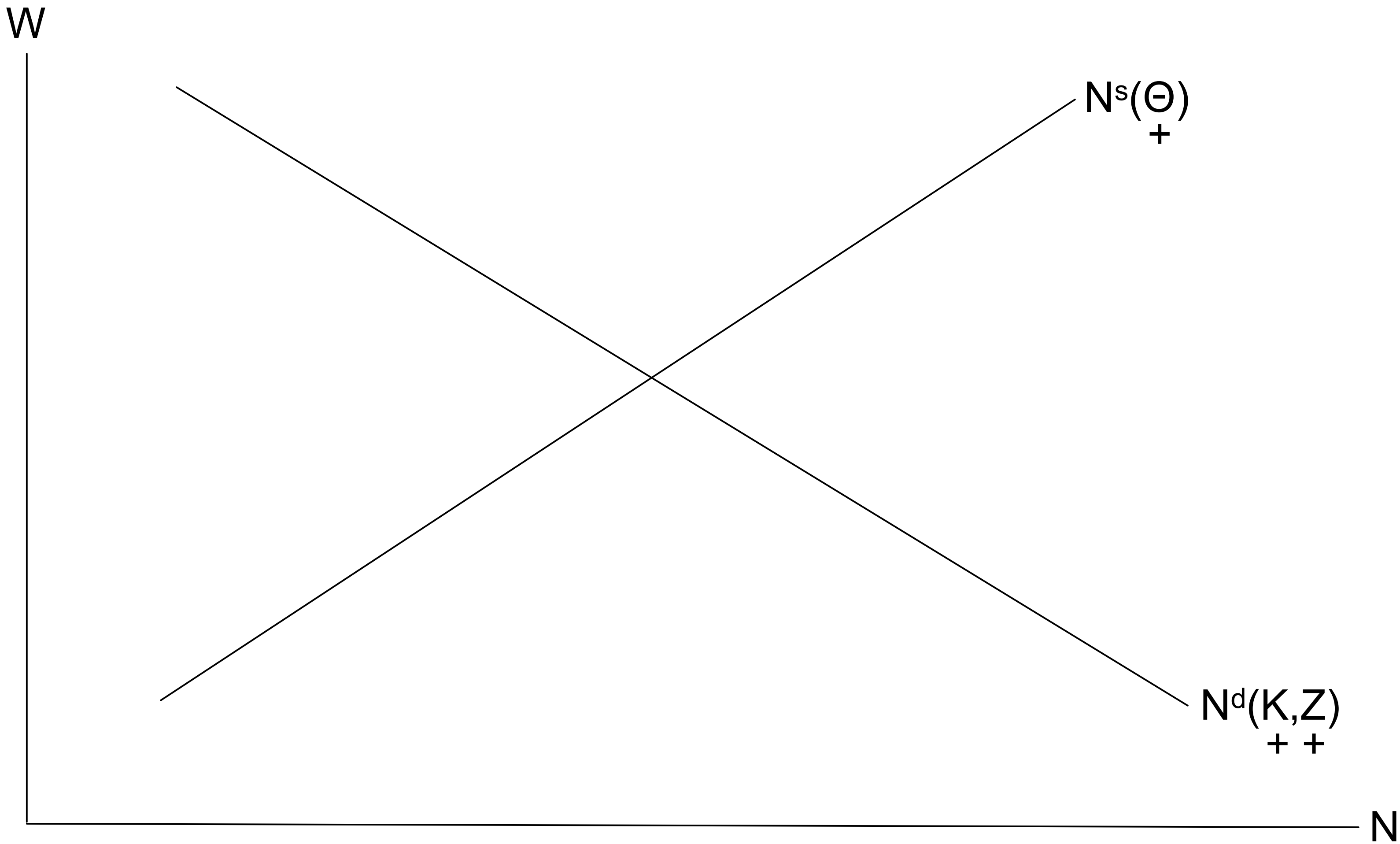

Figure 12: Contemporaneous Preferences and Technology Diagram

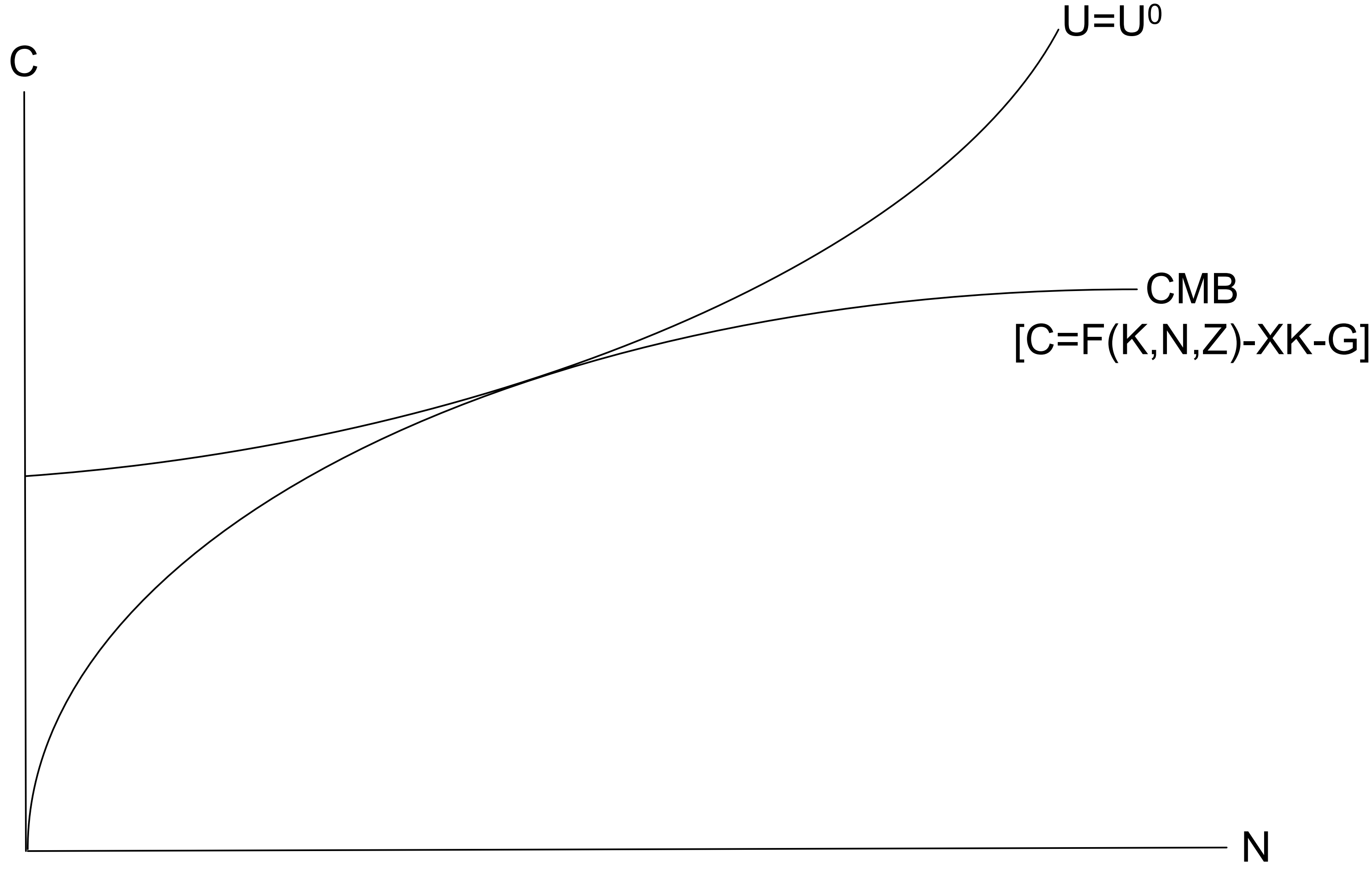

Figure 13: The Investment Demand Curve

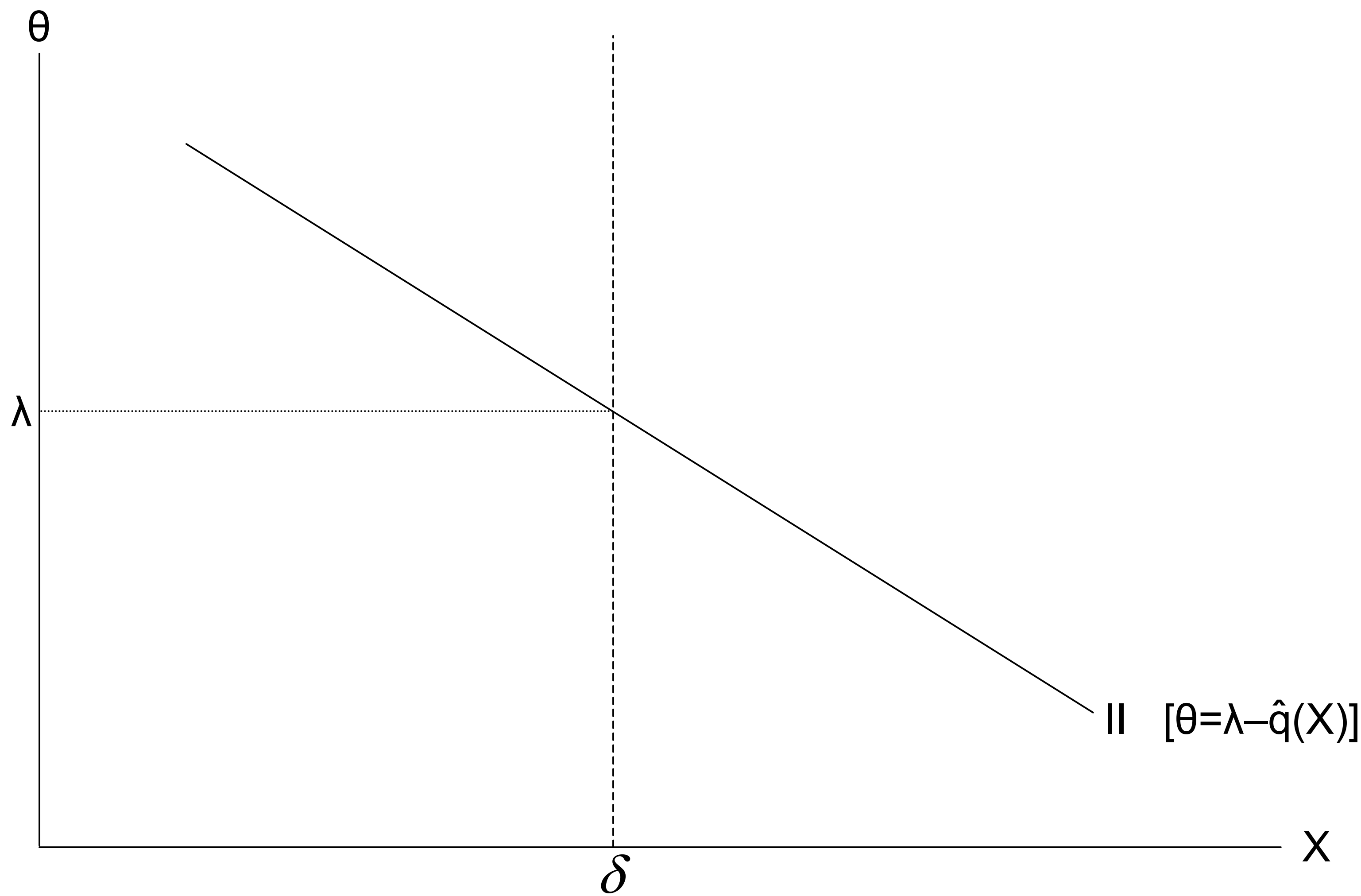

Figure 14: An Increase in Labor Supply

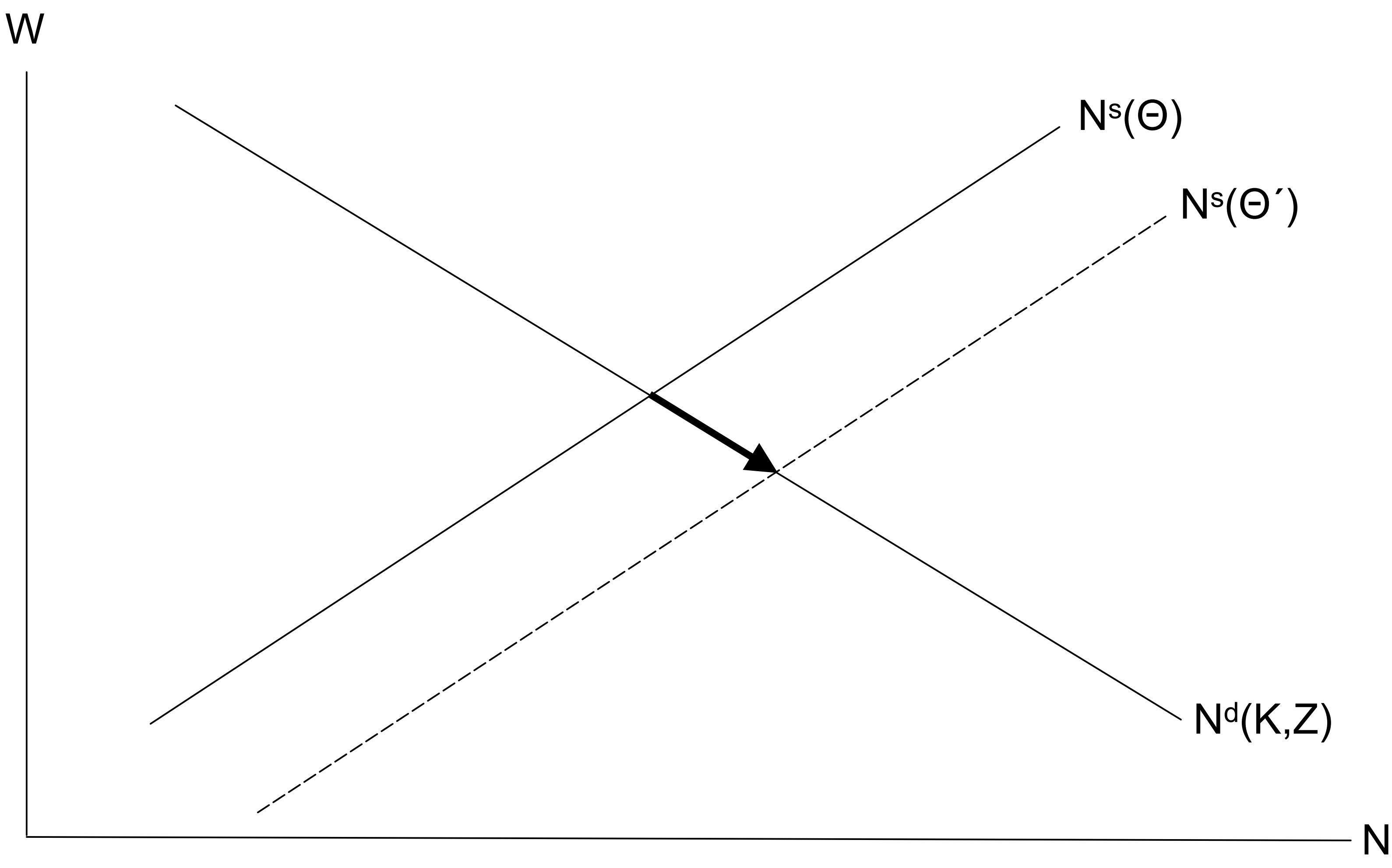

Figure 15: Consumption Falls When Θ Increases

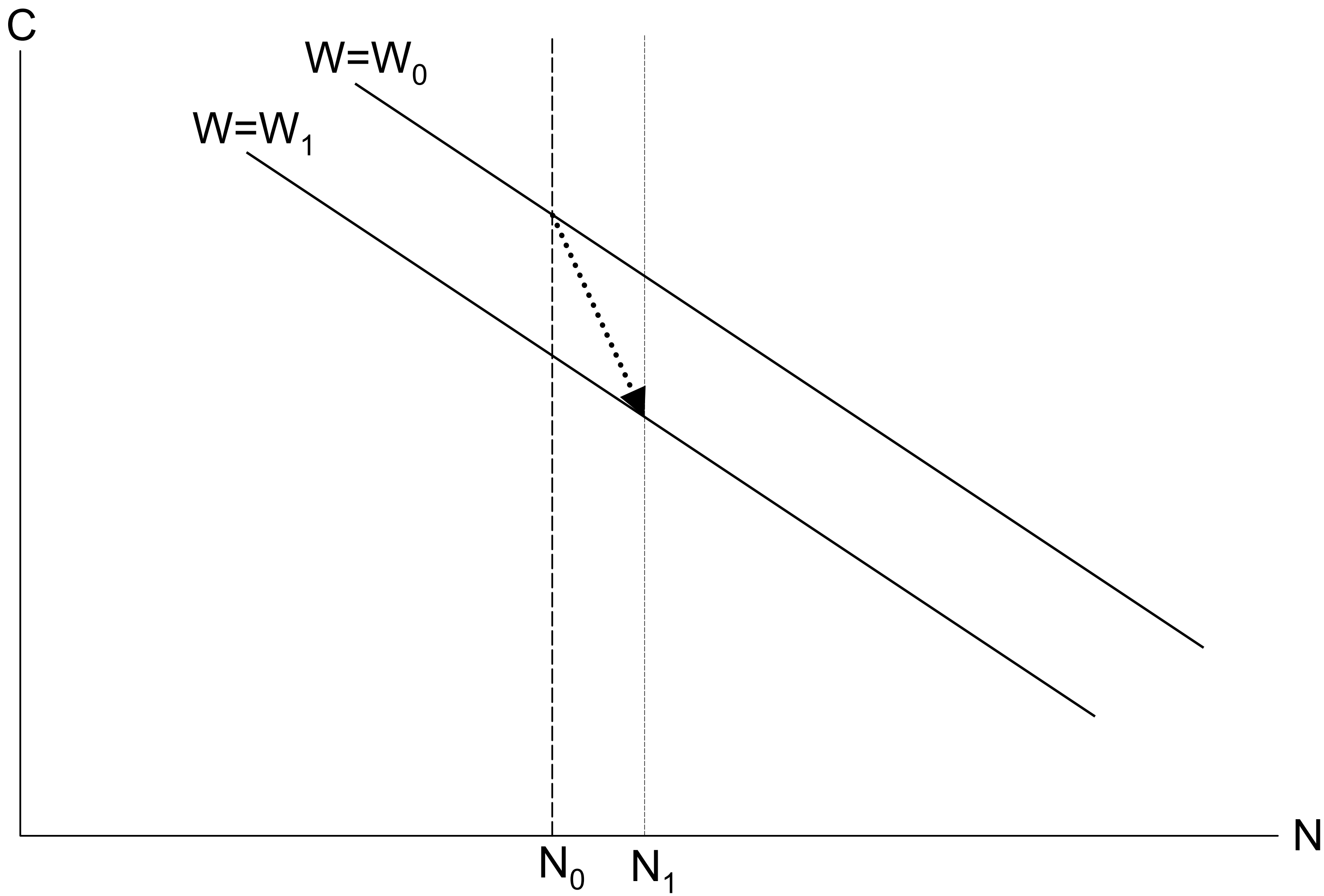

Figure 16: An Increase in K Raises Labor Demand

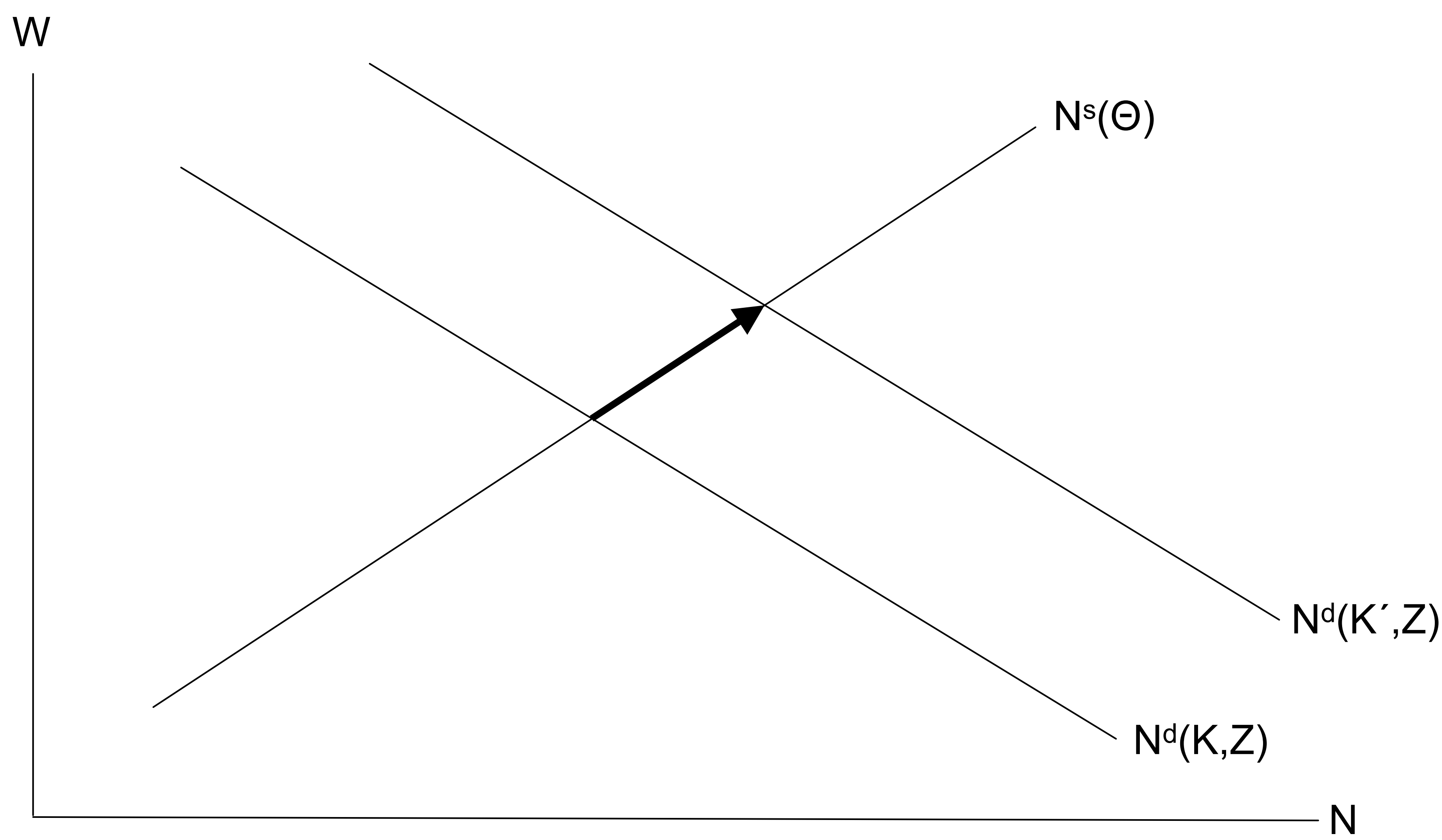

Figure 17: An Increase in K Causes a Movement Along the FPPF

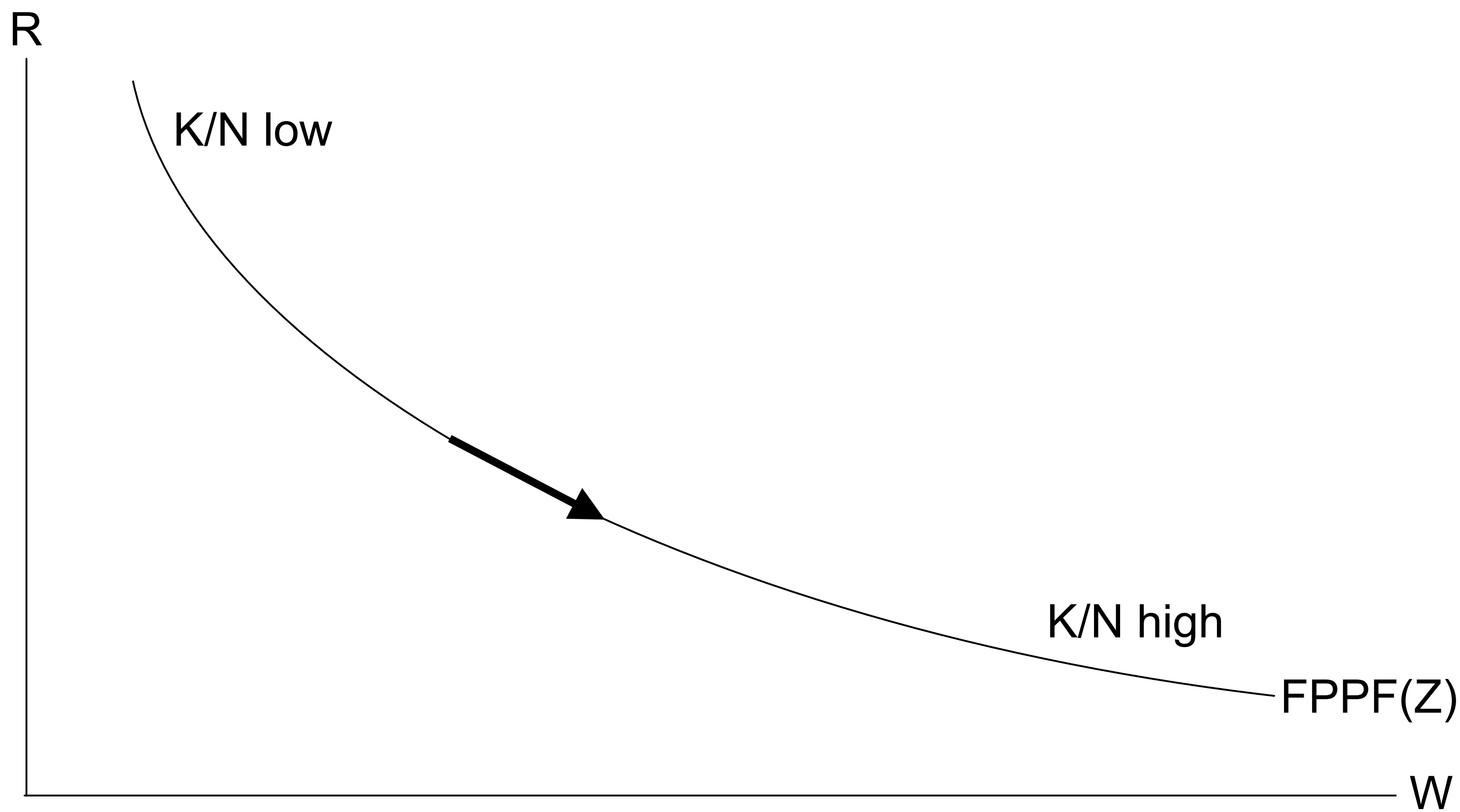

Figure 18: An Improvement in Technology Shifts the FPPF Out

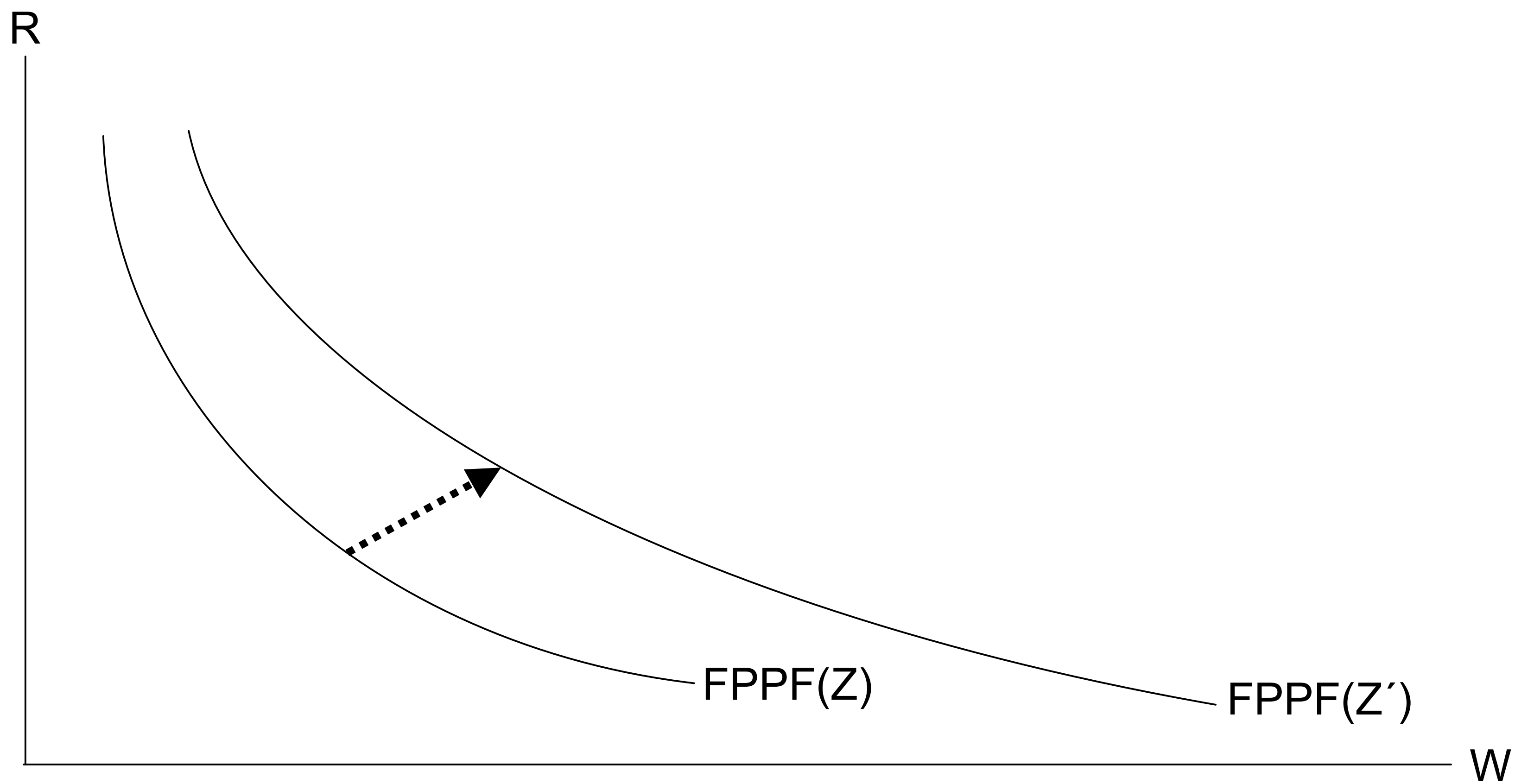

Figure 19: Comparative Statics of the Saving Supply Curve

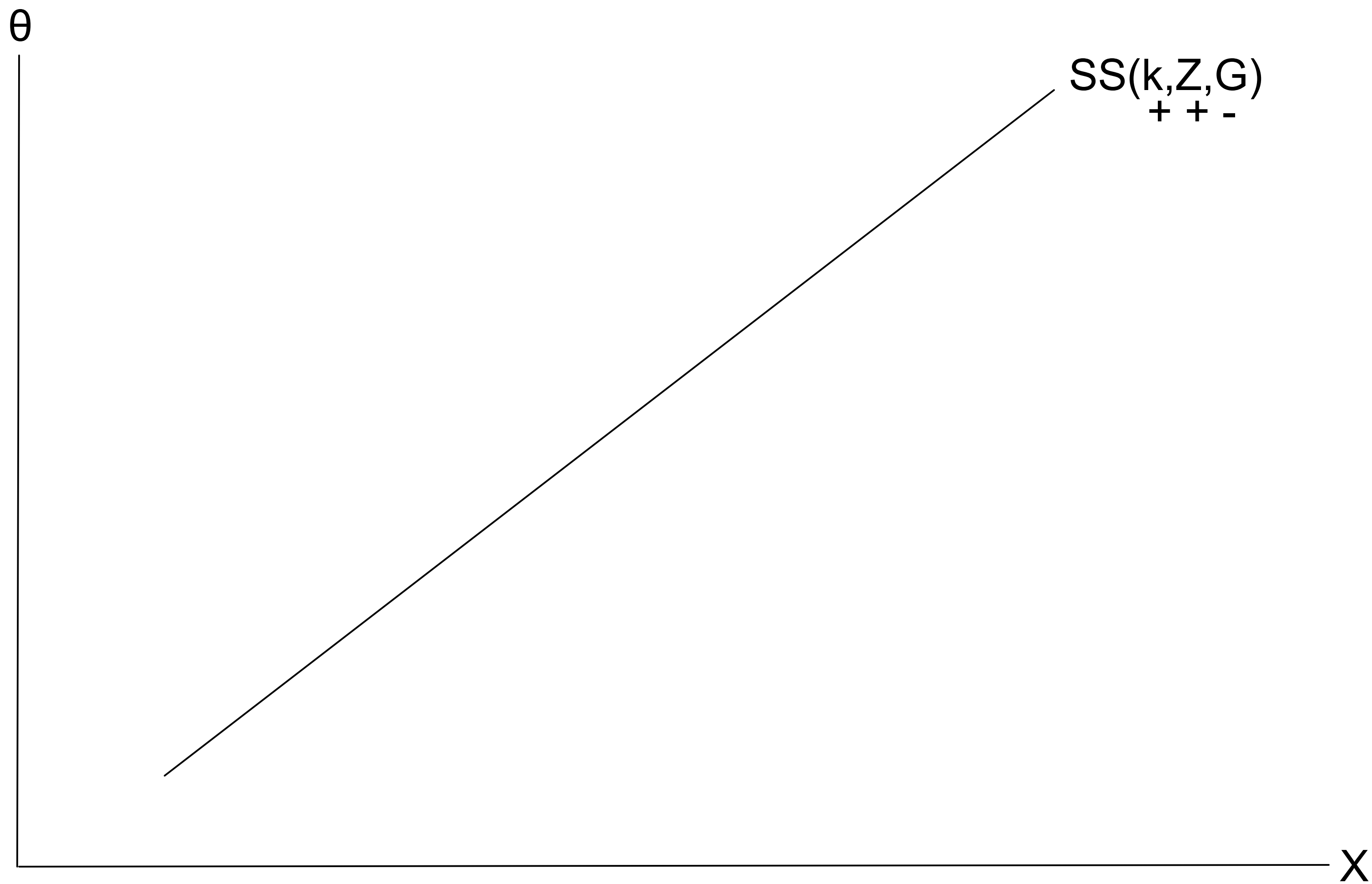

Figure 20: The Effect of an Increase in X, Given K, Z and G

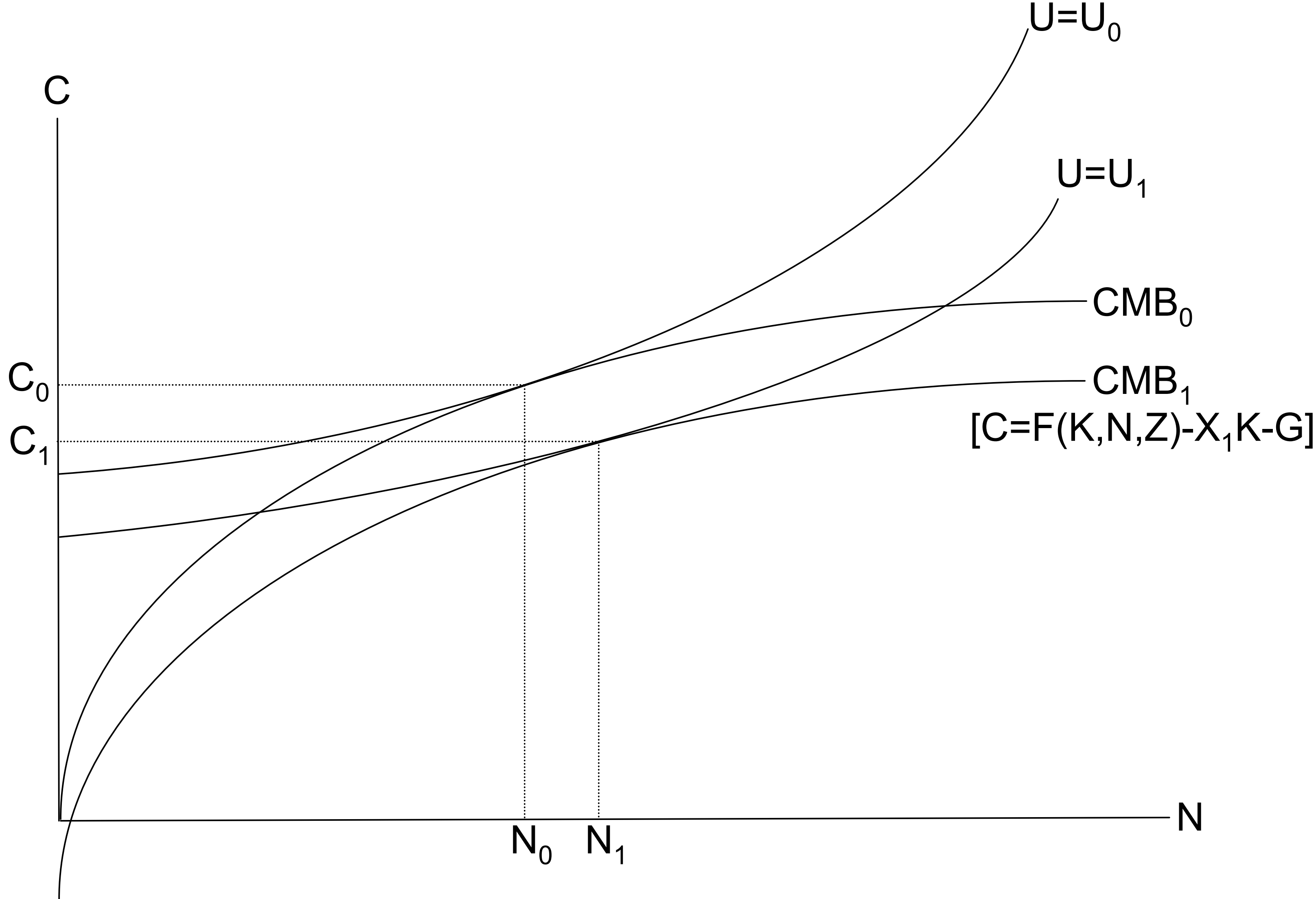

Figure 21: The Effect of an Increase in K, Given X, Z and G

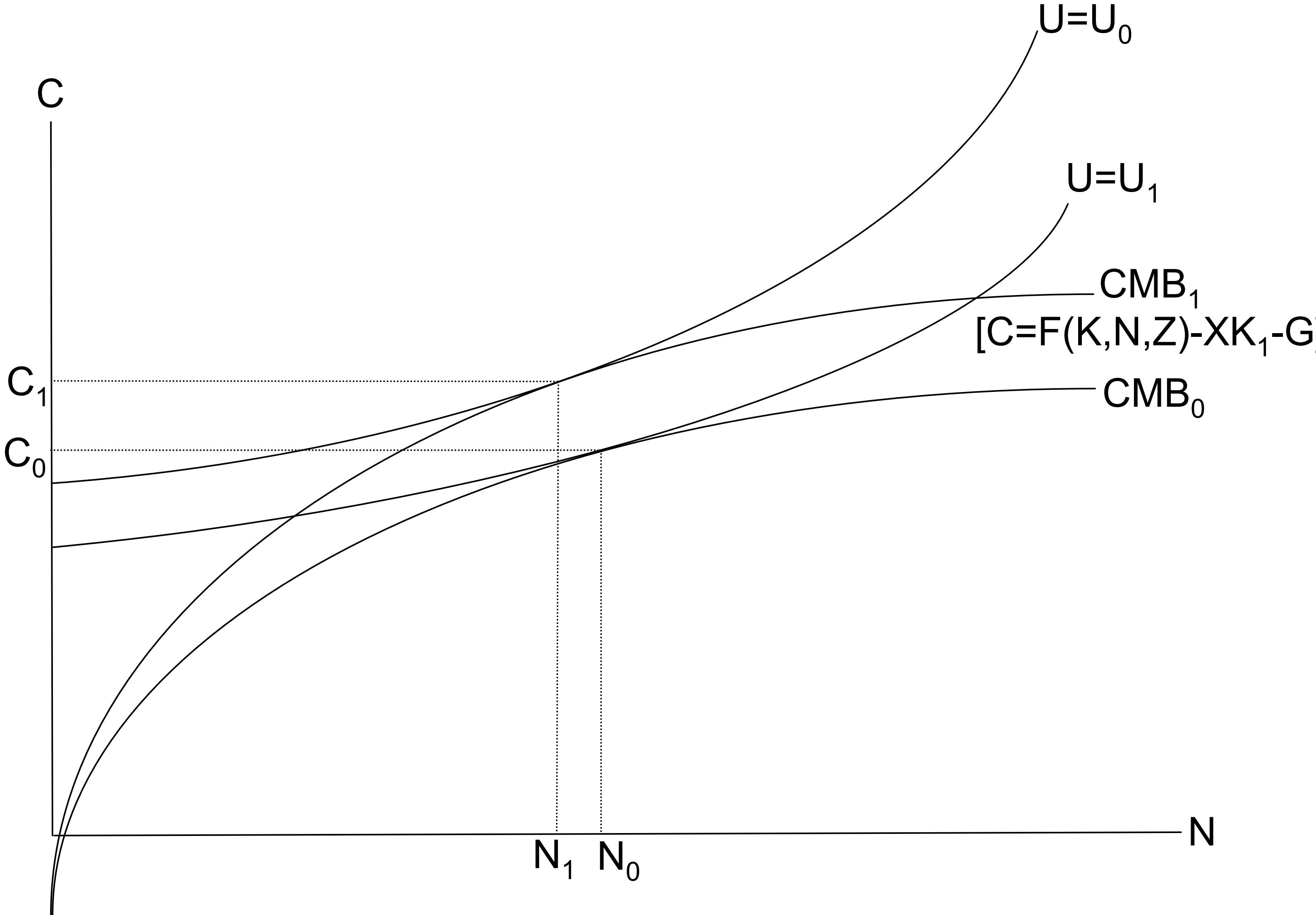

Figure 22: Shifting Out the Investment Demand Curve (λ↑)

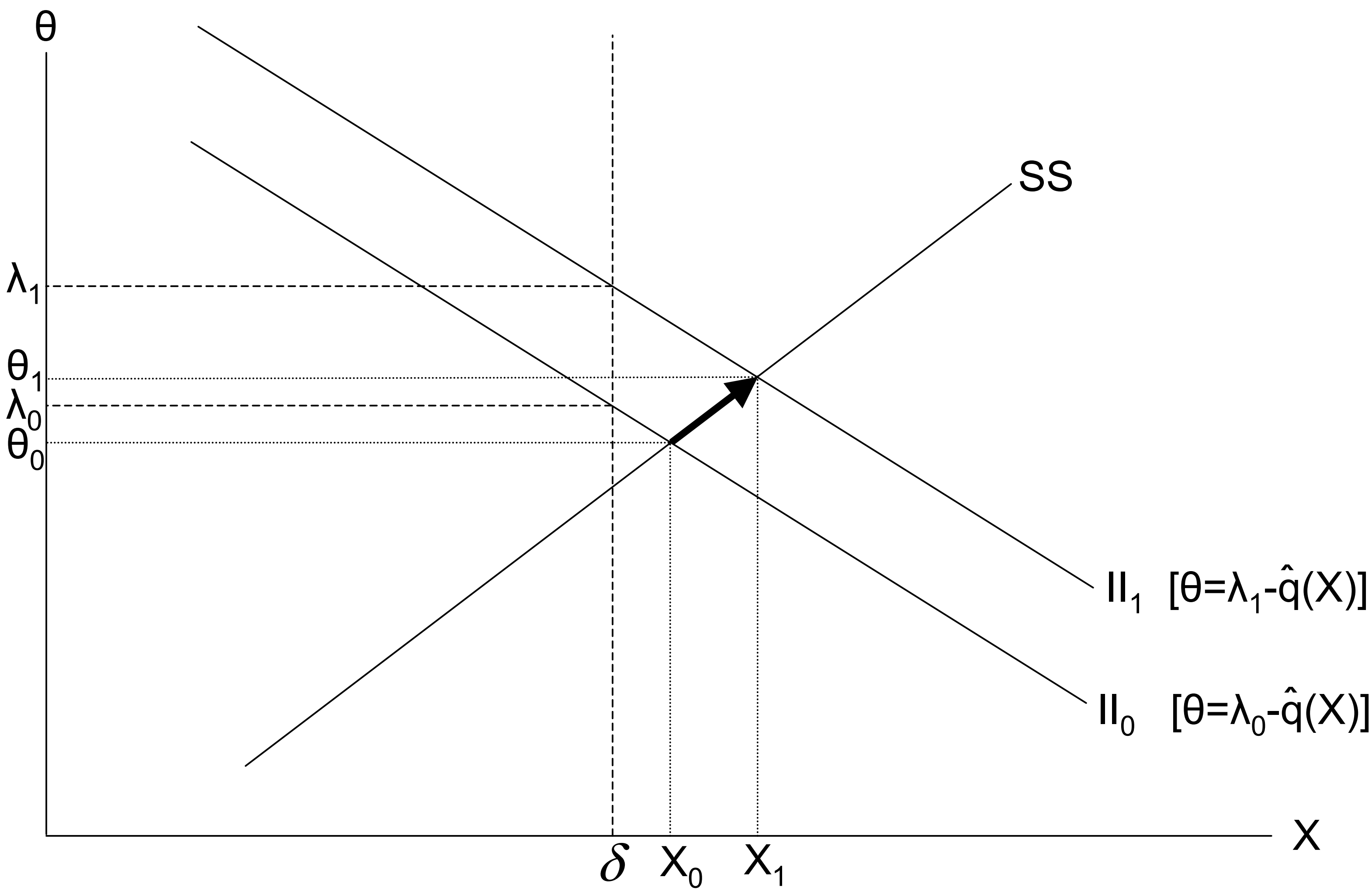

Figure 23: Shifting Out the Saving Supply Curve (K↑, Z↑, or G↓)

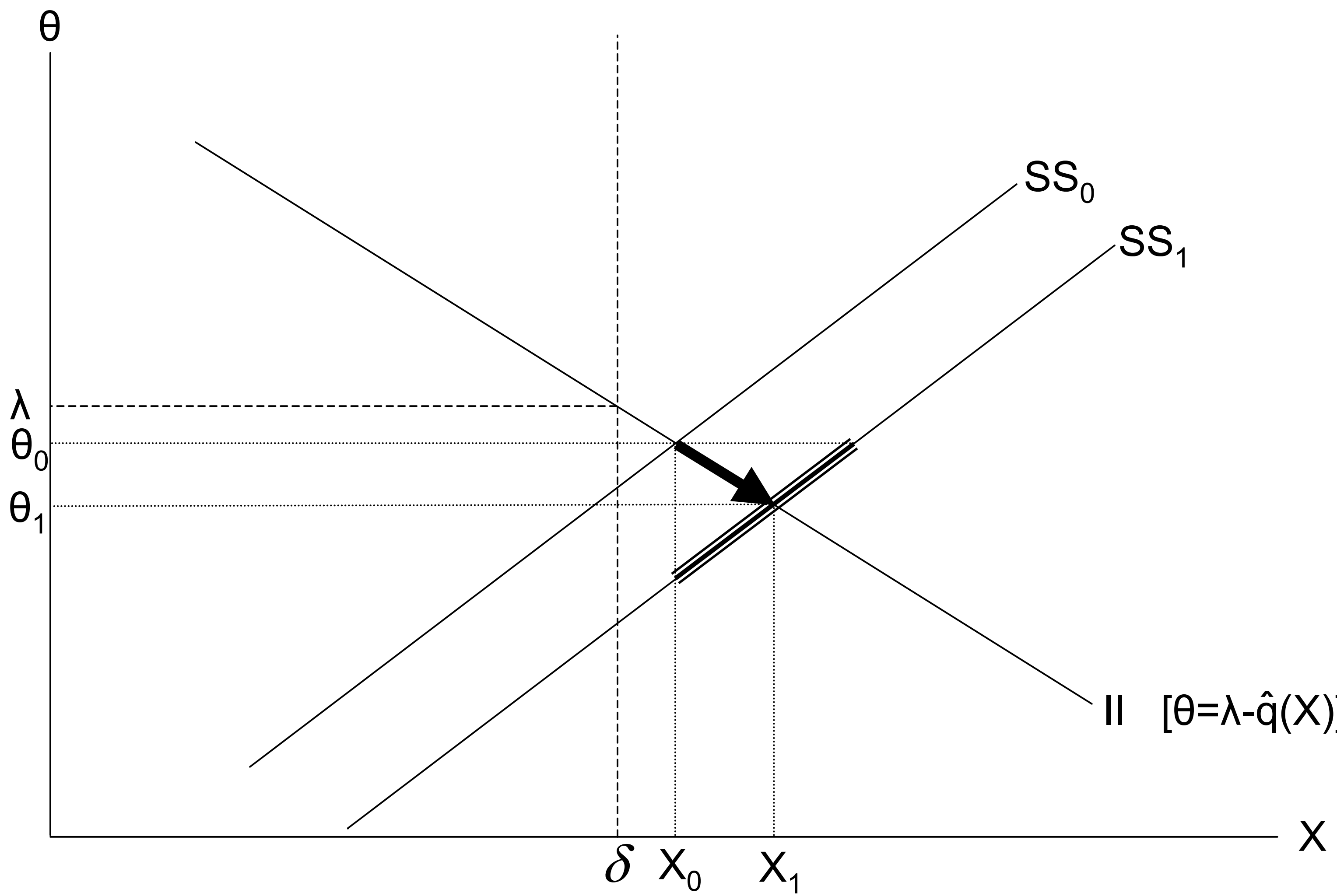

Figure 24a: Phase Diagram with Upward-Sloping $\dot{\lambda}=0$ Locus (Low Adjustment Costs)

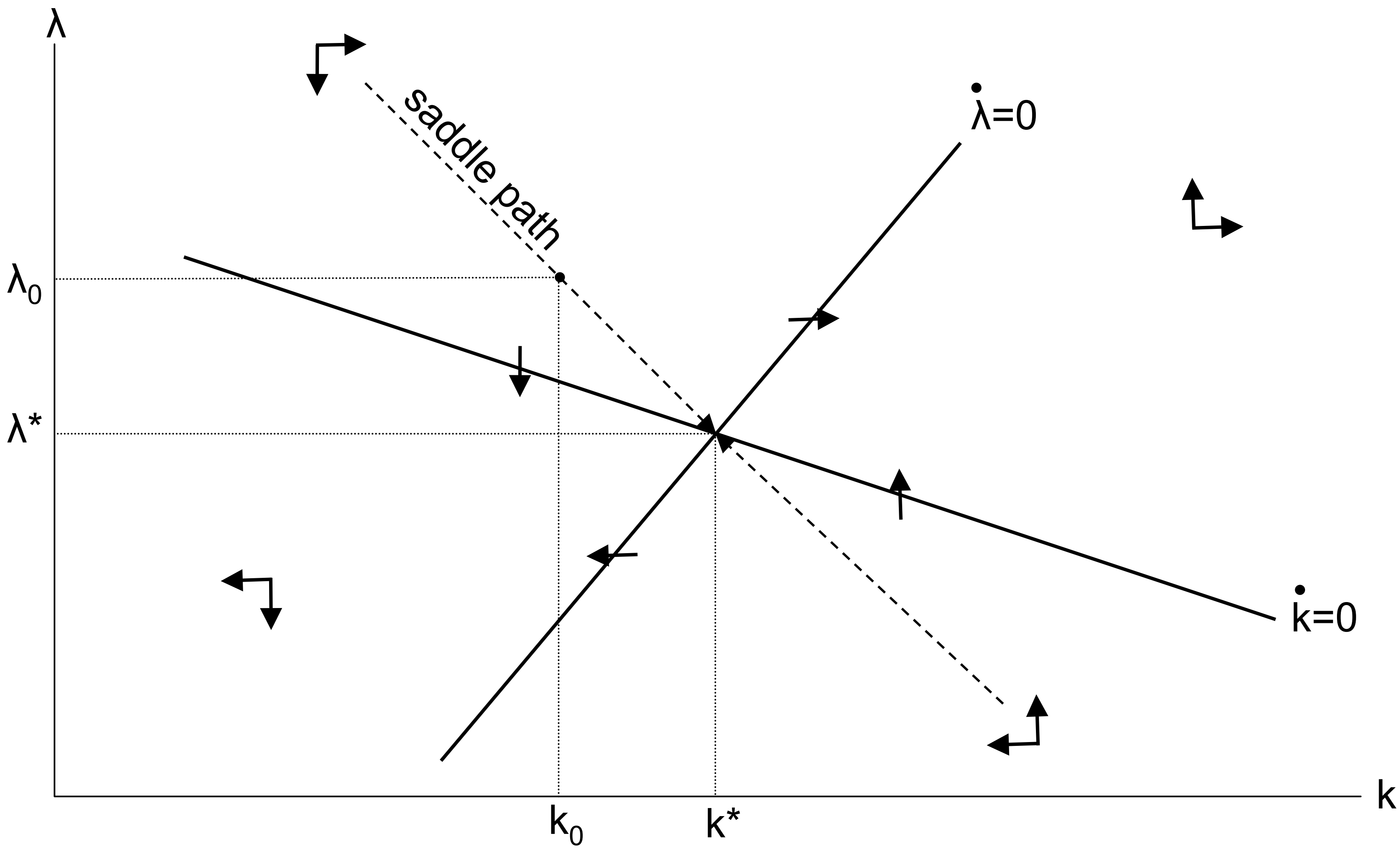

Figure 24b: Phase Diagram with Vertical $\dot{\lambda}=0$ Locus (Medium Adjustment Costs)

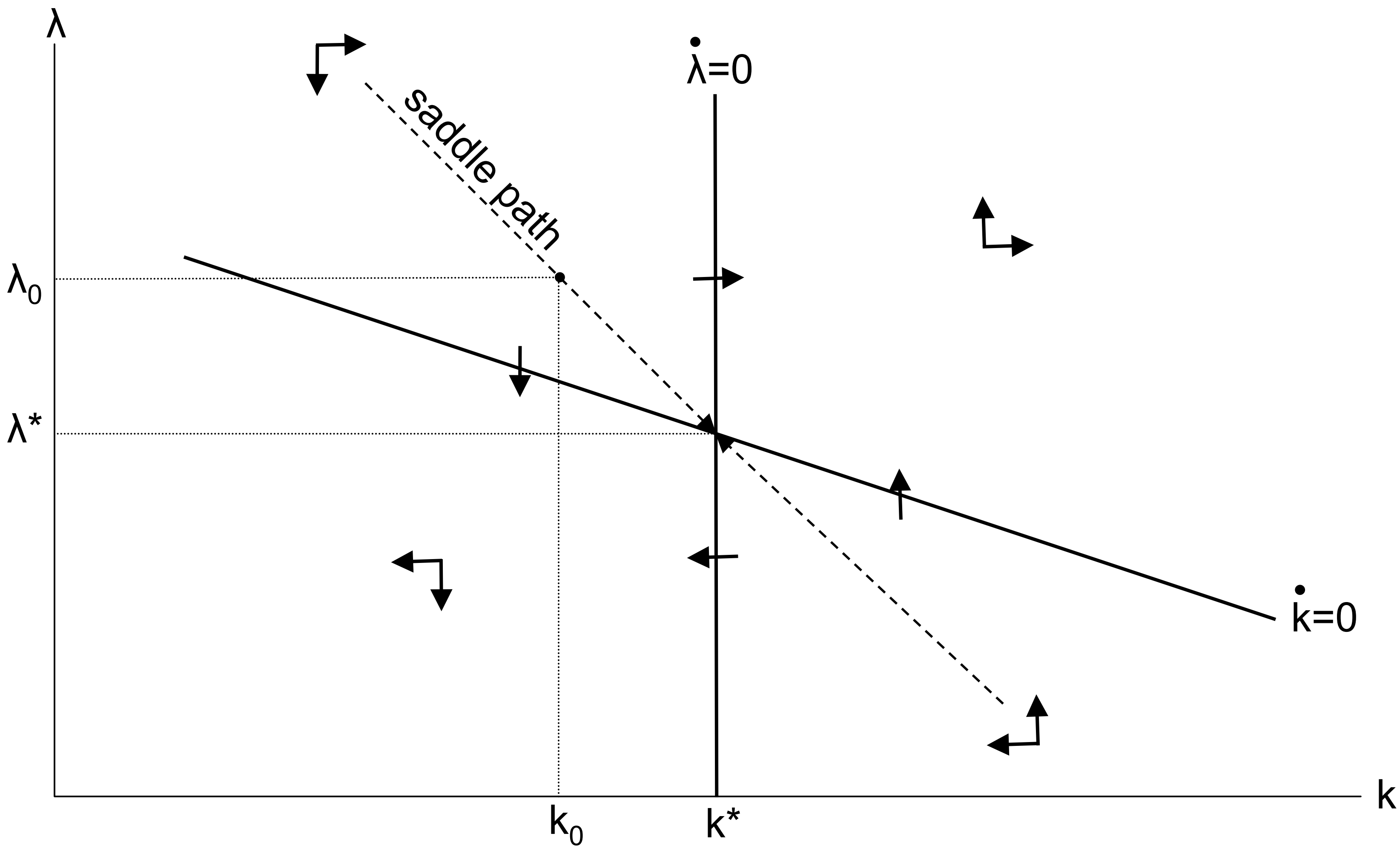

Figure 24c: Phase Diagram with Downward-Sloping $\dot{\lambda}=0$ Locus (High Adjustment Costs)

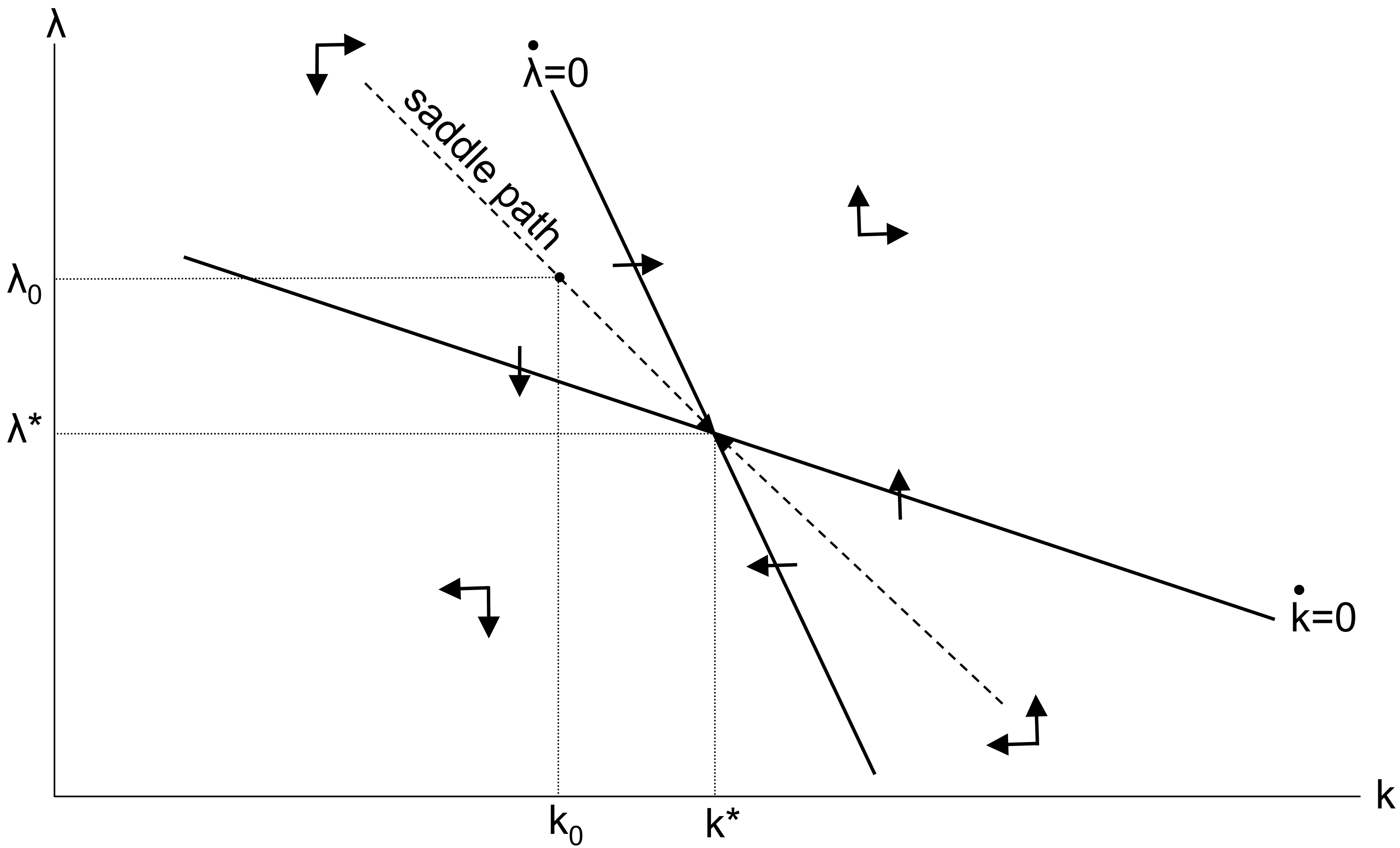

Figure 25: II-SS When Moving Down the Saddle Path (K↑ and λ↓)

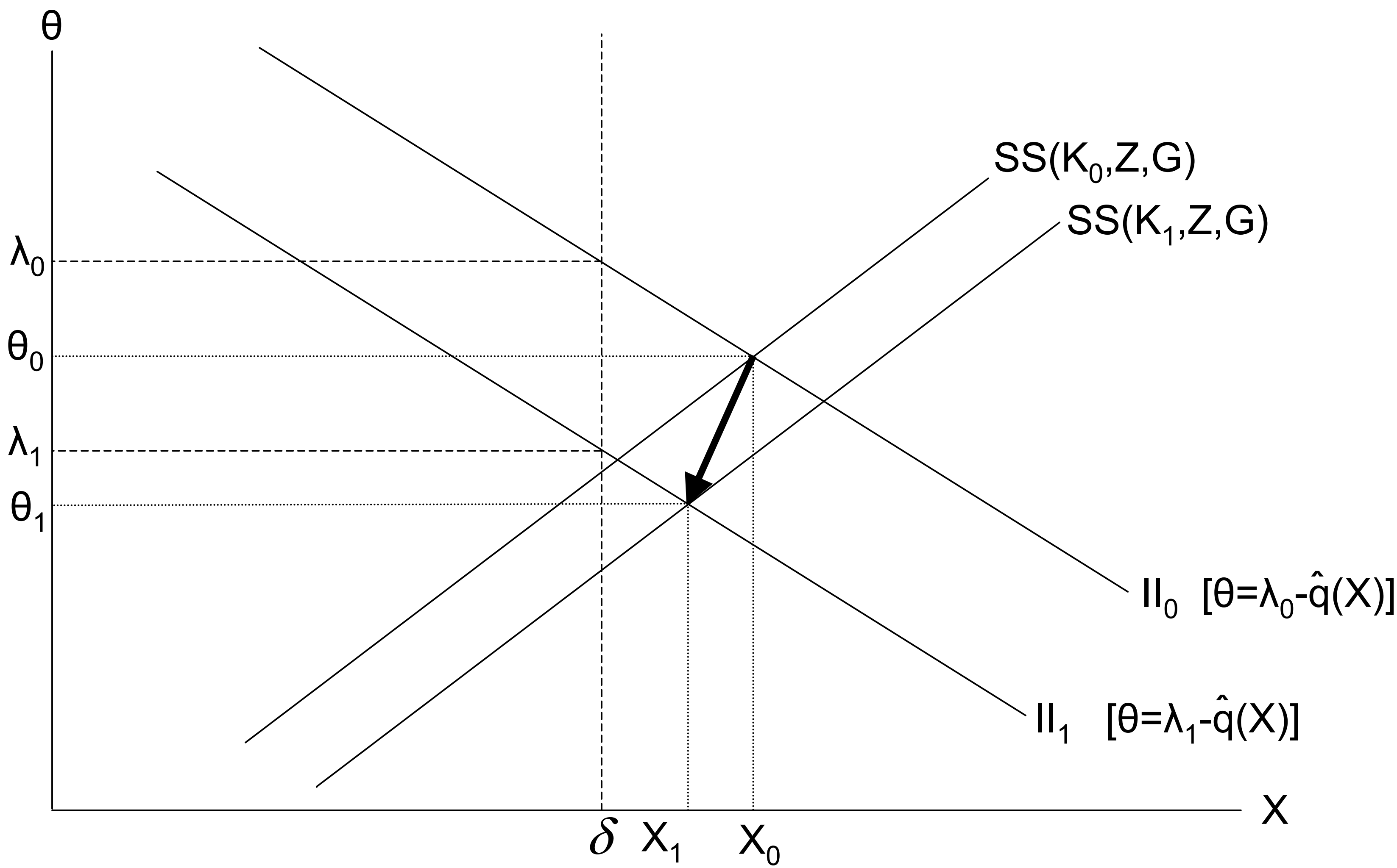

Figure 26a: DGE Effects of a Permanent Increase in G (Low Adjustment Costs)

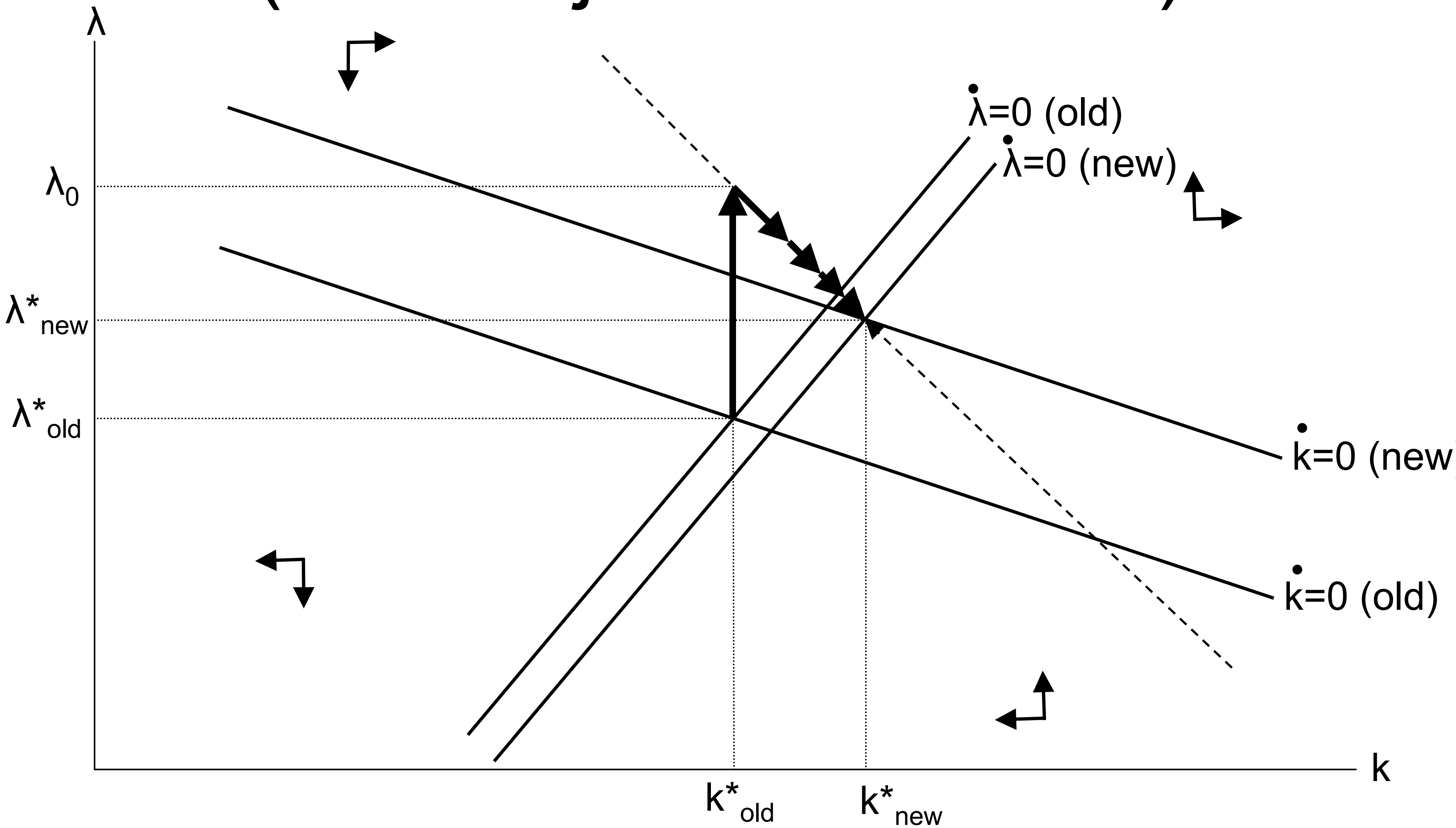

Figure 26b: DGE Effects of a Permanent Increase in G (High Adjustment Costs)

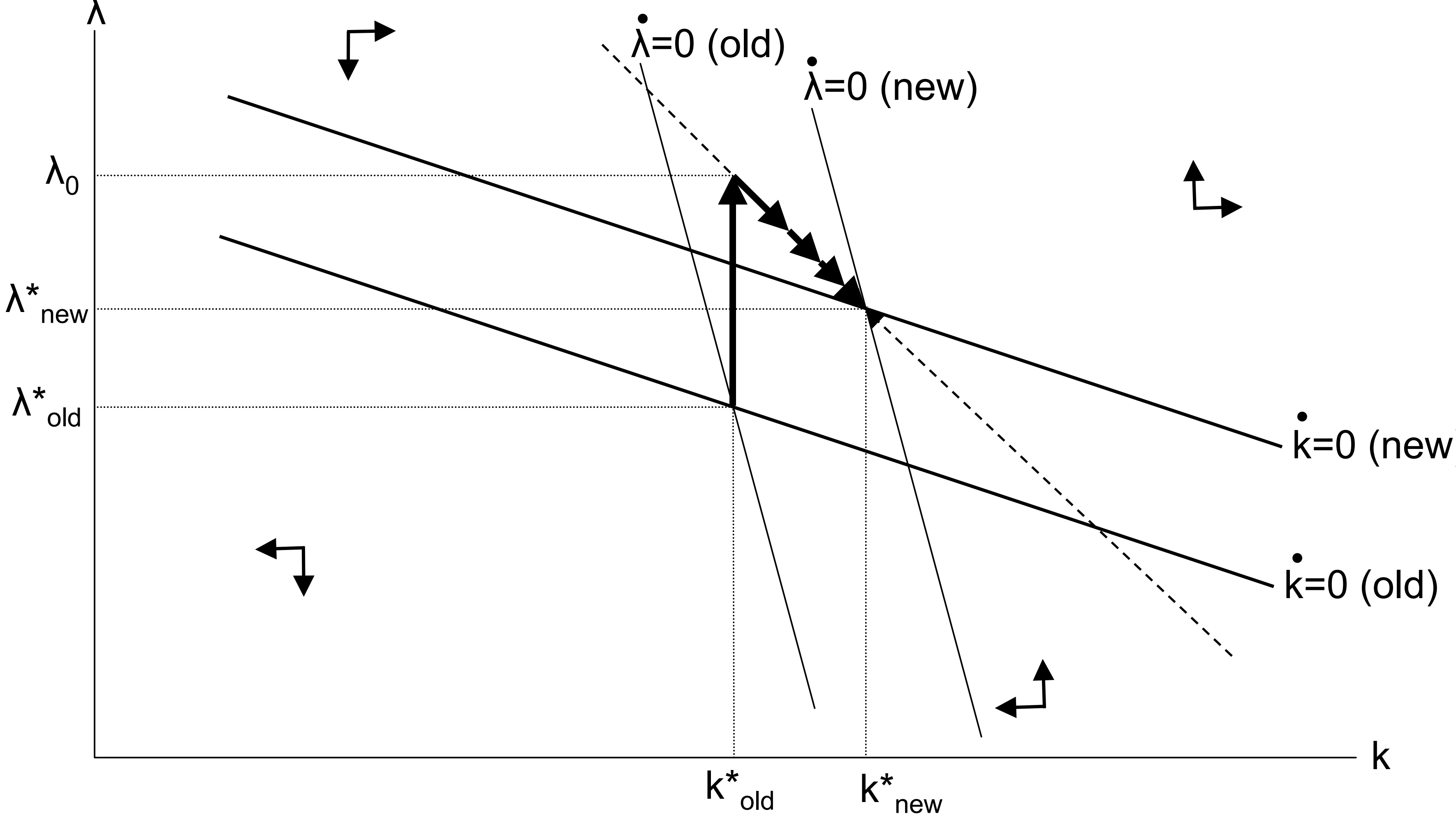

Figure 27a: DGE Effects of a Permanent Increase in Z (Wealth Effect Dominates Rental Rate Effect)

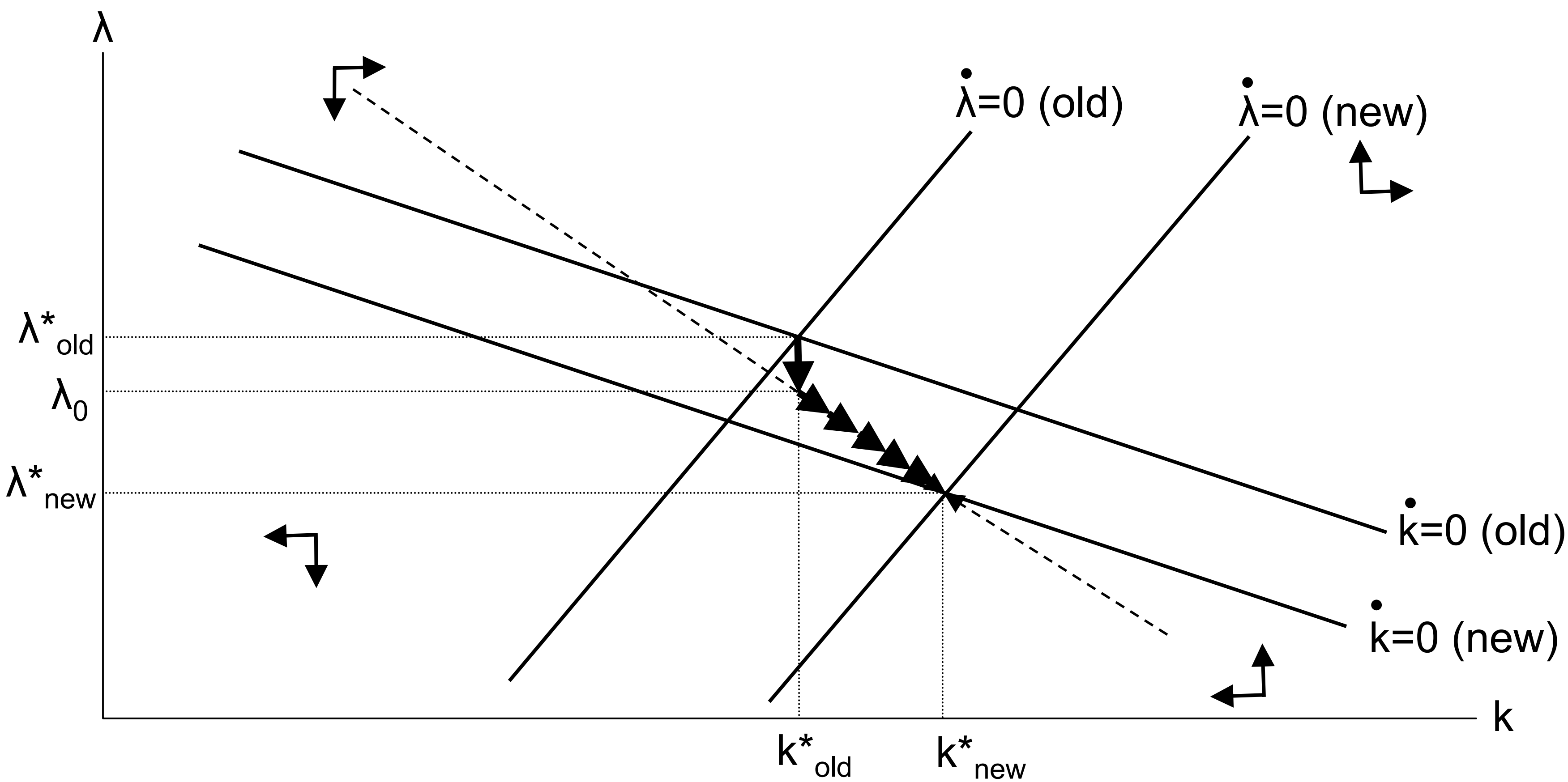

Figure 27b: DGE Effects of a Permanent Increase in Z (Rental Rate Effect Dominates Wealth Effect)

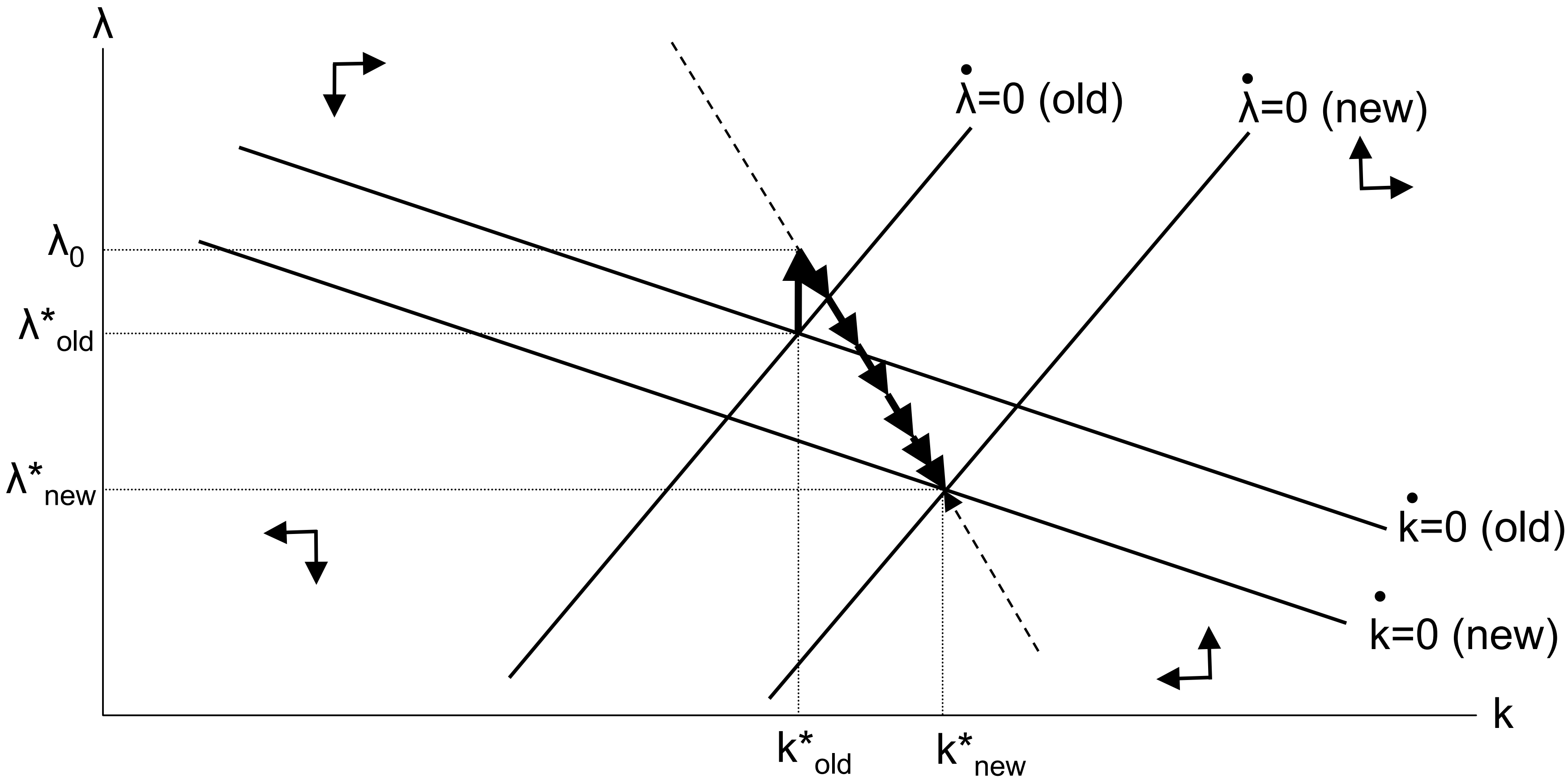

Figure 27c: DGE Effects of a Permanent Increase in Z (Equal and Opposite Wealth and Rental Rate Effects)