Final Exam

Economics 609, Winter 2004

Due: **Tuesday**, March 2, Noon.

This is an open book take home exam, but no collaboration is allowed. Do not talk to anyone about the test before Wednesday, March 3. You can e-mail me questions and I will e-mail back an answer to everyone.

1. Submodularity has the same definition as supermodularity, but with the direction of the inequality reversed. Is submodularity preserved under maximization much as supermodularity is preserved under maximization? Why or why not?

2. Consider an agent with utility displaying habit formation. The agent's Bellman equation is

$$V(B_t, H_t) = \max_{C\in(-\infty,B_t]} U(C, H_t) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(B_t - C) + \tilde{y}_{t+1}, \quad \delta C + (1-\delta)H_t),$$

where the arguments of V^{t+1} are equal to $\tilde{B}_{t+1}$ and H_{t+1} respectively. B is wealth, H is the habit, C is consumption, R is the gross rate of return and y is exogenous labor income. $\delta > 0$ and $\beta > 0$. There is a vector of exogenous state variables in the background governing the process for the rate of return $\tilde{R}$ and labor income $\tilde{y}$, which could be represented explicitly, but need not be for our purposes. Beyond the end of time, $\mathcal{V}^{T+1} \equiv 0$.

a. Define

$$X = \delta C + (1-\delta)H_t$$

$$K_t = \delta B + (1-\delta)H_t$$

and

$$\mathcal{U}(X, H_t) = U(\frac{X - (1-\delta)H_t}{\delta}, H_t)$$

$$\mathcal{V}(K_t, H_t) = V(\frac{K_t - (1-\delta)H_t}{\delta}, H_t)$$

Show that the Bellman equation above is equivalent to

$$\mathcal{V}(K_t, H_t) = \max_{X\in(-\infty,K_t]} \mathcal{U}(X, H_t) + \beta E_t \mathcal{V}^{t+1}(\tilde{R}_{t+1}(K_t - X) + \delta\tilde{y}_{t+1} + (1-\delta)X, \quad X).$$

Correct any mistakes I made along the way. The aim was to make a transformation where X is equal to H_{t+1} and K is defined so that H does not appear explicitly in the first argument of $\mathcal{V}^{t+1}$.

b. It will be most convenient to make assumptions directly on $\mathcal{U}$. Assume that $\mathcal{U}$ is increasing in X, decreasing in H_t, jointly concave in X and H_t, and supermodular in X and H_t. Also, assume that $\tilde{R}_{t+1} \geq 1-\delta$ with probability 1. Prove that $\mathcal{V}$ is increasing in K_t, decreasing in H_t, jointly concave in K_t and H_t, and supermodular in K_t and H_t.

c. Show that the unmaximized right-hand side of the Bellman equation is supermodular in K_t and X, AND supermodular in H_t and X. Conclude that the optimal value of X is increasing in K_t and in H_t.

d. Reinterpret both the assumptions in (b) and the conclusions in (b) and (c) in terms of the original problem in B, C and H.