

Final Exam
Economics 609, Winter 2003

Due: **Friday**, February 21, Noon.

This is an open book take home exam, but no collaboration is allowed. Do not talk to anyone about the test before Saturday. In addition to asking questions in class on Thursday, you can e-mail me questions and I will e-mail back an answer to everyone.

1. Consider an agent with logarithmic utility:

$$v_t = \ln(C_t) + \beta E_t v_{t+1}.$$

The agent has wealth B_t and has available a set of investment options indexed by $i \in 1, \dots, n$ that each have a *gross* return of $R_i^t(Z_t, \omega_{t+1})$ where Z_t is a vector of *exogenous* state variables and ω_{t+1} is a vector of i.i.d. uniform $[0, 1]$ random variables. Denote the share of the portfolio put into each asset by α_i , with $\sum_i \alpha_i = 1$. The agent's only source of income is investment income. You can think of the agent as retired. She will live until time T and $V^{T+1} \equiv 0$. The exogenous state variables obey the transition equation

$$Z_{t+1} = \Gamma(Z_t, \omega_{t+1}).$$

- a. Find the explicit formula for B_{t+1} in terms of B_t , C_t , $\alpha_{i,t}$ and $R_i^t(\cdot, \cdot)$.
- b. Use the symmetry theorem to characterize the value function $V^t(B_t)$ as precisely as possible. What happens to the value function as $T \rightarrow \infty$?
- c. Find the optimal policy function for consumption. Show that the optimal policy function for consumption does not depend on Z_t . This is a remarkable implication of log utility. It does not carry over to other utility functions.
- d. Suppose the vector Z_t is partitioned into two subvectors, \hat{Z}_t and \bar{Z}_t , and the R_i are functions of \hat{Z}_t alone: $R_i^t(\hat{Z}_t, \omega_{t+h})$. The transition function still depends on both \hat{Z} and \bar{Z} :

$$(\hat{Z}_{t+1}, \bar{Z}_{t+1}) = \Gamma(\hat{Z}_t, \bar{Z}_t, \omega_{t+1}).$$

Show that the optimal policy functions for the α_i do not depend on \bar{Z}_t . In words, you are showing that only the current set of functions $R^i(\hat{Z}_t, \omega_{t+h})$ are needed to determine the optimal portfolio shares. (That is, "myopic" portfolio choice is optimal.) These do not depend on the subvector \bar{Z}_t , which says something only about *future* R^i functions. *You do not need to find the exact form of the policy functions for the α_i . Just show that they do not depend on \bar{Z}_t .*

2. Consider a firm's investment problem with Bellman equation

$$V(K_t, \theta, Z_t) = \max_{I_t} R(Z_t)K_t - P_I(Z_t)I_t + \beta(\theta, Z_t)E_t V^{t+h}(K_{t+h}, \theta, Z_{t+h})$$

where

$$K_{t+h} = K_t \Gamma(I_t / K_t)$$

and

$$Z_{t+h} = \Xi(Z_t, \omega_{t+h}).$$

Also, $\Gamma'(\cdot) > 0$, $\Gamma''(\cdot) < 0$, $R(Z_t)$ is a rental rate for capital and the discount factor β is increasing in θ :

$$\frac{\partial \beta}{\partial \theta} = \beta_\theta > 0.$$

- a. Prove that V is supermodular in K and θ . If it helps you, feel free to assume that V is always twice differentiable in K and θ , so that supermodularity can be expressed as $V_{K\theta} \geq 0$.
- b. Using the result to part a, show that the optimal investment policy $I^*(K_t, \theta, Z_t)$ is increasing in θ .
- c. Can you extend the proof you used in part a to cover the case where

$$V(K_t, \theta, Z_t) = \max_{I_t} \Pi(K_t, Z_t) - P_I(Z_t)I_t + \beta(\theta, Z_t)E_t V^{t+h}(K_{t+h}, \theta, Z_{t+h})$$

where $\Pi(K_t, Z_t)$ now has decreasing returns in K , unlike $R(Z_t)K_t$, which has constant returns in K ? Why or why not?

- d. Do you find any of the results of parts a–c surprising? Why? Do you find any of the results intuitive? Why?
- 3. Consider a consumption-savings-portfolio choice problem with terminal condition $V^{T+h}(B_{T+h}) \equiv 0$ and Bellman equation

$$\begin{aligned} V^t(B_t) = \max_{S_t^+, S_t^-} & U(B_t - S_t^+ - S_t^-) \\ & + pe^{-\rho h} E_t V^{t+h}(R_t^+ S_t^+ + \epsilon^t(\omega_{t+h})) \\ & + (1-p)e^{-\rho h} E_t V^{t+h}(R_t^- S_t^- + \epsilon^t(\omega_{t+h})) \end{aligned}$$

where the expectation E_t only applies to ϵ and ϵ always has a mean of zero. Writing the argument of U as C_t , $U'(C_t) > 0$.

- a. Show that V is *strictly* increasing in B . In parts b–f, you may assume this result and treat V as strictly increasing in B . Hint: to prove V is *strictly* increasing, just modify the proof technique of the preser-max theorem by using > 0 instead of ≥ 0 .
- b. Decreasing *relative* risk aversion is defined as the property of $V(\alpha B)$ being an increasing, convex function of $V(B)$. Consider the following property P , that for any real number $\alpha > 1$,

$$[V(\alpha B) - V(B)][V(\alpha^3 B) - V(\alpha^2 B)] - [V(\alpha^2 B) - V(\alpha B)]^2 \geq 0$$

Show that decreasing relative risk aversion in this sense of $V(\alpha B)$ being an increasing convex function of $V(B)$ implies property P .

- c. Argue informally that if $V(B)$ is continuous, property P implies decreasing absolute risk aversion in the sense above.
- d. Show that if V is thrice-differentiable, that decreasing relative risk aversion in the sense of $V(\alpha B)$ being an increasing convex function of $V(B)$ for any $\alpha > 1$ implies that the relative risk aversion index $\frac{-BV''(B)}{V'(B)}$ is decreasing in B .
- e. Show that V given by the Bellman equation above has the property P if the underlying utility function U has decreasing relative risk aversion. You may freely use the fact that the sum of several functions with decreasing relative risk aversion also has decreasing relative risk aversion. That is, if $U_i(C)$ each have decreasing relative risk aversion, then $U(C) = \sum_i U_i(C)$ also has decreasing relative risk aversion.
- f. Can you extend your proof in part e to the case where p , R^+ , R^- and ϵ^t are functions of a vector of exogenous state variables Z_t ? Explain.