Problem Set # 2: The Symmetry Theorem
Economics 609, Winter 2003

Due: Thursday, January 30, 2:30 P.M.

1. Consider the firm problem

$$V^t(K_t) = \max_{U,\mathcal{E},H,N,M,I} E_t \sum_{\substack{\tau=t+jh \\ j\in\{0,1,\ldots\}}} \frac{D_\tau}{D_t}[P_{Y,\tau}F(U_\tau K_\tau, \mathcal{E}_\tau H_\tau N_\tau, M_\tau; Z_\tau) - W_\tau G^\tau(H_\tau, \mathcal{E}_\tau, U)N_\tau - P_{M,\tau}M_\tau - P_{I,\tau}K_\tau\Phi(I_\tau/K_\tau)]$$

s.t.,

$$K_{t+h} = K_t\Gamma(I_\tau/K_\tau, U; Z_\tau).$$

The intertemporal pricing kernel D, the output price P_Y, the factor prices W, P_M and P_I and the technology vector Z are stochastic, but exogenous to the firm. U is the workweek of capital. $\mathcal{E}$ is worker effort per hour. H is weekly hours for each worker. N is the number of workers. M is the quantity of materials used. I is investment. The superscripts on V and G indicate that these functions are time-varying.

   a. Write down the Bellman equation (the recursive equation) for the value function.

   b. Show that if the function F has constant returns to scale in the three arguments other than technology—namely UK, $\mathcal{E}HN$, and M—that the value function has constant returns to scale in K. Show that this guarantees that average Q ($= V(K)/K$) is equal to marginal q ($= V'(K)$). Besides a constant-returns to scale production function, what other assumptions above are critical to this result? Note that the surface complexity of the problem above is meant to indicate a wide variety of things the result of constant returns to scale for $V(K)$ does *not* depend on. If you have trouble with this problem, follow Polya's advice: do a simpler, related problem first, then gradually add back in the extra complexity of the original problem.

2. *Rate of Time Symmetry.* Let preferences in a maximization problem be given by

$$v_t = hU(X_t) + e^{-\rho h} E_t\, v_{t+h},$$

$v_T \equiv 0$. The contemporaneous constraint does not depend on K_t:

$$X_t \in \mathcal{X}$$

The transition equation for K (=intertemporal budget constraint) is given by

$$K_{t+h} = K_t + h[rK_t + f(X_t)] + K_t \sigma g(X_t) \sqrt{h}\, \epsilon_{t+h}$$

where ϵ_{t+h} is an i.i.d. binomial random variable equal to +1 with probability .5 and equal to -1 with probability .5.

Since we will be considering a transformation that alters the rate at which time passes, label everything by the number of periods from the terminal date rather than by a date. Time t corresponds to the $[n+1]$st period from end and time $t+h$ corresponds to the $[n]$th period from the end. After doing this relabeling, consider the following transformation of the constraints and preferences:

$$\begin{aligned}
h &\to \frac{h}{\theta} \\
\rho &\to \theta\rho \\
r &\to \theta r \\
\sigma &\to \sqrt{\theta}\,\sigma \\
v &\to \frac{v}{\theta} \\
K &\to \frac{K}{\theta} \\
X &\to X.
\end{aligned}$$

a. Show that this transformation is a symmetry of the constraints and the preferences.

b. Consider the limit of the original maximization problem as the time interval $h \to 0$ and the end-date $T \to \infty$:

$$V^t(K_t; \rho, r, \sigma) = \max E_t \int_0^\infty e^{-\rho(\tau - t)} U(X_\tau)\, d\tau$$

s.t.,

$$X \in \mathcal{X}$$

and

$$dK_t = [rK_t + f(X_t)]dt + K_t \sigma g(X_t) dz.$$

Argue that with U, r, f and g not having time as an argument, V is not in fact a direct function of time. Then use the symmetry theorem and find what implications there are for the form of V as a result of the symmetry of the value function with respect to the *rate of time* transformation above. This turns out to be a very useful result. It hinges on recognizing the difference between stocks, flows and pure rates.