Problem Set # 1: The Continuous-Time Bellman Equation for General Recursive Utility
Economics 609, Winter 2003

Due: Tuesday, January 21, 2:30

Consider the case of preferences given by the general form

$$v_t = \Psi^t(K_t, X_t, E_t\, v_{t+h}; h).$$

When $h = 0$, the function Ψ satisfies

$$\Psi^t(K_t, X_t, v_t; 0) = v_t$$

Also, the function Ψ has continuous first derivatives. The direct partial derivative with respect to the last argument h after the semicolon has the limit as $h \to 0$ of

$$\Psi_h(K_t, X_t, v_t; 0) = G(K_t, X_t, v_t).$$

The contemporaneous constraint is of standard form:

$$X_t \in \mathcal{X}^t(K_t).$$

The transition equation for K_t (=intertemporal budget constraint) is given by

$$K_{t+h} = K_t + hA^t(K_t, X_t) + \sqrt{h\Omega^t(K_t, X_t)}\epsilon_{t+h}$$

where ϵ_{t+h} is an i.i.d. binomial random variable equal to +1 with probability .5 and equal to -1 with probability .5.

1. Write down the recursion equation for the value function $V^t(K_t)$ in discrete time with time interval h.
2. The continuous-time value function $V^t(K_t)$ satisfies the limit of what this recursion equation becomes when $h \to 0$. Take this limit, using L'Hopital's rule where appropriate. Assume that V has continous second derivatives and find a partial differential equation for $V^t(K_t)$. This partial differential equation will have maximization in the middle of it. Show your derivation.

Hint: Treat $V^{t+h}(K_t)$ as a function of h only through the time superscript $t + h$. The justification for this is that we are solving for the continuous-time differential equation that applies to whatever the continuous-time limit of $V^t(K_t)$ is. So $V^t(K_t)$ is like the unknown we are trying to solve for. And we are only trying to solve for the limit of $V^t(K_t)$ when $h \to 0$. In any case, you will have a hard time doing the problem if you try to treat $V^t(K_t)$ as a function of h, or $V^{t+h}(K_t)$ as a function of h other than through the time superscript $t + h$.