Submission to the *Journal of Answers to Gollier*

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Exercise 47

Prove that $h(x, \theta) = I\ (x \leq \theta)$ *is LSPM.*

Solution 47

Note that $I(\cdot)$ is the indicator function and LSPM is the abbreviation for log-supermodularity.

The solution will use the definition of log-supermodularity. A function h is log-supermodular if for all $\mathbf{z}, \mathbf{y} \epsilon R^n$,

$$h(\mathbf{y} \vee \mathbf{z})\ h(\mathbf{y} \wedge \mathbf{z}) \geq h(\mathbf{y})h(\mathbf{z}), \tag{1}$$

where the $\vee$ and $\wedge$ operators are defined as in Gollier (p.99).

Proof: Since the range of $I(\cdot)$ is $\{0, 1\}$, a sufficient condition for log-supermodularity is that the left-hand side of equation (1) equal one whenever the right-hand side is one. This is intuitive. If the right-hand side equalling one implies that the left-hand side is also one, then the right-hand side can never be larger than the left. Thus, equation (2) states a sufficient condition for the proof, and the solution will use this condition:

$$h(\mathbf{y})h(\mathbf{z}) = 1 \quad \Rightarrow \quad h(\mathbf{y} \vee \mathbf{z})\ h(\mathbf{y} \wedge \mathbf{z}) = 1. \tag{2}$$

Next, note the following:

$$h(\mathbf{y})h(\mathbf{z}) = 1 \quad \Rightarrow \quad h(\mathbf{y}) = 1 \textbf{ and } h(\mathbf{z}) = 1.$$

The notation is suboptimal, but let $\mathbf{y} = (x_y, \theta_y)$ and $\mathbf{z} = (x_z, \theta_z)$. Reconsider the above condition in terms of x and θ:

$$h(\mathbf{y}) = 1 \textbf{ and } h(\mathbf{z}) = 1 \quad \Rightarrow \quad x_y \leq \theta_y \textbf{ and } x_z \leq \theta_z.$$

When both of these conditions hold, it implies the result in terms of the $\vee$ and $\wedge$ operators:

$$\begin{aligned} x_y \leq \theta_y \textbf{ and } x_z \leq \theta_z \quad &\Rightarrow \quad \inf\{x_y, x_z\} \leq \inf\{\theta_y, \theta_z\} \\ &\Rightarrow \quad h(\mathbf{y} \wedge \mathbf{z}) = 1. \end{aligned}$$

Likewise:

$$\begin{aligned} x_y \leq \theta_y \textbf{ and } x_z \leq \theta_z \quad &\Rightarrow \quad \sup\{x_y, x_z\} \leq \sup\{\theta_y, \theta_z\} \\ &\Rightarrow \quad h(\mathbf{y} \vee \mathbf{z}) = 1. \end{aligned}$$

Thus, we have shown that the indicator function $I\ (x \leq \theta)$ satisfies equation (2), which is a sufficient condition for equation (1). *QED*.

Note that this process does not tell us if $h(\mathbf{y} \vee \mathbf{z})\ h(\mathbf{y} \wedge \mathbf{z})$ is one or zero when $h(\mathbf{y}) = 1$ and $h(\mathbf{z}) = 0$ (or vice-versa), but this is not required to satisfy the weak inequality in the definition of log-supermodularity.