Answer to Gollier Problem 113

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(Lehmann 1988) Consider location experiments with $\tilde{m}^i|x = x + \tilde{\varepsilon}^i$. Suppose that $\tilde{\varepsilon}^1$ and $\tilde{\varepsilon}^2$ are both uniformly distributed, with $\tilde{\varepsilon}^1 \sim U[-\rho, \rho]$ and $\tilde{\varepsilon}^2 \sim U[-1, 1]$. Show that structure 1 is finer than structure 2 if ρ is the inverse of a positive integer.

Proof:

By the definition based on the technique of garbling messages, structure 1 is finer than structure 2 if the messages of structure 2 are obtained by garbling the messages coming from structure 1. So, if we can find a garbling machine characterized by a function μ and a random variable $\tilde{\xi}$ independent of $\tilde{m}^1$ s.t. $\tilde{m}^2$ is distributed as $\mu(\tilde{m}^1, \tilde{\xi})$ $\forall x$, then we are done.

Suppose ρ is the inverse of some positive integer, n.

Let $\mu(m, \xi) = m + \xi$ where

$$\xi = \begin{cases} \frac{n-1}{n} & \text{wpr } \frac{1}{n} \\ \frac{n-3}{n} & \text{wpr } \frac{1}{n} \\ \cdot & \cdot \ \cdot \\ \cdot & \cdot \ \cdot \\ -\frac{n-3}{n} & \text{wpr } \frac{1}{n} \\ -\frac{n-1}{n} & \text{wpr } \frac{1}{n} \end{cases}$$

Then, $\tilde{m}^2|x$ is distributed as $\mu(\tilde{m}^1, \tilde{\xi})$ $\forall x$ since $\mu(\tilde{m}^1, \tilde{\xi}) = \tilde{m}^1 + \tilde{\xi} = x + \tilde{\varepsilon}^1 + \tilde{\xi}$ where $\tilde{\varepsilon}^1 \sim U[-\frac{1}{n}, \frac{1}{n}]$. So, we showed the existence of such $\mu(\cdot)$.

Intuitively, we can see that signal $\tilde{m}^2$ is centered at the mean of $\tilde{m}^1$ since $E[\tilde{\xi}] = 0$ by construction, but $\tilde{m}^2$ is more dispersed than $\tilde{m}^1$ since the variability of $\tilde{m}^2$ comes from both $\tilde{m}^1$ and $\tilde{\xi}$. (Alternatively, since μ takes the form $\mu(m, \xi) = m + \xi$, it is sufficient to compare the dispersion of the errors for the comparison of two location experiments, as proved by Boll (1955). In this problem, $\tilde{\varepsilon}^2$ is more dispersed than $\tilde{\varepsilon}^1$ by the assumption of ρ.)

Therefore, structure 1 is finer than structure 2. ■