Submission to the *Journal of Answers to Gollier*

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Exercise 3 (Rank Dependent EU)
Consider the lottery $L = (x_1, p_1; \ldots; x_s, p_s)$ with $x_1 < x_2 < \ldots < x_s$. Quiggin (1982) proposed a generalization of the EU model by introducing a function g of transformation of the ***cumulative*** *distribution:*

$$U(L) = \sum_{s=1}^{S} u(x_s) \left[g\left(\sum_{t=1}^{s} p_t\right) - g\left(\sum_{t=1}^{s-1} p_t\right) \right]$$

with $g(0) = 0$ and $g(1) = 1$.
• *Draw the indifference curves in the Machina triangle for $g(q) = q^2$ and $g(q) = \sqrt{q}$.*
• *Which restriction on g would you consider to solve the Allais paradox?*

Solution 3
Part 1: See figures below.
Part 2: $g(q) = \sqrt{q}$ will "solve" the Allais paradox from Table 1.1. More generally, any **concave** function g will solve the particular Allais paradox represented in the text.
To see this, it is useful to first rewrite Table 1.1 in the following way:

Modified Table 1.1

Probability of Lottery	0	50	250
L_a	0	1	0
L_b	.01	.89	.1
M_a	.89	.11	0
M_b	.9	0	.1

This makes clear the utility of each lottery, in terms of the modified U:
· $U(L_a) = u_{50}$
· $U(L_b) = u_0[g(.01) - g(0)] + u_{50}[g(.9) - g(.01)] + u_{250}[g(1) - g(.9)]$
· $U(M_a) = u_0[g(.89) - g(0)] + u_{50}[g(1) - g(.89)]$
· $U(M_b) = u_0[g(.9) - g(0)] + u_{250}[g(1) - g(.9)]$
We can simplify these terms by normalizing $u_0 = 0$ and by noting that $g(0) = 0$ and $g(1) = 1$. Recall that the Allais paradox demonstrates that no expected utility function can produce both $L_a \succ L_b$ and $M_b \succ M_a$. The generalized model with $g(q) = \sqrt{q}$ does not generate a contradiction:

$$U(L_a) \quad > \quad U(L_b) \Rightarrow$$

$$
\begin{aligned}
u_{50} &> u_{50}[g(.9) - g(.01)] + u_{250}[1 - g(.9)] \Rightarrow \\
.1513\ u_{50} &> .0513\ u_{250} \Rightarrow \\
2.95\ u_{50} &> u_{250}
\end{aligned}
$$

$$
\begin{aligned}
U(M_b) &> U(M_a) \Rightarrow \\
u_{250}[1 - g(.9)] &> u_{50}[1 - g(.89)] \Rightarrow \\
.0513\ u_{250} &> .0566\ u_{50} \Rightarrow \\
u_{250} &> 1.10\ u_{50}
\end{aligned}
$$

Thus, $2.95\ u_{50} > u_{250} > 1.10\ u_{50}$, which generates no contradiction, solving the paradox. Naturally, the square root function is not unique in this respect. In fact, a little manipulation shows that any strictly concave function that is increasing on the interval $[0, 1]$ will work:

$$
\begin{aligned}
U(L_a) &> U(L_b) \Rightarrow \\
u_{50} &> u_{50}[g(.9) - g(.01)] + u_{250}[1 - g(.9)] \Rightarrow \\
u_{50} - u_{250} &> u_{50}[g(.9) - g(.01)] - u_{250}g(.9) \\
U(M_b) &> U(M_a) \Rightarrow \\
u_{250}[1 - g(.9)] &> u_{50}[1 - g(.89)] \Rightarrow \\
u_{50}g(.89) - g(.9)u_{250} &> u_{50} - u_{250}
\end{aligned}
$$

Together, these conditions imply:

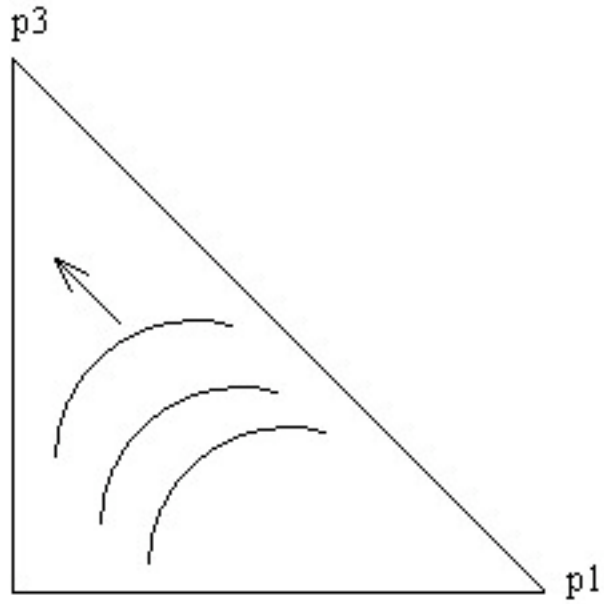
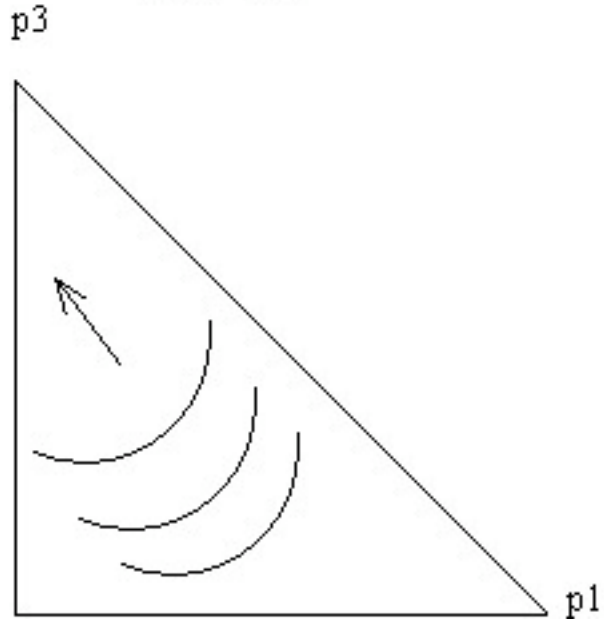
$$u_{50}g(.89) - g(.9)u_{250} > u_{50} - u_{250} > u_{50}[g(.9) - g(.01)] - u_{250}g(.9) \Rightarrow$$

$$
\begin{aligned}
u_{50}g(.89) - g(.9)u_{250} &> u_{50}[g(.9) - g(.01)] - u_{250}g(.9) \Rightarrow \\
u_{50}g(.89) &> u_{50}[g(.9) - g(.01)] \Rightarrow \\
g(.89) + g(.01) &> g(.9) \Rightarrow {}^{1}
\end{aligned}
$$

This will be true of any strictly concave function g that is increasing on the interval [0,1]. The last inequality is in the form $g(a + b) - g(a) > g(0 + b) - g(0), a, b > 0$, which, loosely, says that a b unit increase in the function is larger when the starting value is above zero. The square root function is one example.

[1] This uses $u_{50} > 0$, which must be true because we normalized $u_0 = 0$.

$g(q) = \sqrt{q}$



$g(q) = q^2$