Econ 609 Partial Notes
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1 The Symmetry Theorem

1.1 Statement of Theorem

Suppose we have a Bellman Equation of the following form:

$$V^t(K_t) = \max_X \Psi\left(K_t, X_t, \Omega(V^{t+1}(K_{t+1})\right)$$

- K: vector of state variables.
- X: vector of control variables.
- Ω is any function that maps a distribution function into a scalar.
  - Typically, $\Omega \equiv \mathbb{E}_t$
  - Ω must respect First Order Stochastic Dominance.
- Transition equation is given by $K_{t+1} = \Gamma\left(K_t, X_t, \omega_{t+1}\right)$

If:
(a) There is a symmetry of the contemporaneous constraints

$$(K, X) \in \mathcal{W} \rightleftarrows (T_K(K), T_X(K, X)) \in \mathcal{W}$$

- $\mathcal{W}$ is the constraint set
- T is the transformation
- A triangular transformation can be represented by:

$$\hat{K} = T_K(K)$$
$$\hat{X} = T_X(K, X)$$

- The way you change the control variable may need to depend on the state variable.
- For example,

$$\hat{K} = \theta K$$
$$\hat{X} = \theta X$$

(b) There is a symmetry of the intertemporal constraint

$$K_{t+1} = \Gamma\left(K_t, X_t, \omega_{t+1}\right) \Leftrightarrow T_K(K_{t+1}) = \Gamma\left(T_K(K_t), T_X(K_t, X_t), \omega_{t+1}\right)$$
$$\Leftrightarrow T_K\left(\Gamma\left(K_t, X_t, \omega_{t+1}\right)\right) = \Gamma\left(T_K(K_t), T_X(K_t, X_t), \omega_{t+1}\right)$$

- The transformation *can* depend on time. Just treat the time index as another state variable.

(c) There is a symmetry of preferences associated with the transformation.

$$\nu_t = \Psi\left(K_t, X_t, \Omega(F_{\nu(t+1)})\right)$$

- ν_t is the non-maximized lifetime utility.
- Typically, $\Omega \equiv \mathbb{E}_t$

$$S(K_t, \nu_t) = \Psi\left(T_K(K_t), T_X(X_t), \Omega(F_{S(K,\nu)})\right)$$

For example,

$$\nu_{t+1} \rightarrow \theta^{1-\gamma}\nu_{t+1} \Rightarrow \nu_t \rightarrow \theta^{1-\gamma}\nu_t$$

(d) Terminal condition. At τ, the end of time, we have:

$$V^{\tau}\left(T_K(K_\tau)\right) = S\left(K_\tau, V(K_\tau)\right)$$

If (a)-(d), then $V^t\left(T_K(K)\right) = S\left(K, V(K)\right)$

- "S on the outside does the same thing as T on the inside"
- We want a result that depends only on K; cannot have X in this expression.

1.2 Proof

First show:

$$V^t\left(T_K(K)\right) - S\left(K, V^t(K)\right) \geq 0$$

Resources:

1. Recursion Hypothesis

$$V^{t+1}\left(T_K(K_{t+1})\right) - S\left(K_{t+1}, V^{t+1}(K_{t+1})\right) \geq 0$$
$$V^{t+1}\left(T_K\left(\Gamma\left(K_t, X_t^*, \tilde{\omega}_{t+1}\right)\right)\right) - S\left(\Gamma\left(K_t, X_t^*, \tilde{\omega}_{t+1}\right), V^{t+1}(\Gamma\left(K_t, X_t^*, \tilde{\omega}_{t+1}\right))\right) \geq 0$$

   where X_t^* is the "sloppy" value of the choice variable that maximizes the second term, but not the first.

2. Symmetry of Transition Equation

$$\Gamma\left(T_K(K_t), T_X(K_t, X_t), \tilde{\omega}_{t+1}\right) = T_K\left(\Gamma\left(K_t, X_t, \tilde{\omega}_{t+1}\right)\right)$$

3. Symmetry of Preferences

$$S(K_t, \nu_t) = \Psi\left(T_K(K_t), T_X(K_t, X_t), \Omega\left(S\left(\Gamma\left(K_t, X_t, \tilde{\omega}_{t+1}\right), \nu_{t+1}\right)\right)\right)$$

   where $\nu_t = \Psi\left(K_t, X_t, \Omega(\hat{\nu}_{t+1})\right)$ and $\hat{\nu}_{t+1} = S(K_{t+1}, \nu_{t+1})$

Rewrite using the Bellman Equation:

$$
\begin{aligned}
V^t\left(T_K(K)\right) - S\left(K, V^t(K)\right) = &\max_X \Psi\left(T_K(K), X, \Omega_t\left(V^{t+1}\left(\Gamma\left(T_K(K), X, \tilde{\omega}_{t+1}\right)\right)\right)\right) \\
&- S\left(K, \max_X \Psi\left(K, X, \Omega\left(V^{t+1}\left(\Gamma\left(K, X, \tilde{\omega}_{t+1}\right)\right)\right)\right)\right)
\end{aligned}
$$

Assume $X^* \in \arg\max \Psi$. By the Preser-Max Theorem:

$$
\begin{aligned}
\geq &\Psi\left(T_K(K), T_X(K, X^*), \Omega_t\left(V^{t+1}\left(\Gamma\left(T_K(K), T_X(K, X^*), \tilde{\omega}_{t+1}\right)\right)\right)\right) \\
&- S\left(K, \Psi\left(K, X^*, \Omega\left(V^{t+1}\left(\Gamma\left(K, X^*, \tilde{\omega}_{t+1}\right)\right)\right)\right)\right)
\end{aligned}
$$

By the symmetry of the intertemporal constraint (2) and the symmetry of preferences [lifetime utility] (3):

$$
\begin{aligned}
= &\Psi\left(T_K(K), T_X(K, X^*), \Omega_t\left(V^{t+1}\left(T_K\left(\Gamma(K, X^*, \tilde{\omega}_{t+1})\right)\right)\right)\right) \\
&- \Psi\left(T_K(K), T_X(K, X^*), \Omega_t\left(S\left(\Gamma(K, X^*, \tilde{\omega}_{t+1})\right), V^{t+1}\left(\Gamma(K, X^*, \tilde{\omega}_{t+1})\right)\right)\right) \\
= &\Psi\left(T_K(K), T_X(K, X^*), \Omega_t\left(V^{t+1}\left(T_K\left(K_{t+1}\right)\right)\right)\right) \\
&- \Psi\left(T_K(K), T_X(K, X^*), \Omega_t\left(S\left(K_{t+1}\right), V^{t+1}\left(K_{t+1}\right)\right)\right)
\end{aligned}
$$

So by the recursion hypothesis, and assuming Ψ increasing in ν_{t+1}, the previous expression in ≥ 0.

Next need to show:

$$
\begin{aligned}
&S\left(K, V^t(K)\right) - V^t\left(T_K(K)\right) \geq 0 && \forall K \\
\Leftrightarrow &S\left(T_K^{-1}(K), V^t\left(T_K^{-1}(K)\right)\right) - V^t\left(K\right) \geq 0 && \forall K
\end{aligned}
$$

Resources:

1. Recursion Hypothesis

$$
\begin{aligned}
S\left(T_K^{-1}(K_{t+1}), V^{t+1}\left(T_K^{-1}(K_{t+1})\right)\right) - V^{t+1}\left(K_{t+1}\right) \geq 0 \\
S\left(T_K^{-1}\left(\Gamma\left(K_t, X_t, \tilde{\omega}_{t+1}\right)\right), V^{t+1}\left(T_K^{-1}\left(\Gamma\left(K_t, X_t, \tilde{\omega}_{t+1}\right)\right)\right)\right) - V^{t+1}\left(\Gamma\left(K_t, X_t, \tilde{\omega}_{t+1}\right)\right) \geq 0
\end{aligned}
$$

2. Symmetry of Transition Equation

$$
\begin{aligned}
\Gamma\left(T_K\left(T_K^{-1}(K_t)\right), T_X\left(T_K^{-1}(K_t), T_X^{-1}(K_t, X_t)\right), \tilde{\omega}_{t+1}\right) &= T_K\left(\Gamma\left(T_K^{-1}(K_t), T_X^{-1}(K_t, X_t), \tilde{\omega}_{t+1}\right)\right) \\
T_K^{-1}\left(\Gamma\left(K_t, X_t, \tilde{\omega}_{t+1}\right)\right) &= \Gamma\left(T_K^{-1}(K_t), T_X^{-1}(K_t, X_t), \tilde{\omega}_{t+1}\right)
\end{aligned}
$$

   If T commutes with Γ then T^{-1} also commutes with Γ.

3. Symmetry of Preferences

$$
S\left(T_K^{-1}(K_t), \nu_t\right) = \Psi\left(K_t, X_t, \Omega\left(S\left(\Gamma\left(T_K^{-1}(K_t), T_X^{-1}(K_t, X_t), \tilde{\omega}_{t+1}\right), \nu_{t+1}\right)\right)\right)
$$

Rewrite using the Bellman Equation:

$$
\begin{aligned}
S\left(T_K^{-1}(K), V^t\left(T_K^{-1}(K)\right)\right) - V^t(K) =& S\left(T_K^{-1}(K), \max_X \Psi\left(T_K^{-1}(K), X, \Omega_t\left(V^{t+1}\left(\Gamma\left(T_K^{-1}(K), X, \tilde{\omega}_{t+1}\right)\right)\right)\right)\right) \\
&- \max_X \Psi\left(K, X, \Omega_t\left(V^{t+1}\left(\Gamma\left(K, X, \tilde{\omega}_{t+1}\right)\right)\right)\right)
\end{aligned}
$$

Assume $X^* \in \arg\max \Psi$. By the Preser-Max Theorem:

$$
\begin{aligned}
\geq & S\left(T_K^{-1}(K), \Psi\left(T_K^{-1}(K), T_X^{-1}(K, X^*), \Omega_t\left(V^{t+1}\left(\Gamma\left(T_K^{-1}(K), T_X^{-1}(K, X^*), \tilde{\omega}_{t+1}\right)\right)\right)\right)\right) \\
& - \Psi\left(K, X^*, \Omega_t\left(V^{t+1}\left(\Gamma\left(K, X^*, \tilde{\omega}_{t+1}\right)\right)\right)\right)
\end{aligned}
$$

By the symmetry of the intertemporal constraint and the symmetry of preferences:

$$
\begin{aligned}
=& S\left(T_K^{-1}(K), \Psi\left(T_K^{-1}(K), T_X^{-1}(K, X^*), \Omega_t\left(V^{t+1}\left(T_K^{-1}\left(\Gamma\left(K, X^*, \tilde{\omega}_{t+1}\right)\right)\right)\right)\right)\right) \\
& - \Psi\left(K, X^*, \Omega_t\left(V^{t+1}\left(\Gamma\left(K, X^*, \tilde{\omega}_{t+1}\right)\right)\right)\right) \\
=& \Psi\left[K, X^*, \Omega_t\left\{S\left(\Gamma\left(T_K^{-1}(K), T_X^{-1}(K, X^*), \tilde{\omega}_{t+1}\right), \Omega_t\left(V^{t+1}\left(T_K^{-1}\left(\Gamma\left(K, X^*, \tilde{\omega}_{t+1}\right)\right)\right)\right)\right)\right\}\right] \\
& - \Psi\left(K, X^*, \Omega_t\left(V^{t+1}\left(\Gamma\left(K, X^*, \tilde{\omega}_{t+1}\right)\right)\right)\right) \\
& \geq 0
\end{aligned}
$$

So by the recursion hypothesis, and assuming Ψ increasing in its last argument, the previous expression in ≥ 0.