Economics 609
Winter 2005

Project: what to do when certainty equivalence isn’t okay.

Theoretical Techniques:
1. Difference theorem - this is a key thing of utility expectations.
(Gollier book)
2. Symmetry Method
   - implicitly, this is really how most guess-and-verify methods work.
   - relates to scale symmetry.
   - not so good for breaking ground, but useful as part of solution.
3. Horizontal & Vertical Integration
   (of marginal value function)
   - good example in Miles’ papers w/ Chris Carroll
4. Methods for Proving the Preservation of Properties
   - i.e., go beyond \( V' \geq 0 \), \( V' \leq 0 \)
5. We won’t do this, but Miles’ other favorite method is the perturbation method.

Why you should start w/ simple (1-2) period problem
1. since 2 is a special case of multi, if you can’t get it w/ 2...
2. intuition
3. chance to figure out the tools you need.

Types of Utility Functions

1. Intertemporal Expected Utility Function
   \[
   E_t \sum_{j=0}^{\infty} \beta^j U_{t+j} (C_{t+j}) = V_t
   \]
   Lifetime utility (like values function before it is squared)
2. Kreps-Porteus (recursive)
More Exotic

\[ v_t = \mathbb{E} \left( c_{t+1}, F_{t+1} \right) \]

Tools of this class work \( w/ \) any of these types of utility functions.

Note: can also express (1) in recursive form.

\[ V_t = u_t(c_t) + \beta E_t V_{t+1} \]

Bellman's equation is basically doing this \( w/ \) a maximization. Start solving at \( \text{end} \), then once you've optimized that step, work your way backward. Like end of tree.

-Can deal w/ this even \( w/ \) infinite horizon.

Length of period is very important - might need to vary it to be convincing.

A Few Words about Expected Utility

- Mikes believes it is the "right" way to make decisions
- but not necessarily what people really do.

However, normative sense of EU makes it still important & valid to think about.

see p. 12 of Allais, Collier for commentary on this
- show that violating expected utility theory (i.e. Allais paradox)

\[ Pf \equiv \text{portfolio} \]
So: that picture just changes whether you choose ex post or ex ante.
Suppose in first case, you always choose A. Then you should be happy to make choice before coin toss.

This is really the Independence Axiom. (Independence Axiom is most controversial.)

When Miles talked about "more exotic" utility functions, they may not obey the independence axiom. Kreps-Porteus is in-betweem. KP obeys vNM axioms & if you realize outcome in the short-term but don't obey them if time to realization is long - somehow this has to do w/ discount rate, but I missed the details.

We won't deal w/ problems of information. (Loves is the expert on that.)

Traditional Approach to Risk Aversion.

Defn. of Risk Aversion.

\[ \forall x, E[x] = 0 \Rightarrow E[u(w_0 + x)] \leq u(w_0) \]
(in words, you don't like risk...)
If true \( \neq w_0 \), we have global risk aversion.

To have this be true \( \neq W \), what do you need to have?

Need concavity

Take that result as known. (Can prove that result w/ difference theorem.)

Define \( u_1 \), more risk averse than \( u_2 \) if:

If \( E u_2 (w_0 + \tilde{x}) \leq u_2 (w_0) \), then

\[ E u_1 (w_0 + \tilde{x}) \leq u_1 (w_0). \]

(Risks that \( u_2 \) would reject are also rejected by \( u_1 \).

Why we define it this way: can apply it to utility types that you can't apply to all utility functions.

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Pratt: 1964 - Risk Aversion in the Small & the Large.

Define \( \phi \) by \( u_1 (z) = \phi(u_2(z)) \).

If \( E u_2 (w_0 + \tilde{x}) \leq u_2 (w_0) \), then

\[ E \phi(u_2 (w_0 + \tilde{x})) \leq \phi(u_2 (w)). \]
\[ E\tilde{y} \leq 0 \Rightarrow Eu(w + \tilde{y}) \leq u(w) \]

Define \( y = u_2(\omega_0 + \tilde{x}) - u_2(\omega_0) \)

\[ E\tilde{y} \leq 0 \]

\[ \downarrow \]

\[ E\phi(\tilde{y}) \leq \phi(0) \]

So: \( \phi \) has to be concave, & increasing.

Out of this, you can get Arrow–Pratt.
Answer format:
1. type in question itself
2. pdf file
3. graphs in PPT

Direct $\delta$ function (engineering math)
$\delta(x-a)$

\[
\int_{-\infty}^{\infty} \delta(x-a) \, dx = 1
\]

\[
\int_{-\infty}^{\infty} f(x) \delta(x-a) \, dx = f(a)
\]

\[
\int_{-\infty}^{a} \delta(x-a) \, dx = 0 \quad \text{if} \quad x \leq a
\]

\[
\int_{a}^{\infty} \delta(x-a) \, dx = 1 \quad \text{if} \quad x \geq a
\]

So that integral is a step function.

We sometimes call the step function a Heaviside step function $H(x-a)$.

\[
\int_{-\infty}^{x} H(\frac{\xi}{\delta} - a) \, d\xi = H(x-a)
\]
\[ \int_{-\infty}^{x} J(\xi - a) \, d\xi \]

Why we care: we can construct utility functions from convex combinations of these (or positive linear combinations).

\[ \int_{-\infty}^{x} u'(\xi) \, H(x - \xi) \, d\xi = ? \]

Think of \( u \geq 0 \) here.

Use integration by parts:

\[ \int_{-\infty}^{x} u'(\xi) \, H(x - \xi) \, d\xi = \left| u(\xi) \right|_x^{\infty} + \int_{-\infty}^{x} u(\xi) \, \delta(x - \xi) \, d\xi \]

Use fact that \( H(x - \xi) = \begin{cases} 1 & \text{if } x \geq \xi \\ 0 & \text{if } x \leq \xi \end{cases} \)

\[ \Rightarrow \text{we only need to integrate over region where } x \geq \frac{a}{2} \]

\[ \int_{-\infty}^{x} u'(\xi) \, H(x - \xi) \, d\xi = \int_{x}^{\infty} u'(\xi) \, d\xi \]

\[ = u'(x) \bigg|_{x}^{\infty} - u(x) \]

\[ = u(x) - C \]

So mechanically, any utility function is a positive monotonic combo of heavyside function.
\[ \int_{x}^{\infty} u'(\xi) H(x - \xi) d\xi \propto \frac{\int_{x}^{\infty} u'(\xi) H(x - \xi) d\xi}{\int_{x}^{\infty} u'(\xi) d\xi} \]

the line above just shows you how to get weights that sum to 1.

Weights are determined by \( u'(\xi) \). Higher \( u' \implies \) higher weight.

Why is this useful? If something is true for every step function, it is often true for a monotonically increasing utility function. (works as long as property is preserved by possibly convex combinations.)

\( T \) is just set of monotonically increasing functions

Suppose \( T \) is set of functions that are monotonically increasing & concave.

(Sidenote: suppose I wanted to construct set of convex functions. I could probably do it with something like \( \int_{x}^{\infty} f'(\xi) H(x - \xi) d\xi \).

What we want is to make these functions via linear combinations of min functions.

\[ \int_{x}^{\infty} u'(\xi) \min(x - \xi, 0) d\xi \]

Now we'll use integration by parts

\[ = -u'(\xi) \min(x - \xi, 0) \bigg|_{x}^{\infty} + \int_{x}^{\infty} u'(\xi) \frac{H(x - \xi)}{\xi} d\xi \]

Note that deriv. of min function we'll call...
What we were trying to show: that you can get $u(x)$ back out.

Continuing:

$$= -u'(h)(x-h) + \int_0^x u'(\frac{u}{2})\,d\frac{u}{2}$$

$$= -u'(h)(x-h) + u(x) - u(0)$$

$$= u(x) - u(0) - u'(h)(x-h)$$

Why do we have this? Like an Inada condition. If $u'(\infty) = 0$, the min functions are perfect on their own, but otherwise, need this term.

This is really the same idea as finding everything in space $\Delta$ as the convex combinations of the vertices.

Thinking about a basis.

If we have $\square$ we have a simple basis (6 vertices)

If we have $\circ$, we need an infinite number of points...

Consider set of DARA utility functions.

If we take a convex combo of these, we get a strictly DARA function—we lost something.
\[ u_1(x) \leq \frac{-u''(x) - u''(x)}{u_1'(x) + u_2'(x)} \]

\[ \frac{1}{2} u_1(x) + \frac{1}{2} u_2(x) \Rightarrow \frac{d}{dx} \frac{u_1'(x)}{u_1(x) + u_2(x)} = \frac{u_1''(x)}{u_1(x) + u_2(x)} - \frac{u_1''(x)}{u_1(x) + u_2(x)} \]

\[ \frac{-u''(x)}{u_1'(x)} \geq \frac{-u''(x)}{u_1'(x)} \Rightarrow \frac{d}{dx} \frac{u_1'(x)}{u_1(x)} - \frac{u_1''(x)}{u_1'(x)} - \frac{u_1''(x)}{u_1'(x)} \leq 0 \]

Interestingly, if one's risk aversion, her preferences change when poor, less if rich. Claudio's paper.

Consider the set \( Y^* = \{ u \in Y' : u' \geq 0, u'' \leq 0, u''' = 0, u'''' = 0 \} \)

This set actually has a very simple basis: the set of exponential utility functions. \( e^{x \alpha} \)

DARA functions are actually harder to find a basis for than that set.

Defn. \( x_2 \succeq x_1 \) iff \( u(x_2) \geq u(x_1) \) \( \forall u \in \) set of monotonically increasing utility functions.

\[ u(x) = \int_{x}^{\infty} u(x) d \xi + u(0) \]

Suppose \( E[H(x_2 - \xi)] \geq E[H(x_1 - \xi)] \) \( \forall \xi \)

Then \( E u(x_2) \geq E u(x_1) \)
\textbf{Proof:} \\
\[ Eu(x_2) = \int_0^\infty u(\delta) H(x_2 - \delta) d\delta + u(\infty) \]
\[ = \int_0^\infty u(\delta) E_x H(x_2 - \delta) d\delta \]
\[ \geq \int_0^\infty u(\delta) E_x H(x_1 - \delta) d\delta \]
\[ = Eu(x_1) \]

So, what does it take to satisfy \[ E_x H(x_2 - \delta) \geq E_x H(x_1 - \delta) \]
\[ \int_0^\infty H(x_2 - \delta) dF_{x_2} \geq \int_0^\infty H(x_1 - \delta) dF_{x_1} \]
\[ \int_0^\infty dF(x_2) = 1 - F_2(\infty) \leq 1 - F_1(\infty) \]

This is the familiar condition for first-order stochastic dominance.

1/13/04 Consider the interval \([a, b]\). 
Think of degenerate pdf (all mass at some point). 
Convex combo of these pdfs give you all pdfs that have support \([a, b]\).

\[ E[f(x)] \geq 0 \quad \forall x \]
\[ f(x) \geq 0 \quad \forall x \]

If you want all mean-zero random variables; use all 2-point distributions. Picture to demonstrate this on the next page.
To have mean zero, we need:
\[ p_1 x_1 + p_2 x_2 + (1 - p_1 - p_2) x_3 = 0. \]

\[ x_3 > x_2 > x_1 \]

must be linear.

everything on line is convex combo of these two points (which are both two-point distributions).

If we are checking ANY PROPERTY THAT IS LINEAR IN THE PROBABILITIES (i.e. means), we can do it w/ one-point & two-point distributions.

Note: if \( E[x] = 0 \) and \( x_3 > x_2 > x_1 \), \( x_1 < 0 \). So our picture must look like:
Now we'll prove the likelihood ratio order,
\[
\frac{f_2(x)}{f_1(x)} = \Psi(x)
\]
\[\Psi(x)\] increasing in \(x\)

- one of the strongest conditions you can put on random variables.
- anything that is true for certainty is true under monotone likelihood ratio.

Conjecture: A set of extremals for \(f_2 \geq f_1\) is increasing \(\xi\) is:
\[
g(x, \xi) = \begin{cases} 
\frac{f_1(x)}{1 - f_1(\xi)} & x \geq \xi \\
0 & x < \xi \end{cases}
\]

(Optional: prove this instead of doing a Gollier problem.)

The problem w/ the MLRD property is it is very sensitive to small changes in the probability distribution.

An alternative property, which is easier to have, is the monotone probability ratio order.
\[
F(x) - \Omega(x) = \Psi(x)
\]
Set of extremals for \( \{ F_\ast \mid F_\ast (x)=\Theta(x)F_1(x), \Theta \in \mathbb{R}^3 \} \)

(we know that at \(b\) (end of support) \( \Theta(b)=1 \)
\( F_\ast (x) \leq F_1(x) \))

- this is strictly stronger than first-order stochastic dominance.

\[ \mathbb{E}_x \]

\[ \begin{array}{c}
1 \\
F_1 \\
F_2
\end{array} \]

this graph is okay for FSD, but not for MPRO.

So: \( F_\ast \geq F_1 \) \( \Rightarrow \) \( F_\ast \geq F_1 \)

Set of extremals:
\[ g(x, \bar{\zeta}) = H(x - \bar{\zeta})F_1(x) \]
(Note: this is defined in terms of a particular \( F_1 \).

what this does: it squishes up the mass of the tail at \( \bar{\zeta} \).

Proof (that \( F_\ast (x) = \int_\alpha^b H(x - \bar{\zeta})F_1(x) d\Theta(\bar{\zeta}) \))

What we are proving: choose any \( F_\ast (x) \) in the set that satisfies \( F_\ast (x) = \Theta(x)\Theta(\bar{\zeta})F_1(x) \). Show that it can be constructed from the extremals.
\[ F_2(x) = \int_a^b H(x-\xi) F_1(x) \, d\theta(\xi) \]

want range \( \xi \leq x \)

\[ = F_1(x) \int_a^x \theta(\xi) \, d\theta(\xi) \]

\[ = F_1(x) \theta(x) \]

(\( \theta(a) = 0 \) because \( a \) is the lower bound of our support \( \Rightarrow F_2(a) = F_1(a) = 0 \))

When to apply this: when the question is, Can you get a comparative statics result?

Now: Time to Start Setting Up the Diffidence Theorem.

\[ E[U_1(w + \xi)] \leq u_1(w) \Rightarrow E[U_2(w + \xi)] \leq u_2(w) \]

Think about this being true just for a particular value of \( w \), rather than \( w \).

Also, think of functions in general rather than utility functions.

**Economically Interesting Functions**

1. \( u(w + x) - u(w) = f_1(x, w) \)
2. \( f_2(x,w) = x u'(w + x) \)

Why is this interesting?

Consider the portfolio problem:

\[ \max_{\alpha} E[u(w + \alpha x)] \quad \text{where} \quad \alpha \text{ is return on stock} \]

- return on treasury bills
\[
\frac{\partial^2 u}{\partial x^2} = E x^2 u''(w + ax) e
\]

"normal" case

If \( ax \hat{a} 0 \) \( \Pr = 1 \)

Precautionary Saving

\[ U(c_1) + U(c_2) \]

What is right amount of saving? when 2nd period income

\[
\max_s U(w-s) + E u(s+x)
\]

2 diff functions builds in discounting

\[
\text{FOC: } -U'(w-s) + Eu'(s+x) = 0
\]

\[ U'(w-s) = Eu'(s+x) \]

Back to "Interesting" Functions

(3) \[ f_2(x,w) = u'(w+x) - u'(w) \]
Note: These pictures are also in paper by Kimball & Weil

Suppose \( E\ u'(s + \bar{x}) \geq u'(s) \). (This is what is shown in picture.)

- We can see that optimal amount is higher under uncertainty. Marginal value of saving is higher w/ uncertainty.

\[ E\ u'(s + \bar{x} + \psi^*) = u'(s) \]

meaning: increasing saving cancels out the effect of increased risk on the marginal utility

\[ E\ u'(s + \bar{x} + \psi^*) = u'(s) \geq 0 \]

\[ E\ u'(s + \bar{\gamma}) = u'(s) \geq 0 \]

Next interesting function

\( (4) \ u''(w) u^*(s + \bar{x}) + u''(w) u'(s + \bar{x}) \)

Define: \( \hat{u}(w) = E\ u(w + \bar{x}) \) (might like this if interested in a independent risks)

Suppose \( \bar{x} \& \bar{\gamma} \) are statistically independent,

\[ E\ E\ u(w + \bar{x} + \bar{\gamma}) = E\ u(w + \bar{x} + \bar{\gamma}) \]

When is... \( -\frac{\hat{u}(w)}{\hat{u}'(w)} \geq -\frac{u''(w)}{u'(w)} ? \) This is an interesting question.

\[ -\frac{E u''(w + \bar{x})}{E u'(w + \bar{\gamma})} \geq -\frac{u''(w)}{u'(w)} \]

\[ u' \geq u'(s + \bar{x}) \]
For Thurs: 3 referee reports
Take-home next Thurs

Today: Kimball & Gollier
Not on reading list but available on website
"New Methods in the Classical Economics of Uncertainty"
I. Comparing Risk   II. Characterizing Utility Functions

Also a Handout in big PDF file w/ notes from last year. - Start w/ table in that handout
Table for functions such that:

\[ E(w, x) \leq 0 \Rightarrow E(w, x) \leq 0 \]
What diffidence means economically: you reject all mean-zero risks.

risk aversion $\Rightarrow$ diffidence

but risk aversion tells you more than diffidence.

So: function (1) is about risk/no risk
(2) is about how much risk
(3) is about saving
(4) ... local risks.

Basic Example First

We'll start with Comparative Central Risk Aversion

$E xu'_1 (w + x) \leq 0 \Rightarrow x \geq 0$ because this is what our table says.

Then person 2 chooses $x$ of risky asset than person 1

$\frac{d}{dx} E u(w + ax) = E xu'_1 (w + ax)$

Forbidden region is a convex set.

Use separating hyperplane to divide the 2 convex sets. What we want: CANNOT allow convex

- line in the forbidden region.

Think of set of $x \in \mathbb{R}$ with $(a, b)$ support. $\mathbb{R}$. Remember, degenerate risks are a good basis for this problem.
\[ E \cdot x u'_2(w + x) \leq m E \cdot x u'_1(w + x) \]

(as long as there is a point in quadrant 3, \( m \neq \infty \) (the hyperplane isn't vertical line).

This is a better proof than what is in the existing version of the paper.

\[ \exists \text{ some } m \in [0, \infty) \text{ s.t. the inequality holds} \]

Pretend we know \( m \):

\[ E \cdot x u'_2(w + x) - m \cdot x u'_1(w + x) \leq 0 \]

\[ E h(x) \leq 0 \quad \forall x \in [a, b] \quad (h \text{ is a known function}) \]

So:

\[ \tilde{x} u'_2(w + x) \leq m \cdot \tilde{x} u'_1(w + x) \quad \forall x \in [a, b] \]

and there is some \( m \) for which this is true.

\[ x u'_2(w + x) - m x u'_1(w + x) \leq 0 \]

\[ h(x) \leq 0 \quad (h(0) = 0) \]

Assume \( h \) is differentiable.

We know \( h'(0) = 0, \quad h''(0) = 0 \)

Applying (first deriv)

\[ u'_2(w + x) + a \cdot x u''_2(w + x) \bigg|_{x=0} = m E w_1(w + x) + x u'_2(w + x) \]

(second deriv)

\[ 2u''_2(w + x) + x u''_2(w + x) \leq m \times 2u''_2(w + x) + xu''_2(w + x) \]

when \( x = 0 \), from first deriv.

\[ u'_2(w) \geq m u_1(w) \]

So, you can get \( m \! \). An actual value for \( m \) is...
local necessary condition

\[ \frac{u_2''(\omega)}{u_1''(\omega)} \implies \frac{-u_2''(\omega)}{u_2'(\omega)} \geq \frac{-u_1''(\omega)}{u_1'(\omega)} \]

The necessary & sufficient condition is earlier condition w/ value of \( m \) plugged in.

\[ xu_2'(\omega + x) \leq \frac{u_1'(\omega)}{u_2'(\omega)} - xu_1'(\omega + x) \]

Now, interpret NSC condition.

For a particular \( w \), NSC \( \Rightarrow \) local NC

Rearrange NSC

\[ \frac{u_2'(\omega + x)}{u_2'(\omega)} \leq \frac{u_1'(\omega + x)}{u_1'(\omega)} \quad \forall \ x \geq 0 \]

So:

\[ \frac{u_2'(\omega + x)}{u_2'(\omega)} \leq \frac{u_1'(\omega + x)}{u_1'(\omega)} \quad \forall \ x \leq 0 \]

Define \( u_i'(\omega + x) = \frac{u_i'(\omega + x)}{u_i'(\omega)} \)

so what you have is a single crossing condition on normalized marginal utility
\[ \frac{\ln u_1(w+x) - \ln u_1(w)}{\ln u_2(w+x) - \ln u_2(w)} \]

but: you can have single crossing at \( w \) \& \( w \) without having \( u_1 \) line steeper everywhere.

If, however, \( u_2 \) have single crossing \( \text{true to } w \), \( u_2 \) line is steeper everywhere.
Global Diffidence $\iff$ Global Risk Aversion

BUT

Central Diffidence $\not\iff$ Central Risk Aversion (although I think implication does work at least one way.)

$u_2$ is more diffident than $u_1$

$$E \left[ u_1(w + x) \right] \leq u_1(w) \implies E \left[ u_2(w + x) \right] \leq u_2(w)$$

another way to say this: Forbidden is $E \left[ u_1(w + x) \right] \leq u_1(w)$ \(\text{AND} E \left[ u_2(w + x) \right] > u_2(w)\)

as long as person I don't like everything, we know we have a separating hyperplane (that doesn't have infinite slope)

$$E \left[ u_2(w + x) - u_2(w) \right] \leq m \left[ E \left[ u_1(w + x) - u_1(w) \right] \right]$$

$$u_2(w + x) - u_2(w) \leq m \left[ u_1(w + x) - u_1(w) \right]$$
We also know they are equal at 0 (by observation, look @ functions)

First deriv
\[ u_2'(w + x) \bigg|_{x=0} = \mathcal{M} u_1'(w + x) \bigg|_{x=0} \]

2nd deriv
\[ u_2''(w + x) \bigg|_{x=0} \leq \mathcal{M} u_1''(w + x) \bigg|_{x=0} \]

First deriv \[ \Rightarrow \frac{u_2'(w)}{u_1'(w)} = \mathcal{M} \]

2nd deriv \[ \Rightarrow \frac{u_2''(w)}{u_1''(w)} \leq \mathcal{M} \]

Local necessary condition

Necessary & Sufficient Condition
\[ u_2(w + x) - u_2(w) \leq \mathcal{M} \left( \frac{u_2'(w)}{u_1'(w)} \right) u_1(w + x) - u_1(w) \]

Rearrange:
\[ \frac{u_2(w + x) - u_2(w)}{u_2'(w)} \leq \frac{u_1(w + x) - u_1(w)}{u_1'(w)} \]

Note: \( u_1(w + x) - u_1(w) \) is the "same" utility function as \( u_1'(w) \) all you do is subtract a constant & divide by a constant

\( u_2 \) always less than \( u_1 \) after 0.
this graph: $u_2$ more risk averse than $u_1$

From before, we had:

The single-crossing prop implies that graph.

Somehow, from the graphs, I am supposed to see that central relative risk aversion implies differdence:

$$u_2(w+x) - u_2(w) = \int_0^x \frac{u_2'(w+z)}{u_2'(w)} \, dz$$

$\Longleftrightarrow$

$$\leq \int_0^x \frac{u_1'(w+z)}{u_1'(w)} \, dz$$

$$= \frac{u_1(w+x) - u_1(w)}{u_1'(w)}$$

greater central risk aversion @ $w \Rightarrow$ greater difference
Necessary Condition for Greater Diffidence
\[-u''_2(w) \geq -u''_1(w) \quad \text{(just arrow Pratt)}\]

What if \(-\frac{u''_2(w)}{u'_2(w)} \geq -\frac{u''_1(w)}{u'_1(w)} \neq w\)? Then diffidence \(\Rightarrow\) greater risk aversion.

Central Risk Aversion: at value \(w\), person will choose less of risky asset.
Diffident: at value \(w\) will reject more.

Some risky asset s.t. a person chooses more of a risky asset even if they reject more risks.

\[\text{this increase in risky asset holding is good} \rightarrow \text{Higher MU}\]

but this comparison always works for more diffidence person.

(so squiggles aren't allowed for global)

Another Example: from Table.
Does a New Risk Raise Saving.
- we are off main diagonal, so are dealing w/ only 1 person.
\[E_xu'(\delta + x) = 0 \Rightarrow E_u'(\delta + x) \geq u'(\delta)\]

\[\text{may } U(w-s) + E_u(s+x)\]
What we are thinking of here: a NEW investment opportunity

i.e. closed economy opens to world

- you are changing this is a voluntarily accepted risk

\( x^* \) is amount of risky asset
\( x^* x = x \) is optimal amount of risky asset

Original Problem: \( \max_{x, s} w - s + E u (x + x^* \bar{Z}) \)
if \( x = 0 \), i.e., closed market
\( \max_{x, s} w - s + u(s) \)

2 periods, effectively

this makes you save more if marginal value of saving goes up, \( E x u'(s^* + x^* \bar{Z}) = 0 \Rightarrow E u'(s^* + x^* \bar{Z}) \geq u'(s^*) \)

\( E x u'(s^* + x^* \bar{Z}) \)

maybe that is own convex set (we don't know what it looks like)

separating hyperplane

forbidden region at negative part of y axis — note
\[ Eu'(s+x) - u'(s) \geq mE \cdot u'(s+x) \]
\[ u'(s+x) - u'(s) \geq m \cdot u'(s+x) \]

Now take deriv. w.r.t. \( x \), eval at \( x=0 \)

**First:** \( u''(s+x)|_{x=0} = m \left[ xu''(s+x) + u'(s+x) \right] \bigg|_{x=0} \)

**Second:** \( u''(s+x)|_{x=0} = m \left[ xu''(s+x) + 2u''(s+x) \right] \bigg|_{x=0} \)

**First:** \( u''(s) = m u'(s) \Rightarrow m = \frac{u''(s)}{u'(s)} \)

\( \Rightarrow m \) is negative, so our picture is wrong

\( \frac{d}{ds} u''(s) \geq u'(s) \frac{d}{ds} u'(s) \)

**NSC:** \( u'(s+x) - u'(s) \geq \frac{u''(s)}{u'(s)} \cdot x \cdot u'(s+x) \)

**Local NC:** \( u''(s) \geq \frac{u''(s)}{u'(s)} \cdot 2u''(s) \)

Divide by NSC by \( u'(s) \cdot u'(s+x) \)

\[ \frac{1}{u'(s)} - \frac{1}{u'(s+x)} \geq \frac{x \cdot u''(s)}{(u'(s))^2} \]

\[ \frac{1}{u'(s+x)} - \frac{1}{u'(s)} \leq -\frac{x \cdot u''(s)}{(u'(s))^2} \Rightarrow \]

\[ \frac{1}{u'(s+x)} \leq \frac{1}{u'(s)} - \frac{x \cdot u''(s)}{(u'(s))^2} \]

```
local NC:
\[ \frac{1}{u''(s+x)} \text{ must be locally concave at } s \]

\[ \frac{d}{dx} \frac{-u''(s+x)}{u'(s+x)^2} \leq 0 \Rightarrow \frac{d}{dx} \ln \left( \frac{-u''(s+x)}{u'(s+x)^2} \right) \]

\[ = \frac{d}{dx} \left[ \ln -u''(s+x) - 2 \ln u'(s+x)^2 \right] \]

\[ = \left. \frac{u'''(s+x)}{u''(s+x)} - 2 \frac{u''(s+x)}{u'(s+x)} \right|_{x=0} \leq 0 \]

\[ \Rightarrow \frac{u'''(s)}{u''(s)} \leq \frac{2 u''(s)}{u'(s)} \]

Then \[ \frac{d}{dx} \frac{1}{u'(s)} \leq 0 \]
suppose this holds at s, then \( u(s) \) is globally concave.

Suppose we have \( u(c) = \frac{c^{1-x}}{1-x} \)

\[ u'(c) = c^{-y} \Rightarrow \frac{1}{u'(c)} = cy \]

This is concave if...

New risk \( \rightarrow \) more saving. Need \( cy \) to be concave \( \Rightarrow y < 1 \). But evidence says \( y > 1 \).

So, new risky asset would make saving go...
Things that can be written as linear properties:
(stuff that is preserved under convex combinations)

\[ v' \geq 0 \Rightarrow E_v v' \geq 0 \]

(1) 
\[ v'(w) \geq 0 \Rightarrow E_{x} v'(w+\tilde{x}) \geq 0 \]

(2) 
\[ \frac{v''(w)}{v'(w)} \geq \gamma \quad \forall w, \gamma > 0 \]
 rewritten:
\[ -v'(w) - \gamma V'(w) \geq 0 \]
\[ E_{x} [-v'(w + \tilde{x}) - \gamma V'(w + \tilde{x})] \geq 0 \]

(3) Relative risk aversion
\[ -\frac{V_{ww}(w, \theta) w}{V_w(w, \theta)} \geq \gamma > 0 \quad \forall \theta \]

\[ -V_{ww}(w, \theta) w - \gamma V_{w}(w, \theta) \geq 0 \]
\[ E_{\theta} [-\theta w V_{ww}(w, \theta) - \gamma V_{w}(w, \theta)] \geq 0 \]

(4) DARA

\[ \ln V'(w) \]

\[ \ln V(w) - 2\ln V'(w+\delta) + \ln V'(w+2\delta) \geq 0 \]

rewrite
\[ V'(w) V'(w+2\delta) - V'(w+\delta)^2 \geq 0 \]
\[ \frac{\left[ V'(w) \ V'(w+\delta) \right]}{\left[ V'(w) \ V'(w+\delta) \right]^2} \text{ has a positive determinant} \]
$E \left[ V_w(w, \theta) - V_w(w+S) \right]$ 

$E_{\tilde{M}} \tilde{M}_x \geq 0 \quad A_{\tilde{M}}$ 

$x^c E[\tilde{M}]_x \geq 0 + x$

$
\ln E\ln V_w(w, \theta) + \ln E\ln V_w(w+S, \theta) - 2\ln E\ln V_w(w+S, \theta) \geq 0
$

This, I think, is what we wanted to end up with. Matrix was a means to this end.

(5) Supermodularity (interesting because of comparative statics)

$\forall x \geq 0$

$E\ln x \geq 0$

(6) Log supermodularity? We are going to check to see.

Log supermodularity:

$\ln V(x+S, y+S) - \ln V(x+S, y) - \ln V(x, y+S) + \ln V(x, y)$

Our matrix:

$\begin{bmatrix}
V(x, y) & V(x+S, y) \\
V(x, y+S) & V(x+S, y+S)
\end{bmatrix}$

assume main diagonal positive

& determinant positive.

Is the matrix positive definite?

No, because matrix is not symmetric.

Is

$E\ln V(x, y)V(x+S, y+S) - E\ln V(x+S, y)V(x, y+S) \geq 0$

$E\ln V(x, y)V(x+S, y+S) - \frac{E\sqrt{V(x+S, y)V(x, y+S)}}{??}$

Maybe, but not obviously.

Log supermodularity is preserved if, for

$\ln V(x, y, \theta)$
\[ \frac{\partial^2 \ln V}{\partial x \partial y} \geq 0 \quad \frac{\partial^2 \ln V}{\partial x \partial \theta} \geq 0 \quad \frac{\partial^2 \ln V}{\partial y \partial \theta} \geq 0 \]
then it is preserved. 

(6) Concavity? Yes. 
\( V(x, y, \theta) \) jointly concave in \( x \) and \( y \) 
then \( E_z V(x, y, \theta) \) is jointly concave in \( x \) and \( y \) because ... 

\[
V(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \geq \theta V(x_1, y_1) + (1-\theta) V(x_2, y_2)
\]
Rewrite: 
\[
V(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) - \theta V(x_1, y_1) - (1-\theta) V(x_2, y_2) \geq 0
\]
\[
E_z(\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad ) \geq 0
\]

Miles is currently on p. 30(?) of José's notes. 

Recursive definition of utility: see notes. 

We do this to set up our approach to multi-period problems.

Notation: 
\( U_t = \) unmaximized utility \[ U_t = \Psi^t(K_t, X_t, E_t^t U_{t+h}, \ldots, h) \]
vector of control \( \quad \quad \quad \quad \quad \quad \) state vars vector.

h is basically \( \Delta t \) (length of time interval). 
Sometimes important to test things w/ varying lengths of \( h \).

\( U_{t+h} = 0 \) need this condition to allow "recursive" solution, like a transversality condition.
Could replace $E_t \nu_{t+n}$ w/ any function of the distribution of $\nu_{t+n}$.

Examples:
1. $\nu_t = h U_e^\tau(k_t, x_t) + e^{-\mu t} \nu_{t+h}$
2. $\nu_t = h U_e^\tau(k_t, x_t) + \phi^{-1} E \phi(\nu_{t+n})$
3. $\nu_t = h U_e^\tau(k_t, x_t) + e^{-\mu t} \phi^{-1} E \phi(\nu_{t+n})$

$\nu_t$ is lifetime utility

4. $\nu_t = h U_e^\tau(k_t, x_t) + e^{-\mu t} \psi^{-1}(E \psi^n)$
5. $\nu_t = h U_e^\tau(k_t, x_t) + e^{-\mu t} \phi^{-1} E \phi(\nu_{t+n})$

(Kepp's-Porteus utility) - deals w/ intertemporal substitution issues - keeps it unchanged.

Now: Time to talk about constraints
Contemporaneous Constraints:
- two ways of writing them down
1. $x_t \in K_x(k_t)$
2. $(x_t, k_t) \in W$

Inter-temporal Constraints
Inter-temporal Transition Equation:

$x_{t+h} = \Gamma^\tau(k_t, x_t, \tilde{W}_{t+h})$
Example 1
\[ K_{t+h} = K_t + hA^t(K_t, X_t) \]
As \( h \to 0 \), this is \( K_t = A^t(K_t, X_t) \)

Example 2
\[ K_{t+h} = \left\{ \begin{array}{ll}
K_t + hA^t(K_t, X_t) + \sqrt{h} \Omega^t(K_t, X_t) & \text{w/ chance } \frac{1}{2} \\
2K_t + hA^t(K_t, X_t) - \sqrt{h} \Omega^t(K_t, X_t) & \text{w/ chance } \frac{1}{2}
\end{array} \right. 
\]
This gives us a diffusion process.
Variance of \( K_{t+h} | K_t = h \Omega^t(K_t, X_t) \)
So both mean & variance are of order \( h \).

Define
\[ Z_{t+h} = \left\{ \begin{array}{ll}
Z_t + \sqrt{h} & \text{w/ } p = \frac{1}{2} \\
Z_t - \sqrt{h} & \text{w/ } p = \frac{1}{2}
\end{array} \right. 
\]
\[ \Delta Z = \left\{ \begin{array}{ll}
\sqrt{h} & \text{w/ } p = \frac{1}{2} \\
-\sqrt{h} & \text{w/ } p = \frac{1}{2}
\end{array} \right. 
\]

Example 3
\[ K_{t+h} = K_t + hA^t(K_t, X_t) + \sqrt{h} \Omega 
\]
\[ E_t(Z) = 0 \\
\text{Var}(Z) = 1 
\]

Suppose only uncertainty is determined by a binary random variable.

**Bellman Equation**
\[ V^*(K_t) = \max_{x} \left\{ U(K_t, X), E_t V^{*+h}(K_t, X_t, \omega_{t+h}) \right\} \]
Bellman Equation

\[ V^*(K_t) = \max_{X_t} \Psi^*(K_t, X_t, E_t V^{*+h}(F^t(K_t, X_t, \omega_t))) \]

Note: value function is only a function of the state variables

How to prove properties of \( V \):

- why we care: if you "know" something about the shape of the value function, we have essentially reduced the problem to a 2-period problem. (Sometimes)
- can show 2-period results do on don't carry over

Literature focuses on \( V \) in closed form. Symmetry lets you do this often.
- however, most of the interesting closed-form solutions have already been solved.
- but inequality problems can be quite interesting especially if you know some properties of the value function.

How to prove properties of \( V \):

Suppose \( P \) is the property of interest.

1) \( V^{*+h} \in P \) (\( V^{*+h} \) has property \( P \))

Show: \( V^{*+h} \in P \Rightarrow F^t(K_t, X_t) \in Q \)

\( F^t \) is really the RHS of the Bellman equation before maximization.

\[ V^*(K_t) = \max_{X_t \in X(K_t)} F^t(K_t, X_t) \]

\[ F^t(K_t, X_t) = \Psi(K_t, X_t, \Gamma^t(K_t, X_t, \omega_t)) \]
2) \( F^+ \in \mathcal{O} \Rightarrow V^+ \in \mathcal{P} \)

Combine these 2 steps

\( V^{+th} \in \mathcal{P} \Rightarrow V^+ \in \mathcal{P} \)

(\( V^+ \) has property \( P \) means at every node)

\[
\begin{array}{cc}
V^{+th} & \Rightarrow V^+ \\
\| & \| \\
0 & 0 \\
\end{array}
\]

want 0 to satisfy the property

when we do actual problems, we'll focus on steps 1 & 2; need to remember that reason we do them is for this final combination.

Steps (1) & (2) are essentially. There are also "optional" steps. Steps (3) & (4) can be done in either order and you can do none, 1, or both.

(3) \( T \to \infty \) (going to infinity)

Basically, no big deal. (As limit of finite horizon...). There are other ways to think about it, though.

(4) \( h \to 0 \) (continuous time)

Proof still works. Even less of a big deal, probably.

If, for example, \(-V^+_k(K; T, h) - fV^+_k(K, T, h) \geq 0\) is true \& \( T, h \), then (3) \& (4) are trivial. To find (3) \& (4) are almost always toxic.)
\[ \lim_{h \to 0} \frac{h}{n} = 0 \quad \text{works, } \lim_{h \to 0} \frac{1}{n} > 0 \quad \text{doesn't. (Because first is closed set)} \]

Sort of a Side Comment: Hard to know what i.i.d. really means in continuous time.

\[ h(\bar{y} + \varepsilon) \quad \text{as } h \to 0. \]

When periods are not natural periods (i.e., harvests) you have probably more persistence.

This is a reason to take \( h \) to zero. (Way to argue there is some persistence — stuff isn’t i.i.d.)

(In continuous time w/i.i.d., there is no risk because of Law of Large Numbers.)

Note: Preservation of a property under expectations is quite important for showing step (1). Step (2) requires different techniques.

See examples in last year’s notes. (Section with “trees”)

Another technique:
Suppose you do know some properties of \( V \). Then it can be helpful to draw a risk-adjusted phase diagram.

What happens to Bellman equation when you have a diffusion process:

\[
\rho V^e_{K^e} (K^e_t) - V^e_t = \max_{K^e_x X_t} U^e (K^e_t, X_t) + E \int V^e_{K^e} (K^e_t) A(K^e_t, X_t) + V^e_{K^e} (K^e_t) \frac{\partial}{\partial K^e} \Omega(K^e_t, X_t) \]

\[ \Delta z = \left\{ \begin{array}{ll} \sqrt{h} & w/ \rho = \gamma^2 \\ -\sqrt{h} & w/ \rho = -\gamma^2 \end{array} \right. \]
Standard deviation is of the order \( \sqrt{\Delta t} \) or \( \sqrt{dt} \).

Variance is of order \( h \sim \Delta t \sim dt \).

Where the above (prev. page) came from:

\[
V^t(K_e) = \max_{x_t \in X(K_e)} U(K_e, x_t) + \sum_{n=0}^{\infty} \frac{V^{n+1}(K_e, X_t, \omega)}{V_t + \text{Taylor expansion}}
\]

\[
V^t(K_e) = \frac{V_{k+1}(K_e)}{V_k(K_e)}
\]

Link to the Hamiltonian:

\[
\lambda^t(K_e) = V^t(K_e)
\]

\[
\lambda^t(K_e) = V^{k+1}(K_e)
\]

So:

\[
\rho V^t(K_e) - V_t = \max_{x_t \in X(K_e)} U(K_e, x_t) + \frac{1}{2} \lambda^t(K_e) A(K_e, x_t)
\]

\[
- \frac{1}{2} \lambda^t(K_e) \frac{\partial \bar{U}(K_e, x_t)}{\partial x}
\]

\[
\lambda^t(K_e) [A(K_e, x_t) - \frac{1}{2} \lambda^t(K_e) \frac{\partial \bar{U}(K_e, x_t)}{\partial x}]
\]

This is the form you would get if there was no uncertainty at all — so dynamic programming becomes equivalent to optimal control.

Risk-Adjusted Model: A perfect-forward and dynamic control model that corresponds to a stochastic diffusion model.

So: a qualitative stuff about \( V \) will tell us qualitative stuff about \( A \). When we know something about \( A \), we can do an affine transform.
The point: you have a mean-variance model that came straight out of continuous time (result of using a Taylor expansion, not an assumption about quadratic utility).

Indifference curves determined by $\lambda$ and $\bar{\xi}$ — in 6.11, we just had $\lambda$ — so, they'll change over time, even static. Variable.

Mean-variance possibility frontier: the tradeoff between $A$ and $\bar{\xi}$.

Chattering Principle:
True in continuous time, approximately true in short-interval discrete time.

Chattering principle allows us to convert a nonconvex set by bouncing back and forth between. Averaging sort of thing, so it is like a steady flow.
Today: toward the symmetry theorem

No problem due this week. Reference reports due next week.
Problems due Tues.
As long as value function is twice differentiable, it is basically a mean-variance problem.

Psychologists doubt that people understand fungibility—but maybe people have a good grasp of money versus fungibility?

Symmetries can reduce the dimensionality of the problem you have to solve computationally.
- Scale symmetry is a subset of symmetry.

Most common symmetry: scale symmetry

\[ u(c) = \frac{c^{1-r}}{1-r} \]

\[ V(w) = A(\ell) \frac{W^{1-r}}{1-r} \]

⇒ you have wealth to same power you have consumption

This is on p. 43 of José's notes.

2 pieces:
1) Symmetry of the constraint set
2) Symmetry of preferences

\[ K_t, X_t \]
\[ X_t \in \mathcal{X}_t(K_t) \quad \text{or} \quad (K_t, X_t) \in \mathcal{Y} \]
Think of set of all achievable utilities.

\[ V(K_t) = \{ v_t | \exists X_t \text{ with } (K_t, X_t) \in \mathcal{W} \text{ and } v_t = \psi(K_t, X_t) \} \cap \mathcal{V}^{t,h}(\Gamma(K_t, X_t, w_{t+n})) \]

To define a symmetry, we need to define a transformation

\[ (K_t, X_t) \rightarrow T(K_t, X_t) \]

in particular, it must be a triangular transform

\[ K_t = T^c(K_t) \]
\[ X_t = T^x(K_t, X_t) \]

Intuition: \( K_t \) depends only on \( K_t \), but \( X_t \) depends on \( K_t \) & \( X_t \).

Defn: \( T \) is a symmetry of the constraint set iff

(a) \( (K_t, X_t) \in \mathcal{W} \iff T(K_t, X_t) \in \mathcal{W} \) (need both directions)

(b) \( K_{t+n} = T^c(K_t, X_t, \tilde{\omega}_{t+n}) \Rightarrow T(K_{t+n}) = \Gamma(T(K_t), \tilde{\omega}_n, T(K_t)) \)

Intuition: symmetry doesn't affect the budget constraint.

But: you also need nice preferences.

Detn. \( S \) is a preference.
Defn: S is a preference symmetry corresponding to T iff

\[ v_t = \Psi(K, x, E, v_0) \Rightarrow S(K_v, v_0) = \Psi(T^*(K_v), T^*(K_v), E, S(F(K_v, x, \omega_0), v_{t+1})) \]

\[ S(K_v, v_0) = \Psi(T^*(K_v), T^*(K_v), E, S(F(K_v, x, \omega_0), v_{t+1})) \]

\[ E_x \quad T_v \quad W \rightarrow 2W \]
\[ T^* \quad C \rightarrow 2W \]
\[ \otimes S \quad v \rightarrow 2^{1-x}v \]

Idea of S: If it changes things at (t+1), it changes them at t in the same way.

If \( S(K_v, v_0) \) didn't depend on K, we'd have:

\[ S(v_t) = \Psi(T^*(K_v), T^*(K_v), E, S(v_{t+1})) \]

Symmetry Theorem: (A generalization of Boyd's results)
If T is a symmetry of the constraints and S is a corresponding symmetry of preferences, then

\[ V(T^*(K_v)) = S(K_v, V(K)) \]

(C) If you have symmetry of presets, it carries over to the value function.

Proof Sketch (will do actual proof later)

Note: \( V(T^*(K)) = S(K, V(K)) \)

Strategy: We know what is happening at time (T+1): Bring stuff from inside @ to outside at (T+1). Try to bring inside to outside @ time T.
Proof:

\[ S(K_e^+, \mathcal{Y}^*(K_e)) = \{ x \mid \exists X_e \text{ with } (K_e, x_e) \in \mathcal{Y} \text{ and } \hat{v}_e = S(K_e^+, \mathcal{Y}(K_e, x_e, E_e \mathcal{V}_{e\text{m}}(\Gamma(K_e, x_e, w_{e+m}))) \}
\]

by definition.

\[ = \{ x \mid \exists X_e \text{ with } (K_e, x_e) \in \mathcal{Y} \text{ and } \hat{v}_e = \mathcal{Y}(T^*(K_e), T^*(K_e, x_e), E_e \mathcal{V}_{e\text{m}}(\Gamma(T^*(K_e), T^*(x_e, K_e), w_{e+m}))) \}
\]

by definition of preference symmetry.

\[ = \{ x \mid \exists X_e \text{ with } (K_e^+, x_e) \in \mathcal{Y} \text{ and } \hat{v}_e = \mathcal{Y}(T^*(K_e), T^*(x_e, K_e), E_e \mathcal{V}_{e\text{m}}(\Gamma(T^*(K_e), T^*(x_e, K_e), w_{e+m}))) \}
\]

by intertemporal constraint symmetry.

\[ = \{ x \mid \exists X_e \text{ with } (T^*(K_e), x_e) \in \mathcal{Y} \text{ and } \hat{v}_e = \mathcal{Y}(T^*(K_e), x_e, E_e \mathcal{V}_{e\text{m}}(\Gamma(T^*(K_e), x_e, w_{e+m}))) \}
\]

by contemporaneous constraint symmetry.

\[ = \mathcal{Y}^*(T^*(K_e))
\]

by definition again.

Then: if you transform every element of a set, the supremum of set is also transformed. \( \mathcal{I} \) \( \mathcal{F} \) \( S \) is monotonically increasing.
Today: Examples of the Symmetry Method

Merton Problem (see paper on reading list.)

\[
\max \int_0^\infty e^{-r(t-t')} \frac{c^{1-\delta}}{c'^{\delta}} \, dt
\]

\[\text{s.t. } dW = \left[ rW + \mu - C \right] dt + [\alpha \sigma] dz \]

where \( \Delta z = \pm \sqrt{h} \text{ w/ equal chance} \)
\( \alpha \) = value of risky assets
\( \mu \) = excess return ($/value) (historically, \(-6.7\% /\text{yr}) \)
\( \sigma = \) (.15 /\text{yr}) (according to Campbell & \( \Rightarrow \sigma^2 = .0225 /\text{yr} \))

We want to write this in Bellman equation form:

\[
\rho V(W_{t+1}) - V_t(W_t, t) = \max \frac{c_t^{1-\delta}}{c_t^\delta} + V_w(W_t, t)[rW + \mu - C]
\]

\( \text{LHS: things not functions of } c \text{ & } \alpha \)
\( \rho V(W, t) \) = required rate of return
\( V_t(W_t, t) \) = capital gain term due to passage of time
\( \text{like if there is a change in situation as time passes - suppose university will double in size in 5 yrs} \)
\( \text{think of changes in value of housing over that time} \)
\( \text{RHS: flow utility, return on saving, risk penalty} \)

This is really just a Taylor expansion of the standard Bellman equation.
Now we want a symmetry 

1) Scale Symmetry 

\[ W \rightarrow \theta W, \quad \hat{W} = \theta \hat{W} \]
\[ C \rightarrow \theta C \]
\[ \alpha \rightarrow \theta \alpha \] (if \( \alpha \) were a share, it would remain unchanged)

Check what this does to the budget constraint 

\[ d\hat{W} = [r \hat{W} + \hat{\alpha} \mu - \hat{C}] dt + [\hat{\alpha} \sigma] dz \]

Show that old constraint \( \Leftrightarrow \) new constraint 

\[ \theta d\hat{W} = [r \theta \hat{W} + \theta \alpha \mu - \theta C] dt + [\theta \alpha \sigma] dz \]

Divide through by \( \theta \) and you get old constraint back.

Now think about new Bellman equation 

\[
\hat{V}(W, t) - \hat{V}(W, t) = \max_{C, \alpha} \left\{ \frac{\hat{\alpha}^2}{2} + \hat{V}_w(W, t) [r \hat{W} + \hat{\alpha} \mu - \hat{C}] + \hat{V}_{ww}(W, t) \right\}
\]

Actually, we want to think just about preferences that multiplying \( C \) by \( \theta \) means utility goes up by \( \theta^{1-r} \)

So:

\[ W \rightarrow \theta W \]
\[ C \rightarrow \theta C \]
\[ \alpha \rightarrow \theta \alpha \]
\[ V \rightarrow \theta^{1-r} V \]

Now apply Symmetry Theorem (transform on outside: transform on inside) 

\[
\theta^{1-r} V(W, t) = V(\theta W, t) \quad (\hat{V}(W, t) = V(\hat{W}, t))
\]

this is true \( \forall \theta \)

In particular, true for \( \theta = \frac{1}{\theta} \)

\[
\left(\frac{1}{\theta}\right)^{1-r} V(W, t) = V(1, t)
\]
Solve for \( V(W, t) \):

\[
V(W, t) = W^{d-r} V(1, t)
\]

Symmetry showed us that \( V \) depends on wealth to some power, but after that it doesn't depend on wealth. This is very helpful!!

Before going on, let's do a slightly different problem:

\[
E_{0}^{T} \frac{C^{1-r}}{1-r} \, dt \quad \text{when} \quad W(T) = 0, \quad dW = [r W + \mu - c] + \sigma dZ
\]

NOTHING CHANGES in terms of our symmetries

Plug in to Bellman's eq.

Note:

\[
\begin{align*}
V_{t}(W, t) &= W^{d-r} V_{t}(1, t) \\
V_{W}(W, t) &= (1-r) W^{r-1} V(1, t) \\
V_{WW}(W, t) &= -r (1-r) W^{r-2} V(1, t)
\end{align*}
\]

\[
\rho W^{1-r} V(1, t) - W^{(d-r)} V_{t}(1, t) = \max_{c, \alpha} \left\{ \frac{(1-r)}{1-r} + (1-r) W^{r} V(1, t) \right\}
\]

\[
= -r (1-r) W^{r-1} \frac{V(1, t)}{2} + \alpha^2 \sigma^2 \tau^2
\]

FOC

\[
(1-r) W^{r} V(1, t)
\]

\[
\Rightarrow \quad C = W \left[ (1-r) V(1, t) \right]^{1-r} = A(t) W
\]

\( \text{average propensity to consume} \)

\frac{slope = A(t)}
\[ A(t) = (1 - \gamma) V(1, t) \]
\[ A(t) - r = (1 - \gamma) V(1, t) \]
\[ V(1, t) = \frac{A(t)}{1 - \gamma} \]

So,
\[ V(W, t) = \frac{W^{1 - r}}{1 - \gamma} A(t)^{-r} \]
\[ V_b(W, t) = \frac{W^{1 - r}}{1 - \gamma} (-\gamma A(t)^{-r} A'(t)) \]
\[ V_{w}(W, t) = W^{-r} A(t)^{-r} \]
\[ V_{ww}(W, t) = -\gamma W^{-r} A(t)^{-r} \]

Now: go back to our rewritten Bellman eqn. Find the "right" value of \( \alpha \). Take FOC wrt \( \alpha \).

\[ \text{FOC. } (w + \alpha) \]
\[ V_w - \mu + \sum V_{ww} \alpha \sigma^2 = 0 \]

\[ \alpha = \left( \frac{-V_{ww}}{V_w} \right) \sigma^2 \Rightarrow \frac{\alpha}{W} = \frac{\sigma^2}{W} \left( \frac{-V_{ww}}{V_w} \right)^{-1} \]

depends on absolute risk aversion

\[ -\frac{V_{ww}}{V_w} = \frac{-\sigma^2}{W} \]

depends on relative RA

\( \alpha = \left( \frac{-\sigma^2}{\gamma \sigma^2} \right) W \)

Note: a short-sales constraint (\( \alpha \geq 0 \)) changes result, but does not affect the symmetry.
Also, $\alpha < W$ doesn't change symmetry either $(\Theta x < \Theta W)$.

Now go back to the recursive value function. Substitute in again, and divide by $\frac{W^{1-r}}{1-r}$

$$\rho A^{1-r} + \gamma A^{r-1}\dot{A} = \frac{A^{2r}}{\omega^2} A^{1-r} + \frac{1-r}{W} A^{-r} [r\dot{W} + \sigma \frac{\mu^3}{\sigma^2} W - AW]$$

Keep simplifying

$$\rho + \gamma \frac{\dot{A}}{A} = A + (1-\gamma)(r + \frac{\mu^3}{\sigma^2}) + (\gamma - 1)A$$

Divide by $\gamma$

$$\frac{\rho}{\gamma} + \frac{\dot{A}}{A} = A + (\gamma - 1)(r + \frac{\mu^3}{\sigma^2})$$

Special Case

$\mu, \sigma$ are constant

$T = \infty$

Result we want: value function constant over time.

$$V^{(\infty)}(W, t) = \max_{c, \alpha} \int_{t}^{\infty} e^{\gamma(1-r)} c_{\alpha} dt$$

Time symmetry:

$t \rightarrow t + \Theta$

$\dot{A}(t + \Theta) = \dot{A}(t)$ (no problem if $t$ is constant, just delay everything)

$T \rightarrow T + \Theta$

This is a symmetry of constraint rest
So: \( V(t+\theta, \text{parameters}) \big|_t = V(t) \)

if parameters don't change, \( V(t+\theta) = V(t) \)

\[ A(t+\theta)^t = A(t)^t \]

so, \( A(0) = A(t) = A \)

\[ A = 0 \]

so: \( A = \frac{1}{q} \rho + (1 - \frac{1}{q}) \left( r + \frac{\mu^2}{2\sigma^2} \right) \)

average of \( \rho \) & risk-adjusted real interest rate.

2/8/05

#2 on exam

Constant: \( -e^{-x}, -\frac{1}{2} e^{-2x} \) are CARA

average of them is DARA

but we need just 1 utility function

Graph of Abs. Risk Aversion

\[ u(x) = -e^{-x} - x \]

another ex/ of IARA: \( u(x) = -e^{-x} - x \)
More With Symmetries

- Merton problem continued

Last time we had

$$A(t) = \frac{\nu}{\gamma} + (1 - \frac{1}{\gamma}) (r + \frac{M^2}{2s^2}) + \tilde{A}$$

Discrete version of the problem

$$V(W, t) = \max_{c, \omega} \frac{C^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V(\tilde{R}(s, \omega)(W_t - C_t), t+1)$$

we are doing short here

Symmetry:

- $W \rightarrow \theta W$
- $C \rightarrow \theta C$
- $s \rightarrow s$
- $(V \rightarrow \theta^{-\gamma} V)$ ???? I think

$$A^{-\gamma} W^{1-\gamma} = \max_{c, s} \frac{C^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t [\tilde{R}(s, \omega)(W_t - C_t)]^{-\gamma} A^{-\gamma}$$

(Actually, $\tilde{R}$ requires another whole vector of state variables. Replace $\tilde{R}(s, \omega)$ with $\tilde{R}(s, \omega, k)$.)

FOC is still $c_t = A_t W_t$

To show that:

take FOC of

$$c_t^{-\gamma} = \beta \mathbb{E}_t \tilde{R}_{t+1} \left[ (W_t - c_t) \tilde{R} \right]^{-\gamma} A_{t+1}^{-\gamma}$$

Use envelope theorem: if we are optimizing $$2$$
From envelope

\[ A_t^\gamma W_t^\gamma = \beta E_{t+1} \tilde{R}_{t+1} \left[ (W_t - C_t) \tilde{R}_{t+1} \right]^{-\gamma} A_{t+1}^{-\gamma} \]

Combine FOC & Envelope Theorem result

\[ C_t^\gamma = A_t^\gamma W_t^\gamma \quad \Rightarrow \quad C_t = A_t W_t \]

Plug that back into value function

\[ A_t^{j-\gamma} W_t^{1-\gamma} = A_t^{1-\gamma} W_t^{1-\gamma} + \beta E_{t} R_{t+1}^{-\gamma} A_{t+1}^{-\gamma} \left[ W_t (1-A_t) \right]^{1-\gamma} \]

(this is known @ time \( t \))

Again, symmetry will allow us to eliminate \( W_t \).

Divide through by \( W_t^{1-\gamma} \)

\[ A_t^\gamma = \frac{A_t^{1-\gamma}}{1-\gamma} + \beta \left[ 1 - A_t \right]^{1-\gamma} E_{t} \tilde{R}_{t+1}^{-\gamma} A_{t+1}^{-\gamma} \]

solve for \( A_t \) to get a nice recursion.

\[ (1-A_t) A_t^\gamma = \beta (1-A_t)^{1-\gamma} E_{t} \tilde{R}_{t+1}^{-\gamma} \tilde{A}_{t+1}^{-\gamma} \]

\[ A_t^\gamma = \beta (1-A_t)^{1-\gamma} E_{t} \tilde{R}_{t+1}^{-\gamma} \tilde{A}_{t+1}^{-\gamma} \]

\[ A_t = \beta^{\frac{1-\gamma}{\gamma}} (1-A_t) E_{t} \tilde{R}_{t+1}^{-\gamma} \tilde{A}_{t+1}^{-\gamma} \]

General hint: easier to solve for \( \frac{1}{A_t} \) than \( A_t \)

\[ \frac{1}{A_t} \bigcirc = \bigcirc + \beta \bigcirc \bigcirc \left[ E_{t} \tilde{R}_{t+1}^{-\gamma} A_{t+1}^{-\gamma} \right]^{-\gamma} \]

Define \( B_t = \frac{1}{A_t} \quad (A_t = \text{marginal propensity to consume} \]

\[ B_t = 1 + \beta \frac{1-\gamma}{\gamma} \left[ E_{t} \tilde{R}_{t+1}^{-\gamma} B_{t+1}^{-\gamma} \right]^{-\gamma} \]
\[ A_t^{1-r} W_t^{1-r} = \max_{1-r} A_t^{1-r} W_t^{1-r} + E_t \tilde{R}_t^{1-r} A_t^{1-r} W_t(1-A_t) \]

Optimization Subproblem:
\[ \max_s E_t [ \tilde{R}_t^{1-r} A_t^{1-r} ] \]

Symmetries we've done so far:
1. Scale symmetry
2. Time symmetry (stationarity)

Now we will do the rate-of-time symmetry.

\[ \tilde{V}(W_t, t) = \max_{\rho, \theta, \sigma, \mu, \gamma}(W_t, t) \]
Symmetry in mind: make every rate happen faster:

\[ \rho \rightarrow \Theta \rho \]
\[ r \rightarrow \Theta r \]
\[ \mu \rightarrow \Theta \mu \]
\[ \sigma \rightarrow \Theta \sigma \quad (\text{so } \sigma^2 \rightarrow \Theta \sigma^2) \]

You could also think of this as just changing your units of time. (Measure annually instead of quarterly.) (Matters for units)

\[ \rho V(W_t, t) - V_t(W_t, t) = \max \frac{c_t}{c_t^0} 1-r + V_t(W_t, t) [rw + \mu - \gamma] \]
\[ + V_{ww} (W_t, t) \frac{\sigma^2 t^3}{2} \]

To get symmetry of constraint set.
Generally, have to recalibrate utility function.
\[ U(C, \theta) \quad \text{but since we have CARA we can} \]
\[ U(C, \theta) = U(C, \theta) \quad \text{ignore} \]
so \( c \to \Theta c \)

What happens to \( V \)?

\[ E_0 \int_0^\infty e^{-\rho t} \frac{C_{1+\rho}}{1-P} dt = c \to \Theta c, \text{ BUT } \rho \to \Theta \rho \]

so \( V \to \Theta^{-1} V \)

\[ V(w, t; \Theta) = \Theta^{-1} V(W, t; \Theta) \]

This type of symmetry is especially relevant in stationary problem in continuous time. What matters is generally rates of rates, not level of rates. First rate determines time scale, other rates depend on ratio to this rate.

One More Symmetry (easy)

Extend Merton model.

\[ \max \int_0^\infty e^{-\rho t} U(C_t) dt \]

\[ dW_t = \left[ rw + \alpha \mu + \theta - C_t \right] + \alpha \sigma dz \]

\( \alpha \sigma dz \) non-interest income

Suppose \( \theta = \gamma \Lambda t \).

Capitalization symmetry:

\[ \frac{V}{W} \to \frac{V}{W} - \Theta \]

\[ W_t \to W_t + \theta \int e^{-\rho t} V \ dt \]

idea: I take away non-interest income & give you equivalent wealth.

(assumes you don't have liquidity constraint if you say this is symmetry of constraint set.)

\[ V \to V. \Rightarrow V(W, \frac{V}{W}) = V(W_1, \frac{V}{W_1}, 0) \]
Joe Lupton's paper on habit formation uses "fancy version of capitalization symmetry\textsuperscript{1}

\[ U(C-K) \quad \text{\textcopyright{W}} \]

\[ K = \theta (C - K) \]
\[ dW = [rW - C + Y + \ldots ] dt \]

symmetry only affects budget constraint: \[ V \rightarrow V \]
Note: as soon as you interact w/ liquidity constraint you lose the symmetry

Two More Techniques: in general topic of... Preservation of Inequality Properties of \( V \)
1. Horizontal & Vertical Addition
   - see Carroll & Kimball "On the Concavity of the Consumption Function" & CK, "Liquidity Constraints & Precautionary Saving"

Invariances Properties of Properties (that you might have)
1. Vertical Scale Invariance
   \[ V \in \mathcal{P} \quad (V \text{ has the property } \mathcal{P}) \]
   \[ V \theta \in \mathcal{P} \Rightarrow \theta V(C) \in \mathcal{P} \quad \forall \theta > 0 \]
\[ \frac{V}{V'}, \frac{V''}{V'''; \ldots, K} \]

2. Horizontal Scale Invariance
   \[ V(K) \in \mathcal{P} \Rightarrow V(\theta K) \in \mathcal{P} \quad \forall \theta > 0 \]

3. Vertical Location Invariance
   \[ V(K) \in \mathcal{P} \Rightarrow V(K) + \xi \in \mathcal{P} \]
4. Horizontal Location Invariance
\[ V(K) \in \mathcal{P} \implies V(K + h) \in \mathcal{P} \]

5. Unit-Free Property
\( \mathcal{P} \) has both vertical & horizontal scale invariance.

6. "Immutable Property"
All 4 invariances.

Note: decreasing absolute risk aversion is immutable.
DARA: \( V''(V') - V'' \geq 0 \implies -\frac{V''}{V'} \geq -\frac{V''}{V'} \)

\[
\begin{align*}
\hat{V}(K) &= V(\theta K) \\
\hat{V}'(K) &= \theta V'(\theta K) \\
\hat{V}''(K) &= \theta^2 V''(\theta K) \\
\hat{V}'''(K) &= \theta^3 V'''(\theta K)
\end{align*}
\]

so
\[
\hat{V}''(K) \hat{V}'''(K) - \hat{V}''(K)^2 = 0
\]
\[
\theta^4 \left[ V''(\theta K) V'(\theta K) - V''(\theta K)^2 \right] = 0
\]

we have horizontal scale invariance. If you can write it as
\[
F(V(K), \theta V'(K), K^2 V''(K), K^3 V'''(K)) = 0
\]

If you put vertical & horizontal together, you have to be able to write property in terms of
\[
F \left( \frac{K V}{V'}, \frac{K V''}{V''}, \frac{K V'''}{V'''} \right) \text{ (unit free)}
\]

For (3), you just have to be able to write it in terms of marginal utility (V').

For (4), you need to be able to write it.
So to have an immutable property, you need to be able to write it in terms of
\[
\frac{V''''V'}{V''^2}, \quad \frac{V''''V''}{V''^2}
\]

Example of an immutable property: DARA
\[
\frac{V''''V'}{V''^2} \geq 1
\]

Note: to get concavity of consumption function, need
\[
\frac{u''''(c)u'(c)}{u''''(c)^2} = \alpha \quad \text{and} \quad \frac{V''''(c) u'(c)}{V''(c)} \geq \alpha \geq 0
\]

Note: not both equal. Note that Merton problem will give you a linear consumption function.

Note: it is easier to get immutability if you are comparing to zero, & even easier is sign of derivatives.

General Rule: it is very hard to get 4th order + properties preserved.

Now we'll move to Bivariate Properties
\[
\begin{align*}
V_{HK} &> 0 \quad \text{(supermodularity)} \\
V_{HK} &< 0 \quad \text{submodularity} \\
V_{HK} &= 0 \quad \text{modular (you get this when you are additively separable)} \\
\frac{1}{V''(H; k)} &\geq 0 \quad \text{log supermodularity}
\end{align*}
\]
"preserved": inherit the property from the utility function.

\* if you have prop in T, you have it in T-1

Note: suppose you have \( V(K, y) \) where \( y \) is a parameter. \( y_2 > y_1 \)

\[
\frac{-V''(K, y_2)}{V'(K, y_2)} \geq -\frac{V''(K, y_1)}{V'(K, y_1)}
\]

this one may not be preserved

Consider

\[
\frac{V'''}{V''} \geq \alpha
\]

\[
-\frac{V'''}{V''} \geq \alpha \left(-\frac{V''}{V'}\right)
\]

Suppose you are concerned about whether \( -\frac{V''}{V'} \) is preserved.

What you can guarantee is:

if \( -\frac{V''}{V'} \geq \beta \), \( -\frac{V''}{V'} \geq \alpha \left(-\frac{V''}{V'}\right) \Rightarrow \frac{V''}{V'} \leq \beta \)

Standard Precautionary Saving Problem

\[
V^*(K) = \max_{C_t} U(C_t) + \beta E_{t+1} V^{*+1}(\tilde{R}_{t+1}(w)(K_{t+1} - C_{t+1}), \tilde{y})
\]

liquidity constraint: \( K_t \geq 0 \quad \forall t \)

\[
\beta E_{t+1} V^{*+1}(\tilde{R}_{t+1}(w)(K_{t+1} - C_{t+1}), \tilde{y}) = \Omega^*(K_t - C_t)
\]
$U'(c_e) = \Omega'(k_e - c_e)$

Note: preservation under expectation IS preservation under vertical aggregation.

**Horizontal Aggr.**

Think about: define $f(\lambda) = u'^{-1}(\lambda)$

$\Rightarrow u'(f(\lambda)) = \lambda$

Express your favorite properties in terms of $\lambda$, $f(\lambda)$, $f'(\lambda)$, $f''(\lambda)$, $f'''(\lambda)$.
\[
\max_c U(c) + \beta E_t V_{\text{tri}} \left( \tilde{R}_{\text{tri}}(W_t - C_t) + \tilde{Y}_{t-1} \right)
\]

After vertical integration

\[
\max_c U(c) + \Omega(W_t - C_t)
\]

Book: Rockafellar, Convex Analysis

returns to scale: increasing: \[\frac{KV'(k)}{V(k)} \geq 1\]

decreasing: \[\frac{KV'(k)}{V(k)} \leq 1\]

\[V = U \bigotimes \Omega\]

\[\text{combining (Miles made this symbol up on the spot)}\]

Define

\[
\begin{align*}
\gamma(\lambda) &= U'^{-1}(\lambda) \\
g(\lambda) &= \Omega'^{-1}(\lambda) \\
h(\lambda) &= V'^{-1}(\lambda) \\
h'(\lambda) &= f(\lambda) + g(\lambda)
\end{align*}
\]

\[\lambda = V'(k)\]

\[K = h(\lambda)\]

\[h'(\lambda) = \frac{d}{d\lambda} V'^{-1}(\lambda) = \frac{1}{V''(V'^{-1}(\lambda))} = \frac{1}{V''(k)}\]

\[h''(\lambda) = \frac{d^2}{d\lambda^2} V'^{-1}(\lambda) = \frac{d}{dK} \frac{1}{V''(k)} \frac{dK}{d\lambda} = -V'''(V'^{-1}(\lambda))\]
2/17/05

Ideas want to express things in terms of inverse marginal value function. (That’s what we did on bottom of previous page.)

\[ h''(\lambda) = \frac{d}{dk} \left( \frac{V''(k)}{V''(k)^3} \right) \cdot \frac{dk}{d\lambda} \]

\[ = -\frac{V''''(k)V''(k) + 3V'''(k)^2}{V''(k)^4} \cdot \frac{1}{V''(k)} \]

Consider \( \frac{\lambda h'(\lambda)}{h(\lambda)} = \frac{V'(k)}{KV''(k)} \). Elasticity of \( h = \) relative risk tolerance.

Consider the property \( \frac{-KV''(k)}{V'(k)} = \gamma \iff \frac{\lambda h'(\lambda)}{h(\lambda)} \leq \frac{1}{\gamma} \)

Show this is preserved under horizontal & vertical integration.
We’ve already done vertical — we showed it was preserved under expectation.
Show horizontal:

\[ \lambda h'(\lambda) - \frac{1}{\gamma} h(\lambda) \leq 0 \]

If \( \lambda f''(\lambda) - \frac{1}{\gamma} f(\lambda) \leq 0 \) AND \( \lambda g''(\lambda) - \frac{1}{\gamma} g(\lambda) \leq 0 \)

THEN \( \lambda h'(\lambda) - \frac{1}{\gamma} h(\lambda) \leq 0 \), where \( h = f \circ g \).
What does it mean if it is preserved under horizontal and vertical integration? (In precautionary saving problem) step toward showing something about relative prudence can extend this to include risky assets.

We will ignore labor supply decisions/risks. Assume Publicly Complete Markets (all Arrow-Debreu Securities \( \mathcal{S} = \) complete market).

In publicly complete markets, whenever a security exists, a complete set of options on it also exists. (So, publicly complete is a weaker assumption than complete markets.)

\[
V(W_{t+1}) = \max_{c_t, s_t, s_{t+1}} U(C_t) + \beta \mathbb{E}_{t} \left[ V \left( \frac{c_t}{p_t}, (W_{t+1} - C_{t+1}) + \frac{s_{t+1}}{p_{t+1}}, \frac{s_{t+1}}{p_{t+1}} \right) + \beta \mathbb{E}_{t+1} \left( V(W_{t+2} - C_{t+2}) + \frac{s_{t+2}}{p_{t+2}} \right) \right]
\]

where \( s_t \) is state-contingent security i.e.

thing what you have here is another horizontal integration problem. (Apparently, the complete public market assumption is very important.)

\[
h''(\lambda) = -\frac{V''(K)}{V'(K)^2}
\]

look at \[
\frac{\lambda h''(\lambda)}{h'(\lambda)} = \frac{-V''(K)}{V'(K)^2} - \frac{V'(\frac{V''}{V'x^2})}{(-\frac{1}{x^2})} = -\frac{V''}{V'x^2}
\]

Think about property \[
\frac{V''}{V'^2} \geq \frac{1}{x^3}
\]

positive definiteness of \[
\begin{bmatrix}
\frac{V'}{V'x} & \frac{\sqrt{3} V'}{V'x} \\
\frac{V'}{V'x} & \frac{V'}{V'x}
\end{bmatrix} \Rightarrow E \begin{bmatrix}
\frac{V'}{V'x} & \frac{\sqrt{3} V'}{V'x} \\
\frac{V'}{V'x} & \frac{V'}{V'x}
\end{bmatrix}
\]

2/17/05
What about liquidity constraints?

Suppose \( \frac{-x}{h(x)} = \beta \neq 0 \) and \( h'(x) \neq 0 \). Is this preserved?

Now, what about horizontal aggregation?

\[ \lambda \theta(x) = \beta \]
Miles thinks horizontal & vertical aggregation are very important
- sees Preiser-Max theorem as a competing approach

(sometimes they can be used for same thing, but
H & V integration is generally easier)

however, H & V aggregation rely on concavity assumption
- so Preiser-Max theorem can help you show things are
monotonic, concave, etc.

Preiser-Max Theorem: p.65 of notes
- this is as much a proof strategy as a theorem.

\[ V(K_t) = \max_{x_t} U(K_t, x_t) + \beta E_t \]

\[ V_t(K_t) = \max_{x_t} U(K_t, x_t) + \beta E_t \cdot V^{\ast \ast} (F(K_t, x_t, w_{t+1})) \]

\[ F(K_t, x_t) \]

\[ V(K) = \max_x F(K, x) \]

If I know a property of \( F \), what can I know
about \( V \)?
   i.e., if \( V_{t+1} \in E_P \), then \( K_t \in E_Q \)
   if \( F \in E_Q \), then \( V_t \in E_P \)

   this part is what the Preiser-Max Theorem
   is about.

Preiser-Max Theorem
- Suppose we have a property \( Q \) that can be
  expressed in the following form:
  \[ \Delta (A_t \Rightarrow x^t) B(K_t, x^t) \geq 0 \quad \forall K_t, X^t, K^t, X^t \in E \]
Example: How to express the property "increasing".

\[ F(K_a, X) - F(K_b, X) \geq 0 \quad \forall K_a \geq K_b \]

- A target in Nash function analysis is when a function being big makes it easy to satisfy the property.
- An anchor is when a function being small makes it easy to satisfy the property.

Example: DRTS in \( K \times X \)

\[ F(\theta K, \theta X) - \theta F(K, X) \geq 0 \quad \forall \theta \in [0, 1] \]

Example: Jointly Concave in \( K \times X \)

\[ F(\theta K_1 + (1-\theta)K_2, \theta X_1 + (1-\theta)X_2) - \theta F(K_1, X) - (1-\theta)F(K_2, X) \geq 0 \quad \forall K_1, K_2, X_1, X_2 \]

Note: To use this property, have to write it down in discrete form (like this) — no derivatives.

Example: Value function is quadratic.

\[ V(K, 3S) - 3V(K, 2S) + 3V(K+5) - V(K) = 0 \]

This has 2 targets & 2 anchors — so, the symmetry theorem can't handle it.
Preser - Max Thm

Suppose we have a property \( Q \) that can be expressed as
\[
\Phi(\bigcup_{A^*} K^* \times X^*), B(K^*, X^*) \geq 0
\]
\[\forall K^*, X^*, K^*, X^* \in \Theta \]

If \( K^*, X^* \in \Theta \)

(1) \( (K^*, X^*, K^*, X^*) \in \Theta \)

and

(2) \( \forall K^*, X^*, \Phi(A(K^*, X^*), \hat{X}(K^*, X^*)), B(K^*, X^*) \geq 0 \)

Then: \( \Phi^* \max_{x^*} A(K^*, x^*), \max_{x^*} B(K^*) \geq 0 \)

where \( x^* = \arg\max_{x^*} B(K^*, x^*) \)

\( \Sigma_x V(K) = \max_{x \in \Theta} F(K, X) \)

\( V(K_2) - V(K_1) = \max_{x \in \Theta} F(K_2, X) - \max_{x \in \Theta} F(K_1, X) \)

\[= \max_{x \in \Theta} F(K_2, X) - F(K_1, X) \]

Suppose \( F \) is increasing in \( K \) alone:

\[\geq F(K_2, X_1) - F(K_1, X_1) \]

\[\geq 0 \]

So: \( V \in \mathcal{P}, F \in \mathcal{Q}, F(K_2, X) - F(K_1, X) \geq 0 \forall K_2 \geq K_1 \)
Note: we also need $x_i \in \mathbb{R}$

Example 1: Inequality constraints

$$\sum_{i=1}^{n} x_i \leq k$$

\[ V(\Theta K) - \Theta V(K) = \max_{x \in \text{region}} F(\Theta K, x) - \Theta \max_{x \in \text{region}} F(K, x) \]

\[ = \max_{x} F(\Theta K, x) - \Theta F(K, x^*) \]

\[ \geq F(\Theta K, x^*) - \Theta F(K, x^*) \geq 0 \]

Two possible properties of interest for $F$.

\[ \text{If } F \in Q_1: F(\Theta K, \Theta x) - \Theta F(K, x) \geq 0 \Rightarrow \Theta x \text{ is feasible} \]

\[ \text{If } F \in Q_2: F(\Theta F, x) - \Theta F(K, x) \geq 0 \Rightarrow x \text{ is feasible} \]
Ex 1: Concavity
Assume \( F \) is jointly concave in \( K \& X \). Show \( V \) is jointly concave in \( K \& X \).

\[
V(\theta k_1 + (1-\theta)k_2) = \theta V(k_1) - (1-\theta)V(k_2)
\]

\[
= \max_x F(\theta k_1 + (1-\theta)k_2, x) - \theta \max_x F(k_1, x) - (1-\theta) \max_x F(k_2, x)
\]

\[
\geq F(\theta x_1 + (1-\theta)x_2) - \theta F(x_1) - (1-\theta)F(x_2)
\]

\[
\geq 0
\]

Problem: Given what is \( \theta \)? (must be convex in \( X \) direction)

- if \( X \) can only take on discrete values, this proof doesn't go through & you don't get concavity of \( V \).

Examining this issue:

\( X \in \{1,2,3\} \)

\[ F(k_1, x) \]

\[ F(k_2, 2) \]

\[ F(k_1, 1) \]

Not concave, note that monotonicity is still preserved.
Exam: replace impossible property w/ DRRA.

Hard part is vertical integration

\[ \phi'(x) = \ln U'(c^x) \]
\[ x = \ln c \]
\[ \phi''(x) \]

\[
\begin{bmatrix}
\phi(x) & \phi'(x) \\
\phi'(x) & \phi''(x)
\end{bmatrix}
\]

Problem 2: what is right w/ proof sketch

- where it goes wrong

issue: where is \( RRA \) going down?

Issue: DRRA is a LOCATION - INVARIANT property (DRA is a location - invariant property).

from the Convex Analysis book.

Legendre Conjugate

\[ V^*(x) = \lambda V'^{-1}(x) - V(V'^{-1}(x)) \]

if we had a convex function, would be like this.
if we have concave function, 

\[ V^*(\lambda) = V^{*-1}(\lambda) + \lambda \frac{1}{V''(V^{*-1}(\lambda))} - \frac{V(V^{*-1}(\lambda))}{V''(V^{*-1}(\lambda))} \]

\[ = V^{*-1}(\lambda) \]

idea: if we want first derivatives that are inverses this tells us how the functions themselves must relate to each other.

\[ V(K) = \max_x \Pi(K_x) - C(I_t) + \mathbb{E}_x D_{t+1}V_{t+1}(K_{t+1}) \]

Define \( X = K_{t+1} \) (to emphasize that we are treating it as control variable)

\[ I = x \otimes (1-\delta) K_t \] (standard eqn)

\[ V_t(K_t) = \max_x \Pi(K_t) - C(X - (1-\delta)K_t) + \mathbb{E}_x D_{t+1}V_{t+1}(K_{t+1}) \]

assume \( V \) is concave

\[ \sqrt{\mathbb{E}_t} \sqrt{x} \]

\[ \text{can prove this using ver. aggregator,} \]

- start problem #2 this way:

\[ C'(X - (1-\delta)K_t) = \mathbb{E}^*_t(X) = X \]
\[ X = \Omega^{-1}(\lambda) \]
\[ X - (1-\delta)K = C^{-1}(\lambda) \]

\[ K = \frac{1}{1-\delta} \left[ \Omega^{-1}(\lambda) - C^{-1}(\lambda) \right] \]

**Note:** \[ C'(I) = \lambda \]
\[ I = C'^{-1}(\lambda) \]
\[ -I = -C'^{-1}(\lambda) \]

Say \(-I = B\). Then we are looking at \((C-B)\).

So... the minus sign is not a problem for aggregation.

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**Problem:**

\[ K_t = \varphi_t(\lambda) \]

Use envelope theorem.

\[ V'(K) = \Pi'(K_t) - \Theta'(1-\delta) C'(X - (1-\delta)K_t) \]

\[ V'(K) = \Pi'(K) + (1-\delta) \lambda \]

Want everything as function of \( \lambda \).

\[ V'(\varphi(\lambda)) = \Pi'(\varphi(\lambda)) + (1-\delta) \lambda \]

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**What this has to do with the conjugate:**

\[ V^*(\lambda) = U^*(\lambda) + \Omega^*(\lambda) \]

\[ \lambda V^*(\lambda) - V(\lambda^*) = \lambda U^*(\lambda) - U(U^{*-1}(\lambda)) + \lambda \Omega^*(\lambda) - \Omega(\Omega^{*-1}(\lambda)) \]
consumption / saving problem

\[ V(k^*_t) = \max_c U^*_c(c) + \Pi^*_t(k^*_t - c) \]

Table of Conjugates

\[
\begin{align*}
V^*(k) &= kV(k) - V(k) \\
\lambda V^*(k) - V(k) &= V(k) \\
\lambda V^*(k) &= V(k) \\
V^{**}(k) &= \frac{1}{V''(k)} \\
\frac{1}{V''(k)} &= V''(k)
\end{align*}
\]

Show how this works:

\[ \lambda V^*(k) - V(k) - \left( V'(k)k - V'(k) + V(k) \right) \]

More Conjugates

\[
\begin{align*}
V^{***}(k) &= -\frac{V^{**}(k)}{V''(k)^2} \\
-\frac{V^{**}(k)}{V^{***}(k)^3} &= V''(k) \\
V^{****}(k) &= \frac{3V'''(k)^2 - V^{**}(k)V''(k)}{V''(k)^5}
\end{align*}
\]

Take property of positive third derivative: the property is conjugate to itself (if you look @ convexity sideways, it is still convexity).
returns to scale

risk aversion is conjugate to risk tolerance

prudence

\[ \begin{align*}
\frac{V^c(x)}{\lambda V^c(x)} & \text{ is conjugate to } \frac{V(x)}{V^c(x)} - \frac{V(c)}{k V^c(k)} \\
\frac{V''(x)}{V^c(x)} & \text{ conjugate to } -\frac{V'(k)}{k V''(k)} \\
\frac{V''(k)}{V''(k)} & \text{ conjugate to } \frac{V''(k) V'(k)}{V''(k)^2}
\end{align*} \]

DARA is conjugate to itself.

One other property: could do this for \#2.

\[ \frac{\partial^2 V(k, \beta)}{\partial k \partial \beta} > 0 \]

\[ V^c(x) \]

\[ \beta^* \text{ draws a line sideways - still goes up. So supermodularity is conjugate to itself.} \]

as people are more patient, \( V' \) goes up.
How to look at supermodularity in terms of Prešer-Max theorem:

\[ V(K_2, \beta) = \max_{\beta} \left[ \frac{\text{UC}(K_2, \beta, X)}{F(K_2, \beta, X)} + \Omega(K_2, \beta, X) \right] \]

If \( F(K_2, \beta, X) \) is jointly supermodular in all \( \beta \) arguments, then \( V(K, \beta) \) is supermodular in \( \beta \) arguments.