Interesting Problems in (and Beyond) Apollonian Circle Packings

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Outline

Geometric Construction of ACPs

- Given three circles α, β, γ, pairwise tangent at three points, is it possible to draw a fourth circle δ, tangent to the original three?
- It was known to the Greek mathematician Apollonius that there are always exactly two such circles δ, δ'.
- This suggests the following procedure.
  - Begin with four circles, pairwise tangent at six points.
  - For any three mutually tangent circles, draw the (other) common tangent.
- The resulting picture is called an Apollonian Circle Packing

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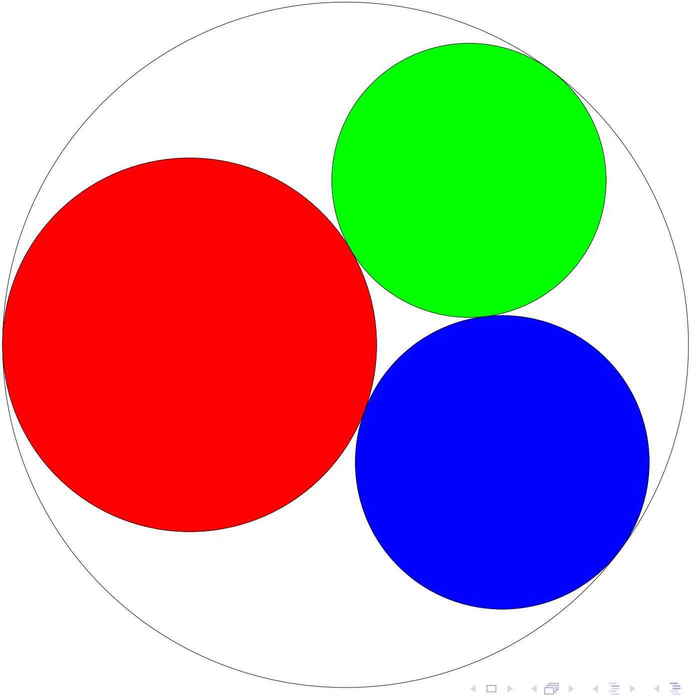

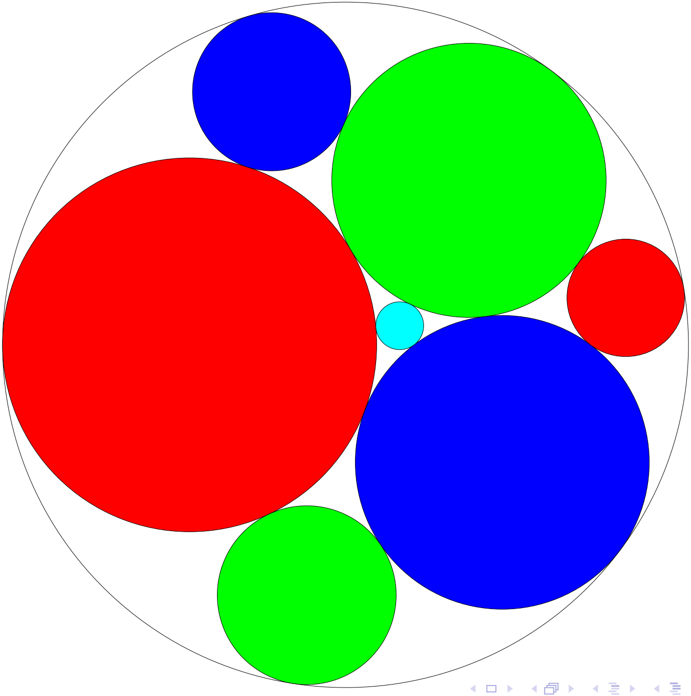

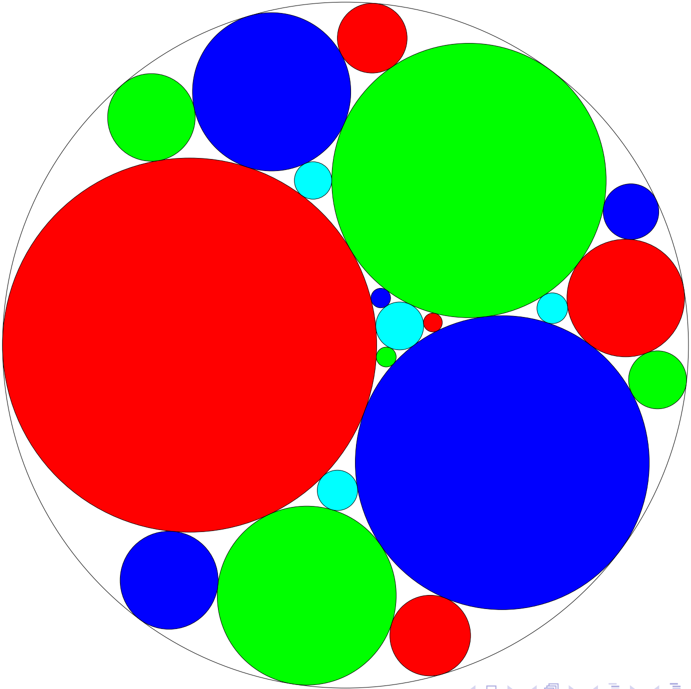

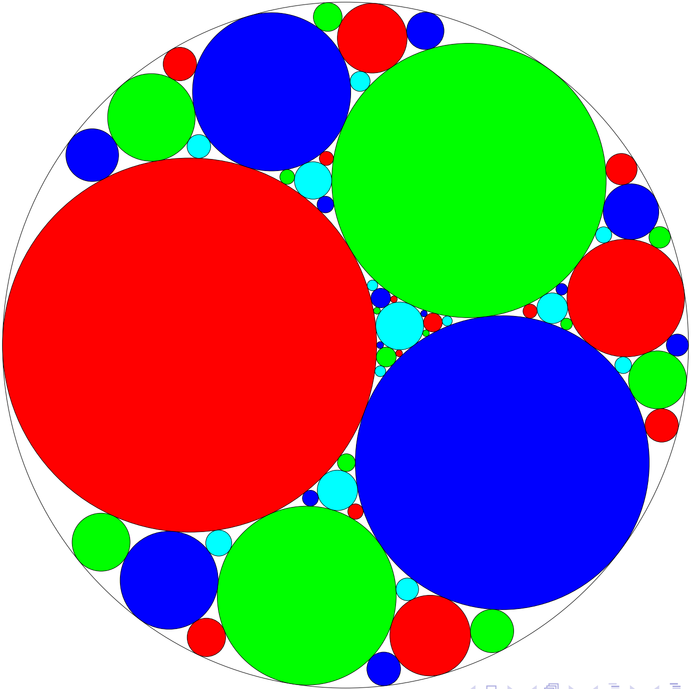

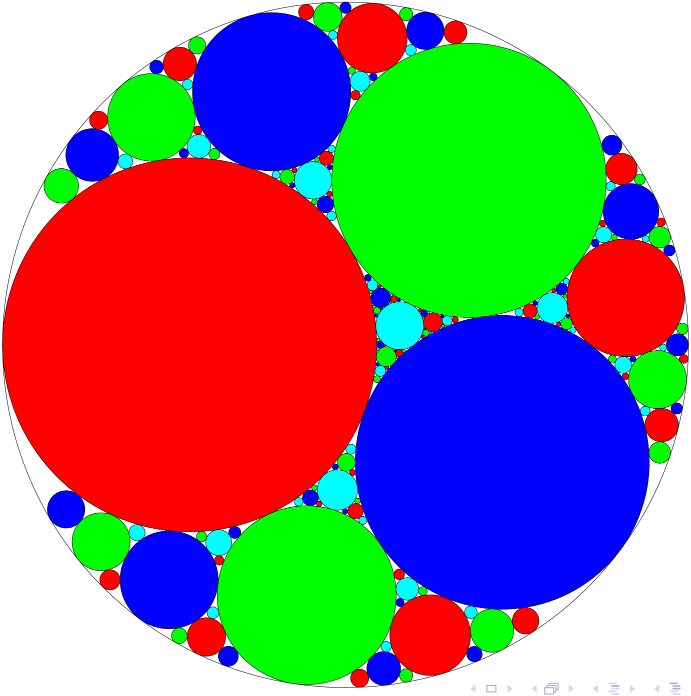

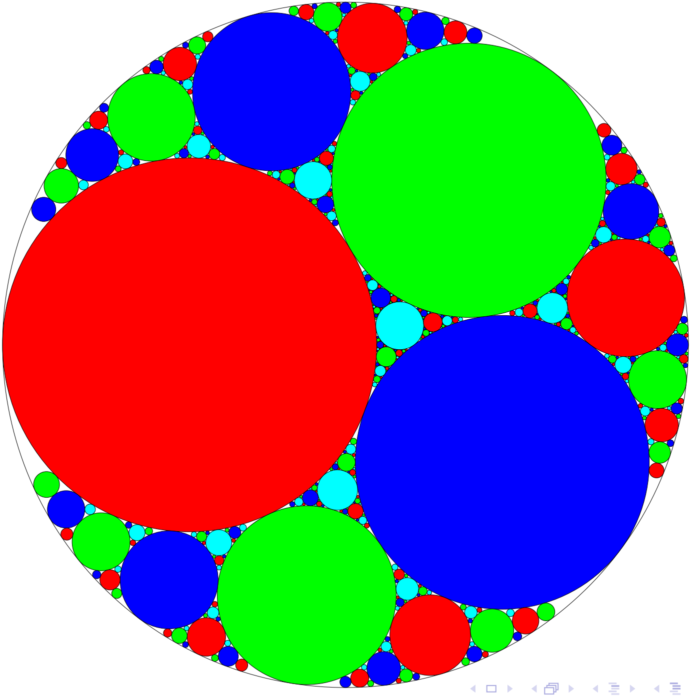

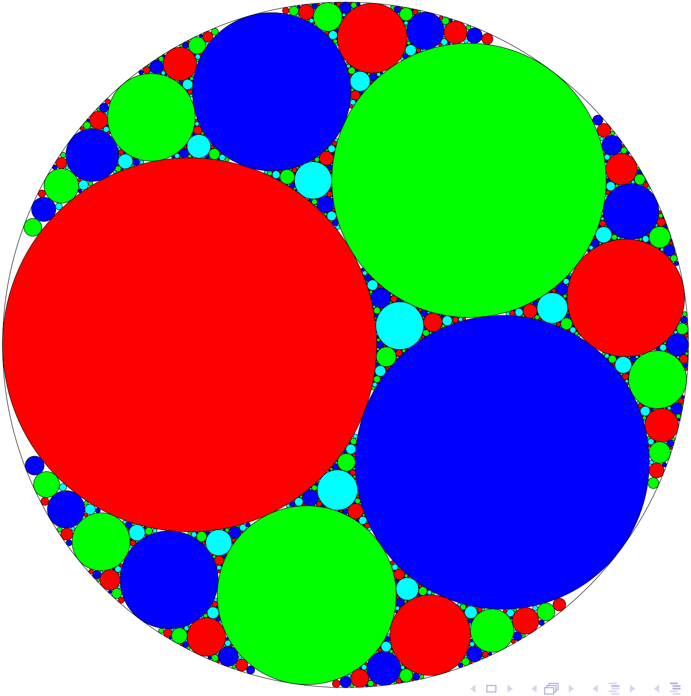

Fundamental Equations

- It turns out to be more convenient to work with the curvature of a circle than the radius.
- It is possible to draw four mutually tangent circles with curvatures a, b, c, d iff $(a+b+c+d)^2 - 2(a^2+b^2+c^2+d^2) = 0$.
- This form $Q(a,b,c,d) = 2(a^2+b^2+c^2+d^2) - (a+b+c+d)^2$ is called the Descartes quadratic form.
- If a, b, c are given, then the possibilities for d are the roots of the quadratic $x^2 - 2(a+b+c)x + (a^2+b^2+c^2-2ab-2ac-2bc)$.
- The roots are $d_\pm = a+b+c \pm 2\sqrt{ab+bc+ca}$.
- In particular $d_+ + d_- = 2(a+b+c)$.

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How Pretty Pictures of Circles Become Number Theory

- If we start with three curvatures a, b, c, we can get all the rest by ring operations and square roots.
- If we start with four curvatures a, b, c, d, we can get all the rest simply by ring operations.
- **Proposition.** If three mutually tangent circles in a packing have integer, then all circles in that packing have curvature in some quadratic extension.
- **Proposition.** If four mutually tangent circles in a packing have integer curvature, then all circles in that packing have integer curvature.
- We don't just have configurations of circles, we have configurations of numbers.

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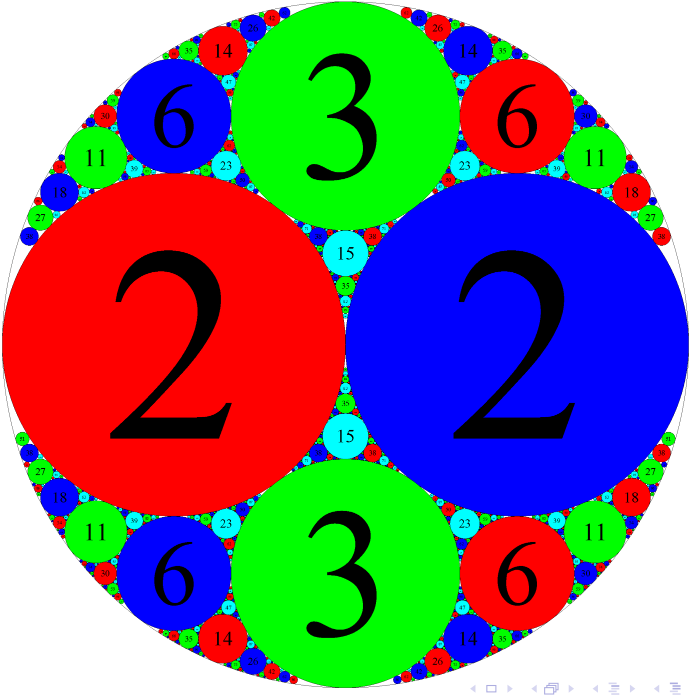

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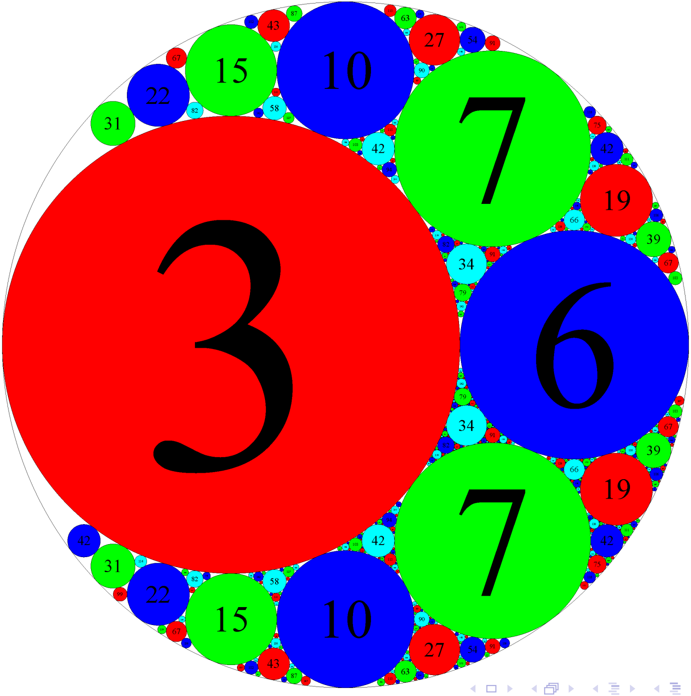
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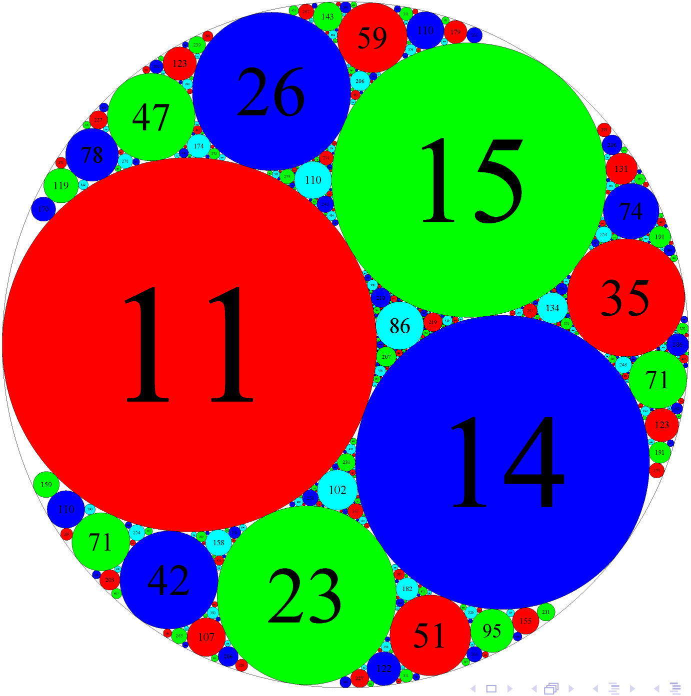
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The Apollonian Group $\mathcal{A}_4$

- It's best to think of an IACP as a configuration of quadruples of circles (or quadruples of numbers).
- View one quadruple as adjacent to the four quadruples obtainable by switching out one circle.
- It is possible to get from any quadruple of circles to any other in this way, essentially uniquely.
- If we let $\mathcal{A}_4 = \langle w, x, y, z \rangle / \langle w^2, x^2, y^2, z^2 \rangle$, then what we are studying is the way this group acts on integers.

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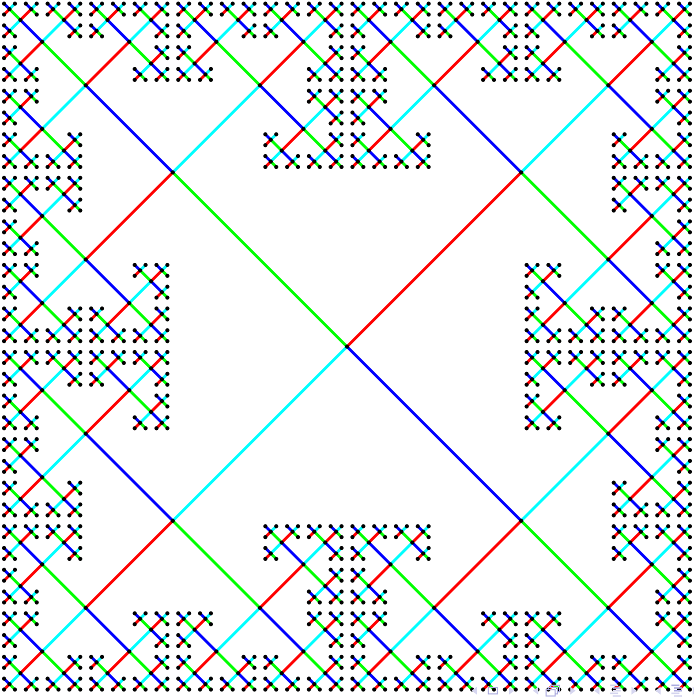

NonEuclidean Packings.

- If we start with four integers a, b, c, d such that $Q(a, b, c, d) \neq 0$, then we can still formally construct a configuration of integers as before (replacing d by $2(a + b + c) - d$ and iterating).
- Such a thing corresponds to packing circles in hyperbolic or elliptic space (according to the sign of Q).
- In particular Q is the same for all quadruples in the generalized packing.

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Counting Problems

- How many packings occur with a given exterior curvature?
- More generally, for an integer a, how many times does it appear in IACPs (up to symmetry)?
- The answer turns out to be unexpectedly pretty; occurrences of a in (primitive) IACPs correspond to (primitive) sublattices of $\mathbb{Z}^2$ with index $|a|$.
- The norms of primitive elements in the lattice correspond to curvatures adjacent to a.
- Occurrences of a also correspond to a certain class of quadratic forms.

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- For a pair of integers a, b, how many times do they appear adjacently in IACPs (up to symmetry)?
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- Counting occurrences of a, b turns out to be the same as counting solutions of $x^2 \equiv ab \pmod{a+b}$.

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Which Curvatures Appear?

- The main open problem in IACPs is knowing which integers occur in a given packing.
- All the curvatures in an IACP must satisfy a certain condition modulo 24.
- **Big Open Question.** Do all but finitely many integers which satisfy the condition modulo 24 appear?
- An affirmative answer was conjectured by Graham, Lagarias, Mallows, Wilks, and Yan. Mathematicians who study ACPs are sometimes classified into optimists and pessimists based on whether they believe the conjecture.
- It is a very recent result of Elena Fuchs that the integers which appear have positive density, but this is still quite far from what is hoped-for.

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Versions of the Main Conjecture

- **Question.** Does the GLMWY Conjecture hold for even one packing?
- **Question.** Can we find a finite list of packings which together contain almost all integers?
- **Question.** Does the GLMWY Conjecture fail for even one packing?
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- The case of arithmetic progressions (i.e. which residue classes appear) is pretty well handled.
- The case of primes was resolved when Peter Sarnak proved the Apollonian Twin Prime Theorem.
- **Question(s).** Unless S is the set of primes or an arithmetic progression, there is still a lot of work to do here. Does a given packing contain infinitely many squares? Cubes? Triangular numbers? Fibonacci numbers?

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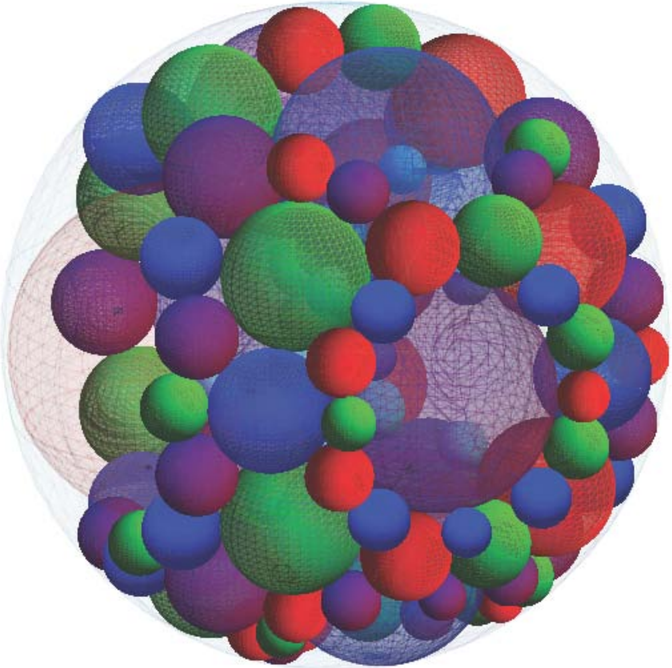

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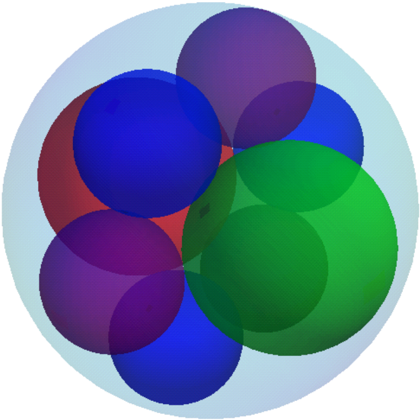

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More General Numbers

- The geometric construction did not require the curvature to be integers or even rationals.
- In fact, if we are only interested in configurations of numbers, we can operate over any ring, whether real or not.
- An ACP over a ring is given by a quadruple in the ring satisfying $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$. extended by iterating the $d \leftrightarrow 2(a+b+c)-d$ action. We are no longer thinking of geometric curvatures.
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