

Isoperimetric regions in surfaces and in surfaces with density

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October 16, 2006

Abstract

We study the isoperimetric problem, the least-perimeter way to enclose given area, in various surfaces. For example, in two-dimensional Twisted Chimney space, a two-dimensional analog of one of the ten flat, orientable models for the universe, we prove that isoperimetric regions are round discs or strips. In the Gauss plane, defined as the Euclidean plane with Gaussian density, we prove that in halfspaces $y \geq a$ vertical rays minimize perimeter. In R^n with radial density and in certain products we provide partial results and conjectures.

1 Introduction

The isoperimetric problem seeks the least-perimeter way to enclose given area or volume. The classical isoperimetric theorem states in R^2 that for given area a round circle uniquely minimizes perimeter. The isoperimetric solution is also known in other locally Euclidean surfaces, such as the torus, cylinder, the klein bottle, and some surfaces with density such as R^2 with density $e^{-cr^2/2}$ (the Gauss plane) or e^{cr^2} .

We will examine the isoperimetric problem in surfaces where the solution is unknown. We are first able to prove solutions to the isoperimetric problem on subsets or quotients of R^2 and S^2 . There are various tools we can use to help solve this problem. Often, regularity results limit the possibilities of candidates for minimizers because, in general, they require a minimizer to have constant curvature and curves of constant-curvature are just circles or lines in locally Euclidean or spherical surfaces. Moreover, we can use proven results in similar locally Euclidean spaces to help determine minimizers in these unknown spaces.

We then examine the isoperimetric problem on surfaces with smooth positive density functions used to weight area and perimeter. Manifolds with density have long appeared throughout mathematics. An example, of much interest to probabilists, is Gauss space

– Euclidean space with density $ce^{-r^2/2}$ (see [M2]). The isoperimetric solution is known for Gauss space. We consider halfspaces, strips and sectors in the Gauss plane, other planes with radial density, and simple products.

The isoperimetric problem becomes more difficult in surfaces with density because less is known about constant-curvature curves in these spaces. One useful tool in spaces with density is symmetrization (see [Ros]). We also often use simple geometric arguments to rule out possible candidates for minimizers.

We now discuss in further detail the spaces we consider and the results.

1.1 Strips of R^2 and S^2

The solutions to the isoperimetric problem in both a strip of R^2 and a strip of S^2 are doubtless known but we provide two more proofs. In a strip in R^2 Proposition 3.1 shows that for small area a semicircle minimizes perimeter and for a larger area a pair of vertical lines minimizes perimeter (Figure 1). Similarly, Proposition 3.2 shows that in a strip of S^2 for small area a circular arc perpendicular to the boundary closer to the equator is minimizing, while for larger area a pair of circular arcs of longitude from one boundary to the other minimizes perimeter (Figure 2).

1.2 Two-dimensional Twisted Chimney Space

Two dimensional Twisted Chimney Space is an infinite horizontal strip of R^2 with the boundaries identified with a flip about the y-axis. This space is a two-dimensional analog of one of the ten flat, orientable models for the universe (see [AS]). Proposition 4.1 shows that in this space for small area a circle minimizes perimeter and for large area a pair of vertical lines minimizes perimeter (Figure 3).

1.3 Gauss Space

Gauss space, G^m , is R^m endowed with Gaussian density $(2\pi)^{-m/2}e^{-r^2/2}$ used to weight both volume and perimeter. We examine the translated half-plane of G^2 and prove that rays perpendicular to the boundary minimize perimeter (Proposition 5.1). We also examine the strip in G^2 . Conjecture 5.5 states that instead of line segments other constant-curvature curves from one boundary to the other minimize perimeter.

1.4 R^n with Radial Density

Section 6 generalizes results about Gauss space to any radial density on R^n . For example, Theorem 6.22 shows that a connected minimizer in a sector of a plane with radial density must be monotonic in its distance from the origin.

1.5 $R^1 \times G^1$

$R^1 \times G^1$ is R^2 endowed with density $(1/\sqrt{2\pi})e^{-y^2/2}$. Conjecture 7.4 says that in $R^1 \times G^1$ for small area some curve from infinity to infinity minimizes perimeter and for large area a pair of vertical lines minimizes perimeter. We also consider $S^1 \times G^1$. Conjecture 7.10 says that in $S^1 \times G^1$ for small area a nonlinear, homotopically nontrivial curve minimizes perimeter and for large area a pair of vertical lines minimizes perimeter.

Acknowledgements. This paper is based on my undergraduate thesis ([L]) at Williams College with Frank Morgan. I would like to thank my advisor Frank Morgan for his input and guidance. I would also like to thank Diana Davis, Rohan Mehra, and Vojislav Šešum for their interest and suggestions.

2 Existence and Regularity

Even the existence of minimizers is often difficult to show. Standard compactness arguments of geometric measure theory (see [M1], 5.5, 9.1) give the compactness of the space of regions of given volume, but a problem arises when there can be a loss of volume at infinity. Theorem 2.1 gives certain cases when the existence of minimizers is known. Remark 2.2 provides known results about the regularity of minimizers. In general, minimizers are smooth surfaces with some exceptions in higher dimensions.

Theorem 2.1. (Existence [[M1], pp. 129-131]) *Let M be a complete Riemannian manifold, possibly with positive continuous density function used to weight both volume and perimeter. If M is compact or of finite volume or if $M/\{\text{isometries}\}$ is compact or of finite volume, then for given volume there is a region of least perimeter.*

Sketch of proof. Consider regions of that volume. The set of their perimeters has an infimum because every set of positive real numbers has an infimum. Take a sequence of regions whose perimeters converge to the infimum. Since the perimeters converge, the perimeters are bounded. Since volume is given it is bounded. Therefore by the local Compactness Theorem (see [M1], 5.5, 9.1) the space of these regions is compact. Thus, there is a subsequence of regions that converges. The limit can have no more perimeter than the perimeters of the regions in the sequence so it must be equal to the infimum of their perimeters. The limit can have no more volume than the volume of the regions in the sequence. The difficulty arises in showing that the limit has the correct volume because

the region could for example have components that go off to infinity. If M is compact or has finite total volume then the region must have the correct volume. If $M/\{\text{isometries}\}$ is compact or $M/\{\text{isometries}\}$ has finite total volume, then if volume disappears at infinity use the isometries to pull some back into a compact region or a region of finite total volume (we are omitting many details here). \square

Remark 2.2. (Regularity) There are several known results about the regularity of perimeter-minimizing enclosures of prescribed volume. In a Riemannian manifold of low dimension $n \leq 7$, minimizers are smooth surfaces ([M3], Corollary 3.7). In higher dimensions minimizers are smooth surfaces except for a set of Hausdorff dimension at most $n - 7$ ([M3], Corollary 3.8). These results also hold for Riemannian manifolds with positive density functions that are as smooth as the metric ([M3], Remark 3.10). If a minimizer is smooth it must have constant mean curvature ([M1], 8.6). In a surface with free boundary (which does not count in the perimeter cost), it is easy to see that a minimizer must meet the boundary orthogonally.

3 Strips in R^2 and S^2

Here we examine strips of R^2 and S^2 . Proposition 3.1 shows that in a strip of R^2 for small area, a semicircle on the boundary minimizes perimeter and for large area a pair of vertical lines minimizes perimeter (Figure 1). Recall that we are working with free boundary, which does not contribute to the perimeter. The proof uses the known minimizers in a cylinder and the correspondence between minimizers in a strip of R^2 and symmetric minimizers in a cylinder. Proposition 3.2 shows that in a strip in S^2 for small area a circular arc that meets one boundary perpendicularly minimizes perimeter and for large area a pair of arcs from one boundary to the other minimizes perimeter. The proof uses existence and regularity results to narrow down the possibilities to circular arcs and then examines the individual cases.

Proposition 3.1. *In an infinite strip $S = \{ -a \leq y \leq a \} \subset R^2$ with free boundary, given $A > 0$, the least-perimeter way to enclose area A is*

1. *a semicircle on a boundary if $0 < A < 8a^2/\pi$,*
2. *a pair of vertical lines if $8a^2/\pi < A$,*
3. *either a semicircle or a pair of vertical lines if $A = a^2/\pi$.*

Proof. Given area A , suppose that a set of curves C enclosing that area has no more length than the semicircle, if it fits, and a pair of vertical lines. Reflect S along one of its boundaries and then identify the other two, creating a cylinder. Then C and its reflection will enclose area $2A$ but have no more length than the circle, if it fits, and a pair of horizontal circles. This is a contradiction because in a cylinder least-perimeter enclosures are small circles for $A \leq 16a^2/\pi$ and two horizontal circles for $A \geq 16a^2/\pi$ [HHM]. \square



Figure 1: In a strip of R^2 for small area a semicircle minimizes perimeter and for large area a pair of vertical lines minimizes perimeter.

Proposition 3.2. *In a strip of S^2 , $S = \{ x^2 + y^2 + z^2 = 1, a \leq z \leq b, b > a \}$, with free boundary, for given area, the least-perimeter way to enclose area is a circular arc on the boundary nearer to the equator or a pair of arcs from one boundary to the other (Figure 2).*

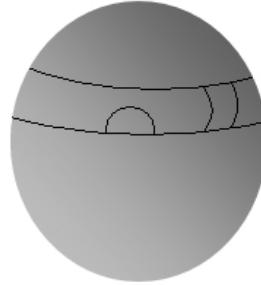


Figure 2: In a strip of S^2 for small area a circular arc from one boundary to itself minimizes perimeter and for large area a pair of circular arcs minimizes perimeter.

Proof. Since the total area of the strip is finite, by Remark 2.1 minimizers must exist. They must also consist of constant curvature curves that meet the boundary perpendicularly (Remark 2.2). A minimizer cannot contain a homotopically trivial curve because otherwise it could be translated to touch the boundary tangentially, contradicting regularity. So the only two possibilities for minimizers are circular arcs that touch one boundary or circular arcs that touch both boundaries. If a circular arc goes from one boundary to itself it must meet the boundary perpendicularly. Since the surface of the sphere is more curved further from the equator, a circular arc on the boundary nearer to the equator will have less length than a circular arc on the boundary further from the equator. If a circular arc goes from one boundary to the other it must be a an arc of longitude because otherwise it would not meet both boundaries perpendicularly. Since the length of a pair of circular arcs of longitude from one boundary to the other remain constant, they will be more efficient for larger areas.

□

4 Two-Dimensional Twisted Chimney Space

The shape of the universe has puzzled and fascinated scientists for centuries. If the universe is a flat orientable 3-manifold, a likely situation, then Adams and Shapiro ([AS]) discuss the ten possibilities. Here, I examine the isoperimetric problem in the 2-dimensional analog of one of these spaces, namely Twisted Chimney Space. Twisted Chimney Space is a cylinder over a parallelogram with both sets of opposite faces glued together, one straight across and the other with a 180 degree twist around a point on the vertical axis of symmetry. Two-dimensional analogs of the other spaces would be the torus, klein bottle, plane, and infinite cylinder. The isoperimetric solution is known in these other two-dimensional analogs.

Proposition 4.1. (*Twisted Chimney Space; see Figure 3*) Let S be an infinite strip $\{ 0 \leq y \leq a \subset \mathbb{R}^2 \}$ with the top boundary glued to the bottom with a flip about the y -axis. Given $A > 0$, the least-perimeter way to enclose area A is

1. a circle if $0 < A < a^2/\pi$,
2. a pair of vertical lines if $a^2/\pi < A$,
3. either a circle or a pair of vertical lines if $A = a^2/\pi$.

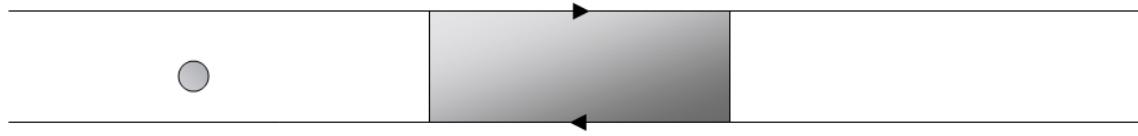


Figure 3: In 2-D Twisted Chimney Space for small area a circle minimizes perimeter and for large area a pair of vertical lines minimizes perimeter.

Proof. Suppose that there exists a different set of curves C that has no more length than a circle or a pair of vertical lines. If C contains a non-trivial component, it must contain at least two non-trivial components in order to enclose area. Since the unique shortest path between two boundaries is a vertical line, C would have length greater than the length of a pair of vertical lines. Therefore, all of the components of C must be homotopically trivial. A circle enclosing equal area will have less length since a circle is the unique solution in \mathbb{R}^2 . If the circle does not fit in S then it must have radius at least $a/2$ and thus length at least πa . This length, however, is greater than the length of a pair of vertical lines, $2a$, a contradiction since C has no more length than a pair of vertical lines. Thus, least-perimeter enclosures in S are either circles or vertical lines. \square

5 Gauss Space

Here we examine the most famous model of a plane with radial density, G^2 , the Gauss plane. G^2 is R^2 endowed with Gaussian density distribution $(1/2\pi)e^{-r^2/2}$. We prove, Proposition 5.1, that in a translated halfspace of G^2 rays perpendicular to the boundary are minimizing. Our proof uses methods used by Corwin et al. ([Co1], Theorem 2.17) to show in G^m that a standard Y is an area-minimizing partition for three nearly equal volumes. It relies on Mehler's 1856 observation that Gauss space is weakly the limit of projections of normalized high-dimensional spheres and the result that in a ball in S^n for given volume the orthogonal intersection with another ball is an isoperimetric region ([BZ], Theorem 18.1.3). We also consider the isoperimetric problem in a strip of G^2 . Conjecture 5.5 says that nonlinear constant curvature curves from one boundary to the other are minimizing.

Proposition 5.1. *In a halfspace of G^m , $H^m = \{y \geq a > 0\}$, for given volume a half hyperplane perpendicular to the boundary minimizes area.*

Proof. We follow Corwin et al. [Co1] to deduce results on G^m from results on S^n for large n . In a ball in S^n , for given volume the orthogonal intersection with another ball is isoperimetric ([BZ], Thm. 18.1.3).

Suppose that in H^m for volume V the half hyperplane is not minimizing, i.e., the half hyperplane has area P and some other surface, S , enclosing the same volume has area $P' < P$ (see Figure 4). Let $\epsilon = P - P' > 0$. By [Co1] Remark 2.15, in a ball of $S^n(\sqrt{n})$, the area of the orthogonal intersection with another ball of volume V converges, as n approaches infinity, to the area of the half hyperplane enclosing volume V in H^m . So for large n , the difference in area between the orthogonal intersection with another ball and the half hyperplane is less than $\epsilon/2$.

Mehler showed that the Gaussian measure on R^m is obtained as the limit as n approaches infinity of projections P_n of the uniform probability density on $S^n(\sqrt{n}) \subset R^{n+1}$ to R^m (see [Co1] Proposition 2.1). Furthermore, by [Co1] Proposition 2.4 the areas of the inverse orthogonal projections to a ball in $S^n(\sqrt{n})$ of a measurable hypersurface $\Sigma \subset H^m$ converge to the area of Σ as n approaches infinity. Thus, the area of S and the area of the preimage of S in the ball of $S^n(\sqrt{n})$ differ by at most $\epsilon/4$ while the volumes differ by at most $\delta/2$.

We want to create a competitor to the proven minimizer in a ball of $S^n(\sqrt{n})$ using the preimage of S . Thus, we need a way to adjust volume at low area cost.

Let $f_n(x)$ be the density function resulting from projecting the uniform probability measure on $S^n(\sqrt{n})$ to R^m . By [Co1] Lemma 2.16, given a compact region $R \in G^m$ and an $\alpha > 0$, there exists $\delta > 0$ and N such that for all $n > N$ and any $\Delta V < \delta$, a ball with f_n -weighted volume ΔV which is split by one hyperplane has f_n -weighted total area (including the area of the hyperplane inside the ball) $\Delta A < \alpha$. This provides a way to adjust small volumes at a low cost.

Observe that there exists a compact region $R \in H^m$ which contains distinct "Lebesgue"

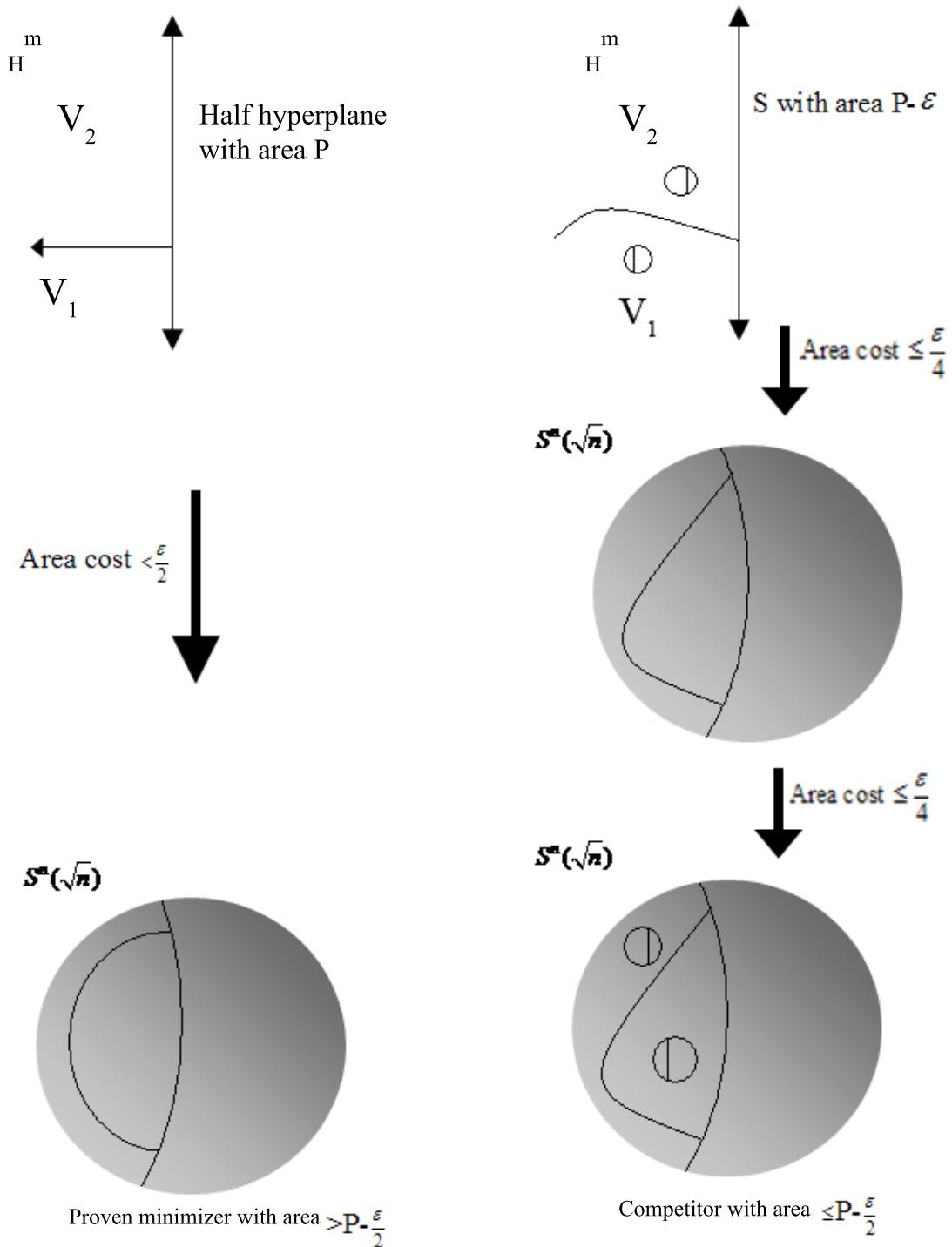


Figure 4: For a given volume V , if there exists a surface S with less area than the half hyperplane in H^m then there exists a surface in a ball in $S^n(\sqrt{n})$ that encloses volume V but has less area than the proven minimizer, a contradiction.

points of density 1 of the region and its complement (i.e. points contained in open balls which are mostly contained in the region or its complement). Using the above result create two balls by choosing $\delta > 0$ for $\alpha = \epsilon/8$ such that the two balls are disjoint and the volume of the region in one ball exceeds half of the volume of the ball and the volume of the complement in the other ball exceeds half of the volume of the ball. The total area cost of constructing two such balls is less than $\epsilon/4$.

Thus, in these balls we can enclose volumes of at least $\delta/2$ at an area cost of at most $\epsilon/4$. Remove the volume from the two balls. Both regions now have less volume than they did in S . Split each ball with a hyperplane and reassign the appropriate amount of volume to each region so that the volume in each region equals the volume in each region of S . This new surface is a competitor on the ball in $S^n(\sqrt{n})$ to the known minimizer. The total area difference between the two is less than $\epsilon/2$ (area difference between the half hyperplane in H^m and the orthogonal intersection with another ball) plus $\epsilon/4$ (area difference between S and the preimage of S) plus $\epsilon/4$ (area cost of adjusting the volumes of the preimage of S). Thus, this new competitor has less area than the proven minimizer, a contradiction. \square

Proposition 5.2. *In an infinite strip $S = \{ -a \leq y \leq a \} \subset G^2$, there exist minimizers that do not intersect a horizontal line $\{ y = y_0 \}$ two or more times.*

Proof. Since the total area of the strip is finite, minimizers must exist (Remark 2.1). Take a minimizer M that intersects a horizontal line two or more times. Then slice the region enclosed by M with horizontal slices and replace each slice with a halfline of equal weighted length. This new region encloses the same area but intersects a horizontal line at most one time. In each slice halflines are minimizing so the perimeter of each slice decreases. This process does not increase perimeter ([Ros], Proposition 7). \square

Proposition 5.3. *In an infinite strip $S = \{ -a \leq y \leq a \} \subset G^2$, if $a < \sqrt{2 \ln 2}$ there exist minimizers that do not intersect a vertical line $\{ x = x_0 \}$ two or more times.*

Proof. Since the total area of the strip is finite, minimizers must exist (Remark 2.1). Consider a vertical line segment in S , $x = x_0$. Since $a < \sqrt{2 \ln 2}$, $e^{-y^2-x_0^2/2} > (1/2)e^{-x_0^2/2}$. In each vertical slice an initial interval will have perimeter less than or equal to $e^{-x_0^2/2}$ and anything else will have greater perimeter since $e^{-y^2-x_0^2/2} > (1/2)e^{-x_0^2/2}$. Take a minimizer M that intersects a vertical line two or more times. Slice the region enclosed by M with vertical slices and replace each slice with an initial interval of equal weighted length. In each slice initial intervals are minimizing so this process does not increase length ([Ros] Proposition 7). \square

Lemma 5.4. *In an infinite strip $S = \{ -a \leq y \leq a \} \subset G^2$, a curve from infinity to infinity cannot be minimizing.*

Proof. Suppose that C a curve from infinity to infinity is minimizing. Then C can be translated along a circular arc until it crosses the boundary not perpendicularly. If it crosses the boundary not perpendicularly, take the portion of the curve, now outside of S and move it

to the other boundary. This curve encloses the same area and has the same perimeter but now contradicts regularity (Remark 2.2). \square

Conjecture 5.5. *In an infinite strip $S = \{ -a \leq y \leq a \} \subset G^2$ with free boundary given area a nonlinear curve from one boundary to the other minimizes perimeter.*

This idea behind this conjecture is similar to the idea behind Proposition 5.1. For low dimensional slabs of R^n , halfspheres and cylinders minimize perimeter, but for $n \geq 10$ cylinder-like surfaces with variable width called unduloids sometimes minimize perimeter for intermediate values of the volume (see [Ros], Theorem 4). Thus, it seems likely that unduloids may solve the isoperimetric problem in slabs of high dimensional spheres. When unduloids are projected down into a strip of the Gauss plane, they would be nonlinear curves from one boundary to the other. Lemma 5.4 should be helpful in proving this conjecture.

6 R^n with Radial Density

We consider Euclidean space R^n with continuous positive density functions $\Psi(r) = e^{\psi(r)}$ used to weight both volume and perimeter. Two classic models have density $ce^{-r^2/2}$, called Gauss space G^2 , and $ce^{+r^2/2}$. Borell and Sudakov-Tsirel'son ([Bor1], [ST]) proved independently that in G^n , for prescribed volumes, halfspaces are perimeter-minimizing enclosures. Carlen and Kerce ([CK]) went on to prove uniqueness. Adams et al. [ACDLV] examined the isoperimetric problem in sectors of the Gauss plane and discovered, numerically, interesting new candidates. Borell ([Bor2], Theorem 4.1) proved that in R^n with density $ce^{r^2/2}$ round balls about the origin minimize perimeter and Rosales et al. ([RCBM], Theorem 5.2) later proved uniqueness.

Here, we generalize various known results for G^m to other radial densities on R^m . We look especially closely at the free boundary isoperimetric problem in α -sectors ($0 \leq \theta \leq \alpha$) of a plane with radial density. Theorem 6.22 shows that connected minimizers in these sectors, if they exist, must be monotonic in their distance from the origin. We show this by first eliminating families of curves that are never monotonic, such as closed curves and curves from infinity to infinity and then showing that the remaining families, curves from one boundary to itself, and one boundary to the other, must be monotonic in their distance from the origin. We also extend some results to R^n with radial density. For instance, we are again able to rule out families of surfaces that are never monotonic in their distance from the origin.

6.1 Existence

Remark 2.1 provided known results on the existence of minimizers when R^n with density is compact or has finite volume. Remark 6.1 provides a necessary but not sufficient condi-

tion for the existence of minimizers when the total measure of R^n with density is infinite. Remarks 6.2, 6.3 provides two examples of spaces in which no minimizers exist.

Remark 6.1. Rosales et al. ([RCBM], Thm. 2.6) show that in planes with increasing, radial density f such that $f(x) \rightarrow \infty$ when $|x| \rightarrow \infty$ minimizers exist for any given area.

Remark 6.2. We provide an example of a plane with nonincreasing radial density where minimizers do not exist for any given area A .

Take a plane with a smooth, non-increasing radial density function such that the region of constant density $1/n$ can fit a circular disc of weighted area n . The perimeter of a circular disc of constant density $1/n$ enclosing area A is $\sqrt{(4\pi A)/n}$. As n approaches infinity the perimeter of a circular disc approaches zero. Thus, there cannot exist a perimeter minimizing region enclosing area A because any possible candidate can be replaced by a circular disc with less perimeter.

Remark 6.3. Rosales et al. ([RCBM] Example 2.7) give an example of R^n with density going to infinity where no minimizers exist. For another example, take R^n with density $f(x) = 1 + |x|^2$ with bumps in the density such that any volume v_k that corresponds to a positive rational can be enclosed with a perimeter of $1/k$. Thus, given any area there is a sequence of regions enclosing this area with arbitrarily small perimeter.

6.2 Constant-curvature curves in planes with radial density

Since in a plane with density minimizers must consist of constant-curvature curves, the study of constant-curvature curves in planes with radial density is interesting. In R^2 (with density 1) the only constant curvature curves are circular arcs and straight lines. More interesting densities yield more interesting constant-curvature curves. We also describe an attempt to find radial symmetric densities such that a given curve has constant curvature.

Definition 6.4. In R^n with density e^ψ , the ψ -curvature κ_ψ of a curve with unit normal vector \mathbf{n} is defined as

$$\kappa_\psi = \kappa - \frac{\partial \psi}{\partial \mathbf{n}},$$

where κ is the Euclidean mean curvature of the curve. Proposition 6.8 justifies this definition.

Remark 6.5. According to computations of [ACDLV], in G^2 , there apparently exist other constant curvature curves, called rounded n-gons (see Figure 5). They satisfy the differential equation

$$-x'(s)y''(s) + y'(s)x''(s) + x(s)y'(s) - y(s)x'(s) = \kappa \quad (1)$$

where s is the arc length with the constraints

$$\begin{aligned} x'(s)^2 + y'(s)^2 &= 1 \\ x(0) &= 0 \\ x'(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Originally, [ACDLV] used a equivalent differential equation in a different form.

$$\sqrt{x''(s)^2 + y''(s)^2} + x'(s)y(s) - x(s)y'(s) = \kappa_\psi \quad (2)$$

where s is the arc length, with the constraints

$$\begin{aligned} x'(s)^2 + y'(s)^2 &= 1 \\ x(0) &= 0 \\ x'(0) &= 1 \\ y'(0) &= 0 \\ x''(0) &= 0 \\ y''(0) &= -y(0) - \kappa_\psi \leq 0. \end{aligned}$$

Equation 2 requires more initial conditions than Equation 1. Moreover, when using Equation 2, although Mathematica still provides a picture of the curve it gives often gives error messages while using Equation 1 produces fewer error messages. [ACDLV] conjecture that these constant-curvature curves are sometimes minimizing in sectors of the Gauss plane.

Conjecture. *There exists an $\alpha_0 \approx 0.58\pi$ such that in an α -sector of the Gauss plane for $\pi/2 \leq \alpha \leq \alpha_0$, minimizers are circular arcs or rays orthogonal to the boundary. For $\alpha > \alpha_0$, minimizers are rays orthogonal to the boundary or emanating from the origin. For $0 < \alpha < \pi/2$, minimizers are circular arcs or half-edges of rounded n -gons.*

Remark 6.6. Inspired by the results of [ACDLV], I started with an interesting curve and looked for a radial symmetric density function such that the curve would have constant curvature. The Euclidean curvature of a curve in polar coordinates is

$$\kappa = \frac{r^2 + 2r_\theta^2 - rr_{\theta\theta}}{(r^2 + r_\theta^2)^{3/2}}$$

The $\partial\psi/\partial n$ term becomes

$$\frac{\partial\psi}{\partial n} = \left(\frac{\partial\psi}{\partial r}, \frac{\partial\psi}{\partial\theta}\right) \cdot \frac{(1, -r'(\theta))}{\sqrt{1 + r'(\theta)^2}}$$

Since we want this density function to be radially symmetric we set $\partial\psi/\partial\theta = 0$. So we get

$$\kappa_\psi = \kappa - \left(\frac{\partial\psi}{\partial r}\right) \left(\frac{1}{\sqrt{1 + r'(\theta)^2}}\right)$$

For a constant-curvature curve, we get the differential equation

$$0 = \kappa' - \frac{\partial^2\psi}{\partial r^2} \left(\frac{1}{\sqrt{1 + r'(\theta)^2}}\right) + \frac{1}{2} \left(\frac{\partial\psi}{\partial r}\right) \frac{1}{(1 + r'(\theta)^2)^{3/2}} r''(\theta) \quad (3)$$

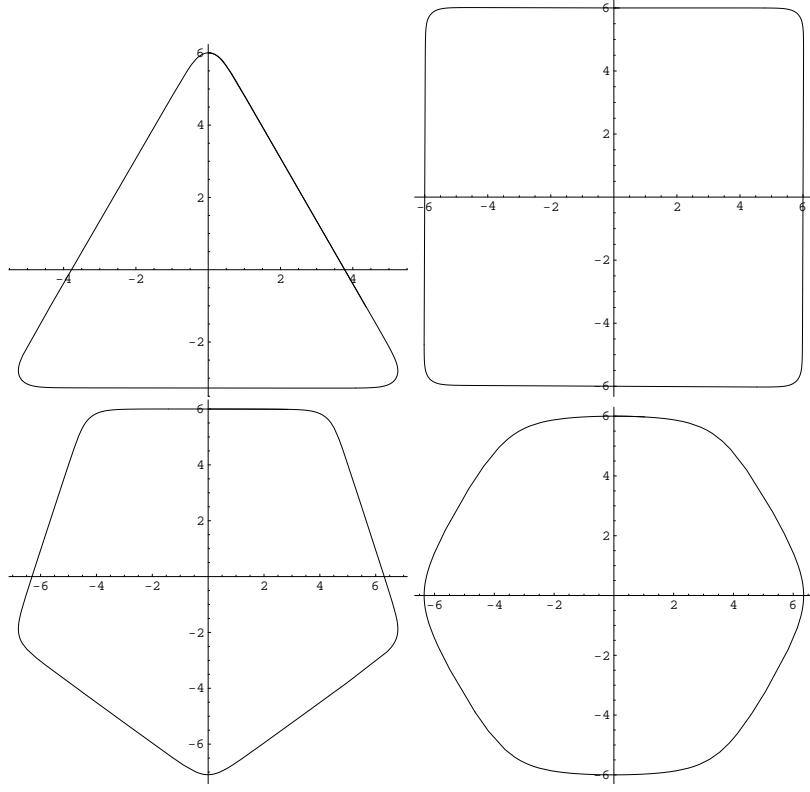


Figure 5: Examples of constant-curvature curves in the Gauss plane, R^2 endowed with density $e^\psi = (2\pi)^{-m/2} e^{-r^2/2}$. Figure taken from [ACDLV] Figure 13.

I tried to solve this differential equation for the following curves

$$\begin{aligned}x^4 + y^4 &= 1 \\x^2 - y^2 &= 1 \\ \frac{x^2}{4} + y^2 &= 1 \\(x - \frac{1}{2})^2 + y^2 &= 1.\end{aligned}$$

Mathematica was unable to solve the differential equation, 3 for the curves $x^4 + y^4 = 1$, $x^2 - y^2 = 1$, and $x^2/4 + y^2 = 1$. The solution to equation 3 for the curve $(x - (1/2))^2 + y^2 = 1$ is

$$\psi = C_2 + C_1(\frac{1}{2}r\sqrt{-2+r^2} - \text{Log}[r + \sqrt{-2+r^2}]).$$

This equation, however, is undefined on $\{ r^2 < 2 \}$ which unfortunately includes the curve.

Take two identical α -sectors and identify the boundaries so that the vertices are identified. This process yields a 2α -cone.

Proposition 6.7. *There is a one-to-one correspondence between symmetric minimizers in a 2α -cone and minimizers in an α -sector.*

Proof. Take a symmetric minimizer M in the 2α -cone. Suppose that one of the halves of M is not a minimizer in the α -sector. Then some other region R in the α -sector with the same area has less perimeter. But then R and its reflection will be more efficient than M , a contradiction.

Conversely, take a minimizer R in the α -sector. Suppose that R and its reflection are not minimizing in the 2α -cone. Then some competing region M in the 2α -cone has less perimeter than R and its reflection. There are antipodal rays dividing the total area of M into halves. The cheaper half will have less perimeter than R , contradicting the fact that R is a minimizer. \square

Proposition 6.8 (Variation formulae, [B], [Co2], [M2]). *The first variation $\delta^1(v) = dL_\psi/dt$ of the length of a smooth curve in a smooth Riemannian surface with smooth density e^ψ under a smooth, compactly supported variation with initial velocity v satisfies*

$$\delta^1(v) = \frac{dL_\psi}{dt} = - \int \kappa_\psi v ds_\psi.$$

If κ_ψ is constant then $\kappa_\psi = dL_\psi/dA_\psi$, where dA_ψ denotes the weighted area on the side of the compactly supported normal. It follows that an isoperimetric curve has constant curvature κ_ψ .

The second variation $\delta^2(v) = d^2L_\psi/dt^2$ of a curve Γ in equilibrium in R^n with density e^ψ for a compactly supported normal variation with initial velocity v and $dA_\psi/dt = 0$ satisfies

$$\delta^2 L(v, v) = - \int_{\Gamma} v \left(\frac{d^2}{ds^2} v + \kappa^2 v \right) - \int_{\Gamma} v \frac{d\psi}{ds} \frac{dv}{ds} + \int_{\Gamma} v^2 \frac{\partial^2 \psi}{\partial n^2}$$

where κ is the Euclidean curvature, s is the Euclidean arc length, and integrals are taken with respect to weighted length.

If $\delta^2 L(v, v)$ is nonnegative, the curve is stable.

Remark 6.9. Circles are unstable in planes with strictly log-concave density (i.e. when ψ is strictly concave).

Proposition 6.10 ([RCBM], Corollary 3.9). *In R^2 endowed with strictly log-concave density, compact minimizing curves are connected.*

Proof. Suppose that a minimizer has two components, Γ_1 and Γ_2 . Choose nonzero, constant initial velocities v_1, v_2 on Γ_1 and Γ_2 such that $A' = 0$. By the second variation formula,

$$\delta^2 L(v, v) = - \int_{\Gamma_1} \kappa^2 v_1^2 + \int_{\Gamma_1} v_1^2 \frac{\partial^2 \psi}{\partial n^2} - \int_{\Gamma_2} \kappa^2 v_2^2 + \int_{\Gamma_2} v_2^2 \frac{\partial^2 \psi}{\partial n^2} < 0$$

because ψ is strictly concave, a contradiction. \square

Corollary 6.11. *In R^2 with radially symmetric, strictly log-concave density, for $\alpha < \pi$, minimizing curves in α -sectors are connected.*

Proof. By Proposition 6.7 it is sufficient to show that symmetric minimizers in cones are connected under symmetric variations. Suppose that a symmetric minimizer in a 2α -cone has two components, Γ_1 and Γ_2 . Since $\alpha < \pi$, neither curve can pass through the vertex by Remark 2.2. Therefore, the second variation formula applies. Choose nonzero, constant initial velocities v_1 and v_2 on Γ_1 and Γ_2 such that $A' = 0$. By Proposition 6.10, minimizing curves must be connected. \square

6.3 Sectors in planes with radial density

Theorem 6.22 shows that in α -sectors, $\alpha < \pi$ connected minimizers must be monotonic in their distance from the origin. Lemmas 6.13-6.20 lead to this result.

Definition 6.12. In R^2 an α -sector is the region enclosed by two rays from the origin at an angle α .

Lemma 6.13. *A closed curve cannot be a minimizer in an α -sector.*

Proof. Suppose C is a closed curve that is minimizing for area A in an α -sector (see Figure 6). Then C must be smooth (Remark 2.2). Maintaining area and perimeter, we rotate C about the origin until C is tangential to the boundary of the sector, a contradiction of regularity (Remark 2.2). \square

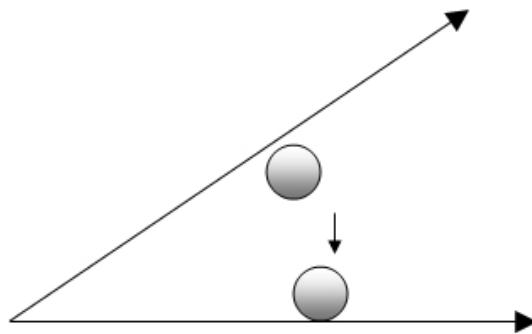


Figure 6: A closed curve can never be minimizing in an α -sector.

Lemma 6.14. *A curve from infinity to infinity cannot be a minimizer in an α -sector.*

Proof. Suppose that a curve C from infinity to infinity as in Figure 7 is minimizing in a sector. Maintaining area and perimeter rotate C about the origin until C crosses the lower boundary of the sector not perpendicularly. Move the portion of C now below the lower

boundary to below the upper boundary. C has the same area and perimeter but now contradicts regularity (Remark 2.2). \square

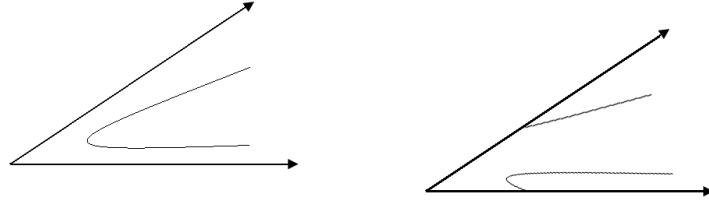


Figure 7: A curve from infinity to infinity can never be minimizing in an α -sector.

Remark 6.15. Lemmas 6.13 and 6.14 should extend to surfaces without boundary in α -cones and α -wedges (see Section 6.4).

Lemma 6.16. *A noncircular minimizer in an α -sector cannot intersect a circular arc three times.*

Proof. Suppose a smooth, noncircular curve C intersects a circular arc at least three times as in Figure 8. Take any three consecutive intersection points. Then we can rearrange C by flipping the portion of the curve between the two outer intersection points across a ray from the origin through their midpoint to obtain a new curve C' . This operation maintains both area and length. C' , however, has sharp corners, so it cannot be minimizing (Remark 2.2). Therefore C cannot be minimizing. \square

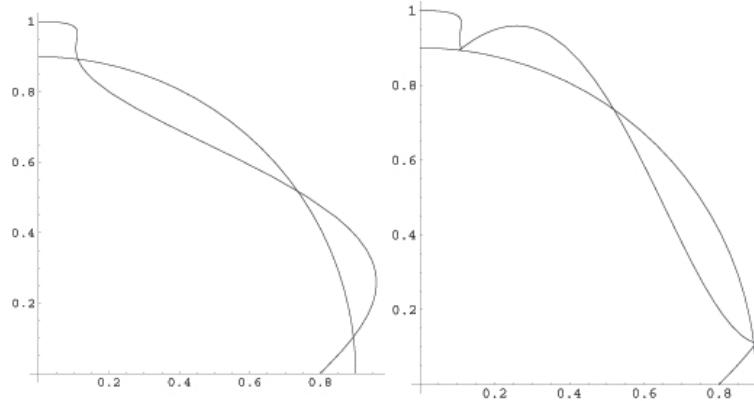


Figure 8: By Lemma 6.16, a minimizers cannot intersect a circular arc three or more times. (Figure taken from [ACDLV] Figure 6, with permission).

Lemma 6.17. *Any constant-curvature curve in an α -sector that is perpendicular to a ray from the origin must be symmetric about the ray.*

Proof. Since the density of the plane is symmetric under reflection across a line through the origin, this result follows from the uniqueness of solutions to differential equations. \square

Lemma 6.18. *In an α -sector, if a minimizer goes from one boundary to infinity, its distance from the origin must be monotonic.*

Proof. By Lemma 6.16, such a minimizer can intersect a circular arc at most twice. If the curve is not monotonic, then there exists at most one point where the curve's distance from the origin is a strict local minimum. At this point, the curve is orthogonal to a ray from the origin, and therefore it must be symmetric about this ray. Reflect the portion of the curve that meets the boundary about this line. If this curve does not hit the other boundary, then at the end of the curve it is again tangential to a circular arc, so reflect again. Repetition of this process yields a curve from one boundary to the other, a contradiction. \square

Lemma 6.19. *In an α -sector, if a minimizer goes from one boundary to the other, its distance from the origin must be monotonic.*

Proof. Suppose there is such a minimizer as in Figure 9. By Lemma 6.16, there is at most one strict local extremum. If such a point exists, by Lemma 6.17 the curve must be symmetric about the ray from the origin through this point. At the edges of the sector, there exist slices of the enclosed region with rays through the origin that consist of only one component. By symmetry, at least some of these slices are repeated. Rearrange all repeated slices at one side of the curve in order of increasing length, thereby decreasing the total tilt of the curve and reducing length, a contradiction. Therefore, there are no strict local extrema and the distance from the origin is monotonic. \square

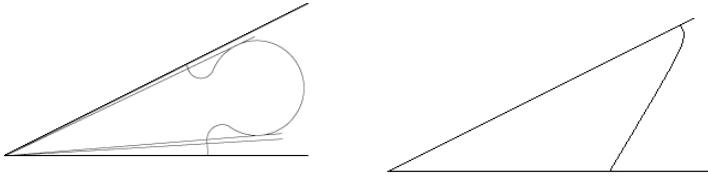


Figure 9: If the distance to the origin is not monotonic, the curve can be rearranged with less perimeter. (Figure taken from [ACDLV], Figure 7, with permission).

Lemma 6.20. *In an α -sector, if a minimizer begins and ends on the same boundary, then its distance from the origin must be monotonic.*

Proof. By Lemma 6.16, we may assume that the point P farthest or closest to the origin is not an endpoint. Then at P the curve is tangent to a circular arc. So by Lemma 6.17 the curve must be symmetric about the ray from the origin through P . Reflect the curve about the ray from the origin through P . If that curve does not hit the other boundary then at

the end of that curve it is tangent again to a circular arc. So reflect that new curve again around the ray from the origin to the endpoint. Repetition of this process yields a curve that goes from one boundary to the other, a contradiction. \square

Proposition 6.21. *In an α -sector if a minimizer begins and ends on the same boundary, it must be concave.*

Proof. By Lemma 6.20, the distance from the minimizing curve to the origin must be monotonic. Suppose the minimizer from a boundary to itself has a portion of convexity. Then there is a point in the concave region and a point in the convex region that are tangent to rays from the origin. At these points, $d\psi/dn$ is zero, so the ψ -curvature is just the Euclidean curvature. The curvature is positive in the concave region and negative in the convex region, a contradiction since a minimizer must have constant curvature. \square

Theorem 6.22. *In an α -sector, $0 < \alpha < \pi$, if a minimizer is connected its distance from the origin is monotonic.*

Proof. By Propositions 6.13 and 6.14, neither a closed curve nor a curve from infinity to infinity can be minimizing. The three remaining possibilities are curves from a boundary to infinity, a boundary to the other boundary, or a boundary to the same boundary. By Lemmas 6.18, 6.19, and 6.20, in each case the curve must be monotonic in its distance from the origin. \square

Lemma 6.23 ([ACDLV], Lemma 3.28). *For a smooth, closed curve enclosing the origin, there are two critical points for distance from the origin not on the same line through the origin.*

Proof. Suppose there is a smooth, closed curve enclosing the origin with all the critical points for distance from the origin on the same line through the origin. Since the curve encloses the origin, the critical point furthest from the origin and the critical point closest to the origin must lie on opposite sides of the origin. Start at one of these critical points and move along the curve. Near the critical point the angles between the ray and the curve are no longer equal; one of them is less than $\pi/2$ and the other is greater than $\pi/2$. Take the angle less than $\pi/2$ and continue traveling along the curve with rays from the origin to the curve. Near the other critical point this angle will be greater than $\pi/2$. So by the Intermediate Value Theorem, there is a ray from the origin that meets the curve perpendicularly at some point in between, a contradiction. \square

Proposition 6.24. *Any closed, constant-curvature curve that encloses the origin has center of mass at the origin.*

Proof. By Lemma 6.23, there are two lines from the origin that meet the curve perpendicularly. By Lemma 6.17, the curve is symmetric under reflection about these two lines. Therefore the center of mass must be at the point where the two lines meet, the origin. \square

Proposition 6.25. *Let $P_\alpha(A)$ denote the minimum perimeter in an α -sector. For $k \geq 1$,*

$$P_{k\alpha}(kA) \leq kP_\alpha(A)$$

Proof. Stretching an α -sector to a $k\alpha$ -sector stretches area by k and length by at most k . \square

Proposition 6.26. *In an α -sector of a plane with finite total area, the ray from the origin provides an upper bound for minimum perimeter.*

Proof. A ray from the origin can enclose all possible areas. \square

6.4 Sectors in R^n with radial density

Remark 6.15 extends some monotonicity results to R^n with radial density. Here we extend some other known results about minimizers in sectors of planes with radial density to sectors of R^n with radial density. In R^n there are two ways to define a sector, an α -wedge and an α -cone.

Definition 6.27. In R^n an α -wedge is the region between two half hyperplanes through the origin meeting at angle α .

Definition 6.28. In R^n let $v_0 = (1, 0, 0, 0, \dots, 0)$. An α -cone in R^n is $\{ v : \angle(v, v_0) \leq \alpha \}$.

Lemma 6.29. *If a half-hyperplane from the origin is a minimizer for a particular α_0 -wedge, it is minimizing for all larger α -wedges in R^n with radial density.*

Proof. Suppose that the minimizer in an α_0 -wedge is a half-hyperplane from the origin and that some surface S encloses a volume V in an α -wedge, $\alpha \geq \alpha_0$ with less area than a half-hyperplane from the origin enclosing volume V . When we shrink the sector to an angle of α_0 , the area of the half-hyperplane does not change. So S must have less area than the half-hyperplane. This contradicts the hypothesis that the half-hyperplane is minimizing in the α_0 -wedge. \square

Lemma 6.30. *If a spherical cap is minimizing for a particular α_0 -cone, it is minimizing for all smaller α -cones in R^n with radial density.*

Proof. Suppose that the minimizer in an α_0 -cone is a spherical cap and that some surface S encloses a volume V in an α -cone with less area than a spherical cap enclosing V . When we stretch the sector out to an angle of α_0 , the spherical cap gains more area than S because all of its area is in the direction of the stretching, so the stretched spherical cap (which is still a spherical cap) has greater area than the stretched surface S . This contradicts the hypothesis that the spherical cap is minimizing in the α_0 -cone. \square

Corollary 6.31. *In R^n endowed with density $e^{r^2/2}$ a spherical cap centered at the origin is minimizing for all α -cones with $\alpha \leq 2\pi$.*

Proof. Rosales, Cañete, Bayle and Morgan ([RCBM], Thm. 5.2) show that in R^n endowed with density $e^{r^2/2}$ balls centered at the origin are minimizing. The result follows immediately from Lemma 6.30. \square

Proposition 6.32. *If in R^n with radial density a hyperplane is uniquely minimizing for volume $2V$, then in the halfspace a half hyperplane perpendicular to the boundary is uniquely minimizing for volume V .*

Proof. Suppose that there exists some other smooth surface S enclosing volume V with no greater area than the half hyperplane perpendicular to the boundary enclosing volume V . Then S with its reflection across the boundary encloses a volume of $2V$ with no greater area than a hyperplane enclosing $2V$, a contradiction. \square

In G^2 lines are minimizing for all areas and thus rays perpendicular to the boundary are minimizing for all areas in the Gauss halfplane. Ros ([Ros] Prop. 1) gives ways to find other spaces where lines are minimizing to enclose half the total area.

Proposition 6.33. *In R^n with radial density in an α -wedge $\alpha < \alpha_1$, if a minimizer in α_1 -wedge can fit in the α -wedge, a minimizer in the alpha-wedge must either come from the α_1 -wedge or touch both boundaries.*

Proof. Otherwise the minimizer would beat minimizers in an α_1 -wedge. \square

7 $R^1 \times G^1$

Here we examine a space with density that is not radially symmetric. $R^1 \times G^1$ is R^2 endowed with density $(1/\sqrt{2\pi})e^{-y^2/2}$. Since in $R^1 \times G^1 / \{\text{integer translations in the horizontal direction}\}$ is an infinite vertical strip of $R^1 \times G^1$ with finite total volume, minimizers exist (Remark 2.1). Conjecture 7.4 states that in $R^1 \times G^1$ constant-curvature curves from infinity to infinity minimize perimeter for small area and a pair of vertical lines minimizes perimeter for large area.

We also examine the isoperimetric problem in $S^1 \times G^1$. Conjecture 7.10 states that in $S^1 \times G^1$ nonlinear, homotopically nontrivial constant-curvature curves minimize perimeter for small area and a pair of vertical lines minimizes perimeter for large area. Proposition 7.5 shows that meridional circles and a pair of vertical lines are not always minimizing.

Lemma 7.1. *In $R^1 \times G^1$ there exist minimizers that are not closed curves.*

Proof. Take a closed curve in $R^1 \times G^1$ and slice the region with vertical slices. Replace each slice with a halfline of equal weighted length. Each slice is G^1 and in G^1 halflines are minimizing. This process does not increase perimeter. ([Ros], Proposition 7). \square

Lemma 7.2. *In $R^1 \times G^1$ there exist minimizers that have reflectional symmetry about the y -axis.*

Proof. Take a minimizer M in $R^1 \times G^1$. Then there is a vertical line that slices the area enclosed by M in half. Take one of the halves and it and its reflection about the vertical

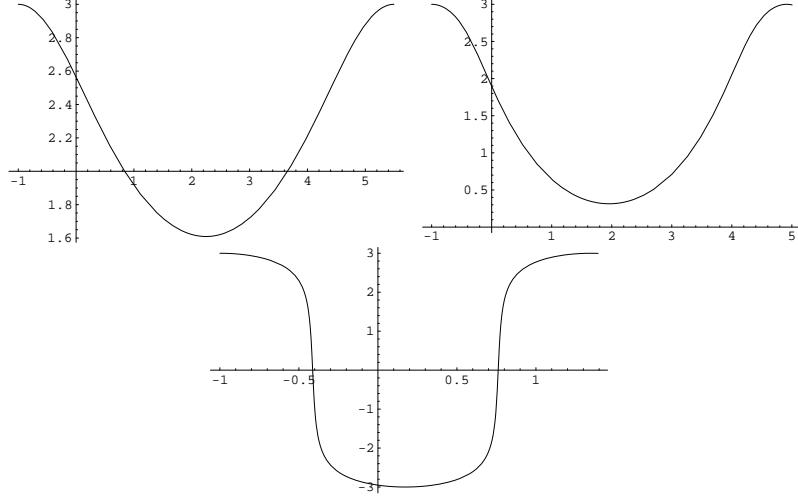


Figure 10: Examples of constant curvature curves in $R^1 \times G^1$

line will still be minimizing. Translate this new curve until the line of symmetry is the y-axis. \square

Proposition 7.3. *In $R^1 \times G^1$, all smooth constant-curvature curves with downward unit normal satisfy the following differential equation:*

$$x'(s)y''(s) - y'(s)x''(s) + x'(s)y(s) = \kappa_\psi \quad (4)$$

where s is the arc length, with the constraints

$$\begin{aligned} x'(s)^2 + y'(s)^2 &= 1 \\ x(0) &= -1 \\ x'(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Proof. Equation 4 follows from the definition of ψ – curvature and the fact that for arc length parameterization Euclidean curvature equals $x'(s)y''(s) - y'(s)x''(s)$ (see Figure 10 for examples). \square

Conjecture 7.4. *In $R^1 \times G^1$ for small area a curve from infinity to infinity minimizes perimeter and for large area a pair of vertical lines minimize perimeter.*

By Lemma 7.1 there exist minimizing curves that are not closed. A pair of vertical lines has constant perimeter 2 and can enclose any given area. Any other curve will have to get longer as it encloses more area.

Proposition 7.5. *In a strip of $R^1 \times G^1$, a horizontal line segment or a vertical line are not always minimizing.*

Proof. Take the strip $S = \{|x| \leq 3\}$. A horizontal line always has length 1 and can enclose any given area. The curve $x = 50\sin^{-1}(10(y - 2))$ encloses area above the curve at most 0.13650467624771713. The perimeter of this line is at most 0.323952. The length of the horizontal line enclosing area at least 0.13650467624771713 has length at least 0.323954. Thus in this strip, a minimizer cannot be always a horizontal line segment or a vertical line. \square

7.1 $S^1 \times G^1$

Here we examine a related space, $S^1 \times G^1$. Since the total area is finite, minimizers must exist (Remark 2.1).

Lemma 7.6. *In $S^1 \times G^1$ there exist minimizers that are not homotopically trivial closed curves.*

Proof. Take a homotopically trivial closed curve C in $S^1 \times G^1$. Then slice the region enclosed by C with vertical slices and replace each slice with a halfline of equal weighted length. Each slice is G^1 and in G^1 halflines are minimizing. This process creates a new curve that is no longer closed but encloses the same area. This process does not increase perimeter ([Ros], Proposition 7). \square

Lemma 7.7. *In $S^1 \times G^1$ there exist symmetric minimizers.*

Proof. Take a minimizer C in $S^1 \times G^1$. Then there is a pair of antipodal vertical line that slices the area in half. Take one of the halves and it and its reflection will still be minimizing. \square

Lemma 7.8. *In $S^1 \times G^1$ there exist minimizers that do not intersect a horizontal line more than twice.*

Proof. Take a curve C that encloses an area in $S^1 \times G^1$ that intersects a horizontal line more than twice. Then slice the region enclosed horizontally. Since the density is constant in a horizontal line, single segments or the whole line are minimizing. So replace each slice with a single segment or whole line with equal weighted length. This process preserves area and does not increase perimeter ([Ros], Proposition 7). \square

Proposition 7.9. *There is a one-to-one correspondence between symmetric minimizers in a $S^1 \times G^1$ and minimizers in an infinite strip $S = \{ -a \leq y \leq a \} \subset R^1 \times G^1$.*

Proof. Take a symmetric minimizer M in the $S^1 \times G^1$. Suppose that one of the halves of M is not a minimizer in the S . Then some other region R in the S with the same area has less perimeter. But then R and its reflection will be more efficient than M , a contradiction.

Conversely, take a minimizer R in the S . Suppose that R and its reflection are not minimizing in the $S^1 \times G^1$. Then some competing region M in the $S^1 \times G^1$ has less perimeter

than R and its reflection. There are antipodal lines dividing the total area of M into halves. The cheaper half will have less perimeter than R , contradicting the fact that R is a minimizer. \square

Conjecture 7.10. *In $S^1 \times G^1$, for small and large area a homotopically nontrivial, nonlinear constant curvature curve minimizes perimeter and for area near a half the total area a pair of vertical lines minimizes perimeter (see Figure 11).*

By Proposition 7.5 in a strip of $R^1 \times G^1$, a horizontal line or a vertical line segment are not always minimizing. Thus, in $S^1 \times G^1$ a meridional circle or a pair of horizontal lines are not always minimizing.

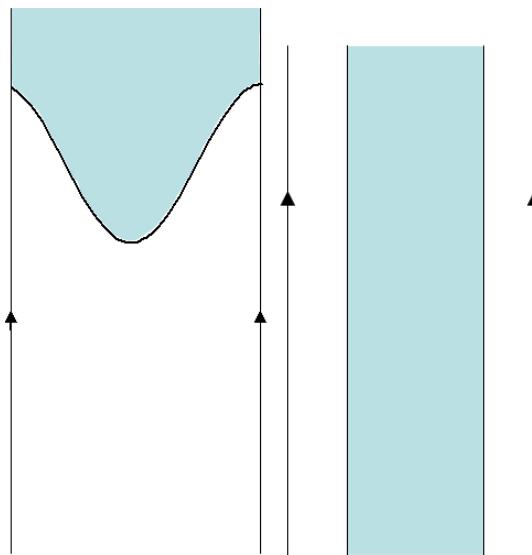


Figure 11: We conjecture that in $S^1 \times G^1$ for small area a nontrivial, nonlinear constant curvature curve minimizes perimeter and for large area a pair of vertical lines minimizes perimeter.

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