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DOUBLE BUBBLES IN GAUSS SPACE AND SPHERES

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Abstract. We prove that a standard Y is an area-minimizing partition of Gauss space into three given volumes, provided that the standard double bubble is an area-minimizing partition of high-dimensional spheres. We prove that the standard double bubble is the area-minimizing partition of spheres of any dimension where the volumes differ by at most 4%.

1. Introduction

In 2002, Hutchings, Morgan, Ritoré, and Ros ([HMRR], or see [Mo1]) proved the Double Bubble Conjecture, which says that the area-minimizing way to enclose and separate two given volumes in R^3 is a standard double bubble, defined as three spherical caps meeting in threes at 120 degrees. There have been partial extensions to R^n, the sphere S^3, and hyperbolic space H^3 ([Rei],[CF], [CH]). We provide extensions to S^n and Gauss space G^m for the case where each volume is approximately 1/3 of the total volume of the space.

1.1. Single and Double Bubbles in Gauss Space.

Gauss space G^m, of much interest to probabilists (see e.g. [McK] and [Str]), is R^m endowed with density

$$(2\pi)^{-m/2}e^{-x^2/2},$$

with total volume 1. Given any three positive volumes $V_1 + V_2 + V_3 = 1$, there exists a standard Y, unique up to symmetries of G^m, consisting of three half-hyperplanes meeting at 120 degrees, that partitions G^m into three regions of those prescribed volumes (Proposition 2.11). Note that the center $m-2$ dimensional meeting plane of a Y passes through the origin if and only if the three volumes are equal. Our main theorem proves that the standard Y is area minimizing for certain cases:

> **Theorem 2.14**: *The standard Y is an area-minimizing partition of G^m for three given positive volumes $V_1 + V_2 + V_3 = 1$, under the hypothesis that the standard double bubble for the same volume fractions is area minimizing in S^n for infinitely many n.*

Section 3 proves that the standard double bubble is area minimizing in S^n for all n for nearly equal volumes. Using Brakke's Surface Evolver, Section 2.3 provides corroborating computer evidence that a standard Y is area minimizing for volumes $(1/3, 1/3, 1/3)$ (as we prove) and for volumes $(1/3, 1/10, 17/30)$ (which is outside our region of proof). This evidence makes it likely that the standard Y is minimizing for all volumes (see Section 1.2 below). It is worthwhile to note that while Section 3 proves a *unique* area-minimizing partition for certain volumes in S^n, the uniqueness of the corresponding partition in G^m is an open question.

The Proof. The main idea comes from Mehler's 1856 observation that Gauss space is weakly the limit of projections of suitably normalized high-dimensional spheres (Proposition 2.1) as used by Borell [Bo] and Sudakov-Tsirel'son [ST] to prove hyperplanes optimal single bubbles in G^m. The proof requires stronger convergence of perimeter for fixed volumes (Proposition 2.12), including a technical Lemma 2.13 on how to repair small volume discrepancies at low area cost.

1.2. **Double Bubbles in S^n.**

We prove the Double Bubble Conjecture in the sphere S^n for volume fractions near $1/3$:

> **Theorem 3.7**: *For all n, the $n-$dimensional standard double bubble is the unique area-minimizing partition of S^n into any three volume fractions within* $.04$ *of* $1/3$.

This, in conjunction with Theorem 2.14, proves that the standard Y is an area minimizer in G^m if all three volumes are within .04 of 1/3. Section 3.3 shows how to extend this result to other volumes where the Hutchings function is positive. To extend to regions where the Hutchings function is negative, a different method of proof would be necessary. Reichardt et al. [Rei] proved that in R^n it suffices to show that one region is connected. Extending this to S^n would be a big step towards proving our result for all volumes.

The Proof. By Hutchings [Hu] and Cotton and Freeman [CF], it suffices to show that the Hutchings function $F_{S^n}(V_1, V_2, V_3)$ (Definition 4) of the prescribed volumes is positive. We consider more tractable lower bounds $\tilde{F}$ on the Hutchings function (Definition 5). By a monotonicity result of Barthe, it suffices to prove $\tilde{F}$ positive on high-dimensional spheres (Lemma 3.6). At this moment, we reuse the fact that Gauss space G^2 is the limit of projections of high-dimensional spheres, so it suffices to prove $\tilde{F}$ positive in G^2 (Lemma 3.2), which we prove analytically (Lemma 3.3).

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Lectures by Antonio Ros at the Clay Mathematics Institute Summer School [Ros] provided the initial inspiration for the investigation presented in this paper, which was first pursued by Corneli and others in the 2003 SMALL Geometry Group. The computer experiments of Section 2.3 stemmed from conversations with and help from Ken Brakke in Paris.

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2. The Double Bubble Conjecture in Gauss Space

Our main result, Theorem 2.14, proves that the standard Y is an area-minimizing partition of Gauss space G^m into three prescribed volumes, as long as the standard double bubble is area minimizing in high-dimensional spheres for the same volume fractions (as shown for certain volume fractions in Section 3). Proposition 2.1 provides the main idea: that Gauss space is the limit of projections of high-dimensional spheres. Section 2.2 discusses the standard Y, and Proposition 2.12 proves the convergence of standard double bubbles to the standard Y.

2.1. **Area and Volume Convergence in Gauss Space.** Gauss space G^m is R^m endowed with Gaussian density $f(x) = (2\pi)^{-m/2}e^{-x^2/2}$. We recall that volume and area in G^m and in manifolds with density in general, are given by integrating the density with respect to the Euclidean volume and area (Hausdorff measure). Propositions 2.1 and 2.2 show that this density can be obtained by projecting high-dimensional spheres $S^n(\sqrt{n})$ of radius $\sqrt{n}$ into R^m. Proposition 2.4 shows that the areas of the inverse projections in S^n of a hypersurface in G^m converge to the area of the hypersurface as n approaches infinity.

The following proposition, often attributed to Poincaré, is due to Mehler (see Stroock [Str], Exercise 2.1.40, page 76 and footnote page 77).

Proposition 2.1 (Mehler, 1856)**.** *Gaussian measure on R^m is obtained as the limit as n approaches infinity of (orthogonal) projections P_n of uniform probability density on $S^n(\sqrt{n}) \subset R^{n+1}$ to R^m.*

PROOF. Observe that G^m is the projection of G^n onto its first m coordinates. The coordinates $X_1, \ldots, X_n$ of G^n are Gaussian random variables. By the Central Limit Theorem, the distance to the origin $\sqrt{X_1^2 + \cdots + X_n^2}$ converges to $\sqrt{n}$ as $n \to \infty$. By the spherical symmetry of G^n, the volume of G^n concentrates uniformly on $S^{n-1}(\sqrt{n})$ for large n. Therefore, G^m can be obtained as the limit as $n \to \infty$ of projections P_n of the uniform probability density on $S^{n-1}(\sqrt{n})$ or $S^n(\sqrt{n})$ to R^m. □

Furthermore, the induced density converges pointwise, not just in measure, to the Gaussian density.

Proposition 2.2. *The projections P_n of the uniform probability density on $S^n(\sqrt{n})$ to R^m for fixed m converge pointwise to Gaussian density on R^m.*

PROOF. For a fixed m, let $f_n(x)$ be the density function resulting from projecting the uniform probability measure on $S^n(\sqrt{n})$ to R^m. By symmetry, we may assume that x lies on the nonnegative first coordinate axis. Let θ be the angle between $S^n(\sqrt{n})$ and R^m; then $\sin\theta = x/\sqrt{n}$. A volume element dV of S^n is projected to a weighted volume element $dV \cos\theta$ on R^m. Therefore, a volume element dV on R^m has weighted volume $dV/(\cos\theta)$. Observe that $P_n^{-1}(x)$ is the $(n-m)$-sphere with radius $(\cos\theta)\sqrt{n}$ and thus has area proportional to $(\cos\theta)^{n-m-1}$. Therefore,

$$f_n(x) = c_n(2\pi)^{-m/2}(\cos\theta)^{n-m-2}$$

where c_n is some normalization constant. Since $\cos^2\theta = 1 - \sin^2\theta = 1 - x^2/n$,

$$f_n(x) = c_n(2\pi)^{-m/2}(1 - x^2/n)^{(n-m-2)/2}.$$

Since $(1 - x^2/n)^{(n-m-2)/2}$ converges to $e^{-x^2/2}$ as $n \to \infty$, $c_n \to 1$, because the $f_n(x)$ converge to $f(x)$, the Gaussian density, in measure by Proposition 2.1. □

Lemma 2.3. *Let $g_n(x) = (1+x/n)^n$ and $g(x) = e^x$. Then $g_1(x) \leq g_2(x) \leq \cdots \leq g(x)$ for all $x \in R$.*

PROOF. Note that $g_1(x) = 1 + x \leq e^x$ for all x by comparison of the derivatives.

Let g' denote the derivative of $g_n(x)$ in n; we wish to show that $g' \geq 0$ for all x. Differentiating,

$$g' = \exp\left(n \log\left(1 + \frac{x}{n}\right)\right)\left(\log\left(1 + \frac{x}{n}\right) - \frac{x}{n+x}\right).$$

Since $\exp(n\log(1+x/n)) > 0$, we need only show that $\log(1+x/n)-x/(n+x) \geq 0$.

When x is nonnegative,

$$\log\left(1+\frac{x}{n}\right) = \log(x+n) - \log(n) = \int_n^{x+n} \frac{dt}{t} \geq \int_n^{x+n} \frac{dt}{x+n} = \frac{x}{x+n}$$

and therefore $\log(1+x/n) - x/(n+x) \geq 0$.

When x is negative, $n+x < n$, so

$$-\log\left(1+\frac{x}{n}\right) = \log(n) - \log(x+n) = \int_{n+x}^{n} \frac{dt}{t} \leq \int_{n+x}^{n} \frac{dt}{x+n} = -\frac{x}{x+n}$$

and therefore $\log(1+x/n) - x/(n+x) \geq 0$.

Thus, the $g_n(x)$ are nondecreasing in n. □

Proposition 2.4. *The areas of the inverse projections to $S^n(\sqrt{n})$ of a measurable hypersurface $\Sigma \subset G^m$ converge to the area of Σ as n approaches infinity.*

Proof. Define asymptotically equivalent density functions

$$F_n(x) = \frac{f_n(x)}{c_n} = (2\pi)^{-m/2}\left(1-\frac{x^2}{n}\right)^{n-m-2}$$

which differ from $f_n(x)$ by proportional constants c_n, which approach 1 as n approaches ∞ (see proof of Proposition 2.2). By Proposition 2.2 and Lemma 2.3, $F_n(x)$ converges to $f(x)$ monotonically from below. By Lebesgue's Monotone Convergence Theorem, $\int_\Sigma F_n(x)$ converges to $\int_\Sigma f(x)$. Therefore, the areas of the inverse projections, $\int_\Sigma F_n(x)$, converge to the area of Σ, $\int_\Sigma f(x)$. □

2.2. **The Y and the Standard Y.** This section introduces the standard Y and its properties. Proposition 2.12 proves that the area of the standard double bubble on $S^n(\sqrt{n})$ converges to the area of the standard Y in G^m of the same volume fractions as n grows large.

Definition 1. A Y in G^m consists of three half-hyperplanes meeting along an $(m-2)$-dimensional plane. Let β_i denote the angles between the hyperplanes. A *standard Y* is a Y with each $\beta_i = 2\pi/3$. Let θ denote the angle with the horizontal as in Figure 1. By $O(m)$ symmetry, we may assume that the $(m-2)$-dimensional plane lies normal to the x-y plane and contains the nonnegative x-axis. The Y partitions G^m into three volumes V_1, V_2, and V_3. The volumes partitioned by the standard Y satisfy $V_1 \geq V_2 \geq V_3$ as in Figure 1. Let $\mathcal{V}_i(x, \theta, \beta_1, \beta_2)$ and $\mathcal{A}_i(x, \theta, \beta_1, \beta_2)$ denote the volume and area, respectively, of the i^{th} region as functions of x, θ, β_1, and β_2.

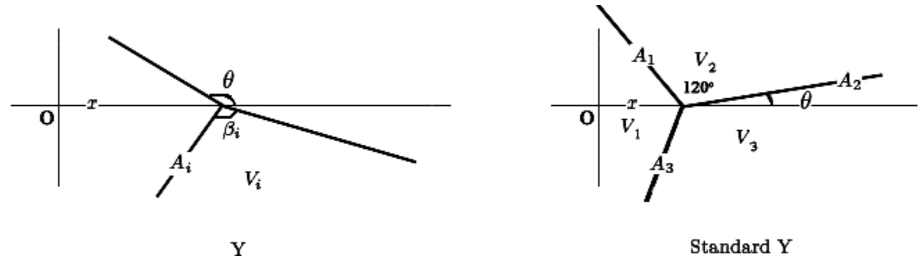

FIGURE 1. We may assume that the Y is centered along the nonnegative x-axis. The standard Y partitions G^m into volumes $V_1 \geq V_2 \geq V_3$.

Lemma 2.5. *In G^m, the values of $\mathcal{V}_i$ and $\mathcal{A}_i$ are independent of m for $m \geq 2$.*

PROOF. This follows from the fact that Gauss space is a product measure and a Y has translational symmetry along an $(m-2)$-dimensional hyperplane. □

We state the following five minor lemmas without proof, the last two of which are used in Section 3.3. The proofs of these lemmas follow from simple geometric arguments.

Lemma 2.6. *In G^m, $\mathcal{V}_i(x, \theta, \beta_1, \beta_2)$ is continuous in x, θ, β_1, and β_2, and for compact regions of x and θ, $\mathcal{V}_i$ is uniformly continuous in β_1 and β_2.*

Lemma 2.7. *In G^m, $\mathcal{A}_i(x, \theta, \beta_1, \beta_2)$ is continuous in x, θ, β_1, and β_2, and for compact regions of x and θ, $\mathcal{A}_i$ is uniformly continuous in β_1 and β_2.*

Note that for a standard Y, we need only consider $0 \leq \theta \leq \pi/3$ due to $O(m)$ symmetry. Let $\mathcal{V}_i(x, \theta)$ denote $\mathcal{V}_i(x, \theta, 2\pi/3, 2\pi/3)$.

Lemma 2.8. *For a standard Y in G^m with $x > 0$, $\mathcal{V}_1$ is strictly increasing in x and strictly decreasing in θ; $\mathcal{V}_3$ is strictly decreasing in x and strictly decreasing in θ.*

Lemma 2.9. *For standard Ys in G^m with V_1 fixed, increasing V_2 increases the signed distance from the origin to the line between the regions of volume V_2 and V_3.*

Lemma 2.10. *For standard Ys in G^m with $V_2 = V_3$, increasing V_1 increases the signed distance from the origin to the lines bounding the region of volume V_1.*

Proposition 2.11. *Given three volumes V_1, V_2 and V_3 such that $V_1+V_2+V_3=1$ and $V_1 \geq V_2 \geq V_3$, there exists a unique standard Y, up to $O(m)$ symmetry, which partitions G^m into volumes V_1, V_2 and V_3.*

PROOF. By Lemma 2.5, we need consider only the case $m = 2$. We exclude the trivial case $V_1 = V_2 = V_3$. Since $\mathcal{V}_3(0, \pi/3) = 1/3$ and $\lim_{x\to\infty} \mathcal{V}_3(x, \pi/3) = 0$, by Lemma 2.6 there is a unique $x_0 > 0$ such that $\mathcal{V}_3(x_0, \pi/3) = V_3$. Similarly, there is a unique $x_1 > x_0$ such that $\mathcal{V}_3(x_1, 0) = V_3$.

For every $x_0 \leq x \leq x_1$, there is a unique $\theta(x)$ such that $\mathcal{V}_3(x, \theta(x)) = V_3$. Also, for $x < x_0$ or $x > x_1$, there exists no $\theta(x)$ for which $\mathcal{V}_3(x, \theta(x)) = V_3$. By Lemma 2.6 and the fact that $\theta(x)$ is continuous and strictly decreasing in x, $\mathcal{V}_1(x, \theta(x))$ is continuous and strictly increasing in x. Since $\mathcal{V}_1(x_0, \pi/3) \leq V_1 \leq \mathcal{V}_1(x_1, 0)$, there is a unique $x_0 \leq x \leq x_1$ such that $\mathcal{V}_1(x, \theta(x)) = V_1$ and, consequently, $\mathcal{V}_2(x, \theta(x)) = V_2$, proving the uniqueness of Y (Figure 2). □

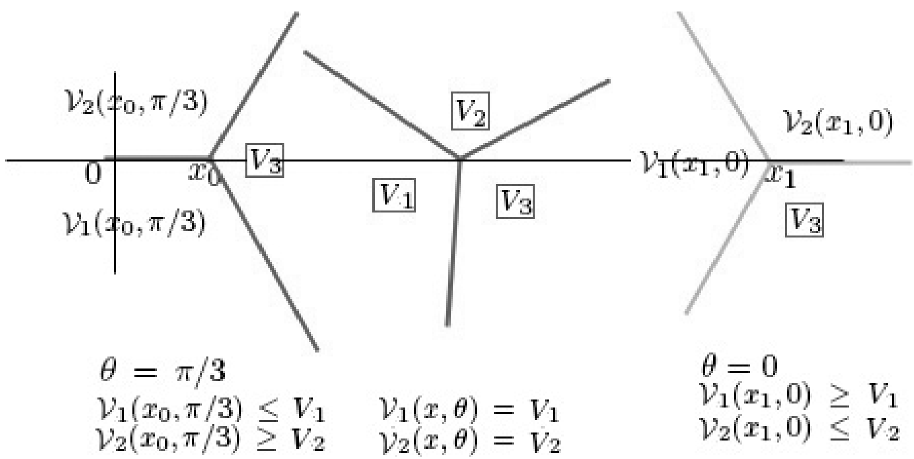

FIGURE 2. By the Intermediate Value Theorem, there exists a unique standard Y that partitions G^2 into given volumes.

Definition 2. A *standard double bubble* partitioning S^n into three prescribed volumes consists of three $(n-1)$-dimensional spherical caps meeting along an $(n-2)$-sphere at 120 degrees. There exists a standard double bubble for any given set of prescribed volumes ([CF], Proposition 2.6.), unique up to isometries of S^n.

Proposition 2.12. *The area of the standard double bubble on $S^n(\sqrt{n})$ converges to the area of the standard Y in G^m as n approaches infinity for the same volume fractions.*

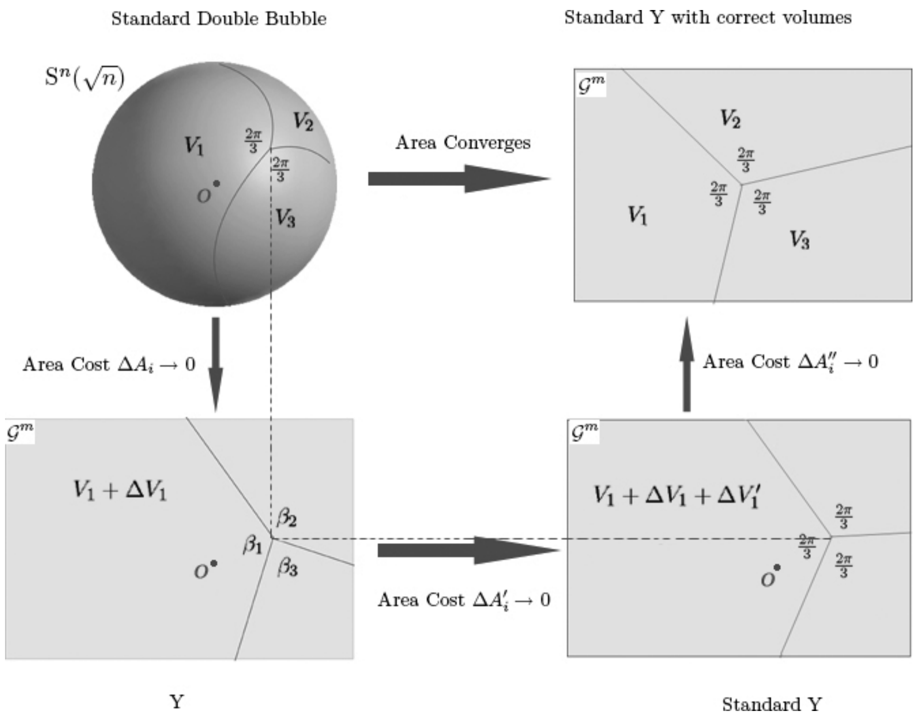

FIGURE 3. The Y projected from the standard double bubble in $S^n(\sqrt{n})$ can be transformed to the standard Y of the correct volume fractions with an area difference that approaches 0 as n approaches ∞.

PROOF. Since the inverse image by the projection of a plane in G^m is a sphere in S^n, the inverse image of a Y (three half-hyperplanes meeting along an $(m-2)$-plane) is three spherical caps meeting along an $(n-2)$-sphere at some angles. Given volumes V_1, V_2 and V_3, choose Y_n in G^m (not necessarily with angles $2\pi/3$) with inverse image the standard double bubble (with angles $2\pi/3$) in S^n with the given volumes.

The distance of the vertex of the Y_n from the origin is bounded above as $n \to \infty$, since too great a distance implies that V_3 is too small. As n increases, the radius of $S^n(\sqrt{n})$ increases, and the section of the sphere above the vertex of Y_n grows increasingly flat. Thus, the angles of the Y_n approach the angles of the preimage—namely, $2\pi/3$.

Let the volume and area differences between the standard double bubble in S^n and the Y_n be denoted ΔV_n and ΔA_n as in Figure 3. By Propositions 2.1 and 2.4, the ΔV_n and ΔA_n approach 0 as $n \to \infty$. Rotate two rays of the Y_n so that the β_i are all $2\pi/3$ (so that it is a standard Y, though not necessarily of prescribed volume), and denote the area and volume changes in the Y as $\Delta V_n'$ and $\Delta A_n'$. Since the changes in β_i go to 0 as $n \to \infty$ and x and θ are restricted to a compact region, Lemmas 2.6 and 2.7 imply that $\Delta V_n'$ and $\Delta A_n'$ both go to 0 as $n \to \infty$. By Proposition 2.11, this standard Y can then be rotated and translated so that it has the correct volumes V_1, V_2 and V_3. From Lemma 2.8, V_1 and V_3 are locally invertible in x and θ. Therefore, for the small volume change which goes to 0 as $n \to \infty$, Lemma 2.7 implies that there is a correspondingly small area change $\Delta A_n''$ which similarly goes to 0 as $n \to \infty$. Therefore, for prescribed volumes V_1, V_2 and V_3, the total area difference between the standard double bubble on S^n and the standard Y is thus $\Delta A_n + \Delta A_n' + \Delta A_n''$ which goes to 0 as $n \to \infty$.

□

Remark. The proof above can be easily modified to show that the area of the single bubble of volume V_1 on $S^n(\sqrt{n})$ converges, as n approaches infinity, to the area of the hyperplane partitioning G^m into volumes V_1 and $1 - V_1$.

2.3. **An Area-Minimizing Partition of Gauss Space.** Theorem 2.14 proves that a standard Y is area minimizing in G^m under the assumption that the standard double bubble is area minimizing in $S^n(\sqrt{n})$ for infinitely many n. The proof uses Lemma 2.13 to adjust volumes with controlled area cost. As in Proposition 2.2, let f_n be the density of R^m from the projection of $S^n(\sqrt{n})$:

$$f_n = c_n(2\pi)^{-m/2}(1 - x^2/n)^{(n-m-2)/2}$$

Lemma 2.13 (Volume balancing at low area cost)**.** *Given a compact region $R \subset R^m$ and an $\epsilon > 0$, there exists $\delta > 0$ and N such that for all $n > N$ and any $\Delta V < \delta$, a ball with f_n-weighted volume ΔV which is split by two hyperplanes has f_n-weighted total surface area (including the area of the hyperplanes inside the ball) $\Delta A < \epsilon$.*

Proof. Lemma 2.3 implies that $F_n = (2\pi)^{-m/2}(1 - x^2/n)^{(n-m-2)/2}$ converges monotonically from below to $(2\pi)^{-m/2}e^{-x^2/2}$ and assumes positive values for all

$x < \sqrt{n}$. By Proposition 2.1, the c_n converge monotonically from above to 1. Hence, for $n \geq N$, f_n is bounded below by $\min_R\{F_N\} > 0$ and bounded above by $\max_R(c_N(2\pi)^{-m/2}e^{-x^2/2}) \leq c_N(2\pi)^{-m/2}$. Therefore the surface area of the weighted ball goes to 0 with the volume. □

Theorem 2.14 (An area-minimizing partition of Gauss space)**.** *The standard Y is an area-minimizing partition of G^m for three given positive volumes $V_1 + V_2 + V_3 = 1$, under the hypothesis that the standard double bubble for the same volume fractions is area minimizing in S^n for infinitely many n.*

Remark. Theorem 3.7 proves that the standard double bubble enclosing nearly equal volumes is area minimizing in $S^n(\sqrt{n})$ for all n. Corollary 3.8 combines that theorem with Theorem 2.14 above to prove that the standard Y is an area-minimizing partition of G^m for nearly equal volumes.

Proof. Note that from scaling, an equivalent hypothesis to that of the theorem is that the standard double bubble for the same volume fractions is area minimizing in $S^n(\sqrt{n})$ for infinitely many n.

Let Y denote the standard Y with area A and assume, to obtain a contradiction, that there exists a Y′ with area $A' = A - \alpha$ where $\alpha > 0$ (Figure 4).

By Proposition 2.12, for large n, the difference in area between the standard double bubble in $S^n(\sqrt{n})$ with the same volume fractions and the Y is less than $\alpha/2$.

Observe that there exists a compact region $R \subset R^m$ which contains distinct "Lebesgue" points of density 1 of each of the three regions. For the i^{th} region, choose $\delta > 0$ as in Lemma 2.13 for $\epsilon = \alpha/12$ such that the balls are disjoint and the volume of V_i in the ball exceeds half of the volume of the ball. The total surface area cost of constructing three such balls is less than $\alpha/4$.

By Propositions 2.1 and 2.4, for large n, the area of Y′ and the area of the preimage of Y′ on $S^n(\sqrt{n})$ differ by at most $\alpha/4$ while the volumes differ by at most $\Delta V'/2$. By Lemma 2.13, we can enclose in our balls at least $\Delta V'/2$ of each volume partition at area change of at most $\alpha/4$. Delete the volume from these balls and observe that every region now has less volume than it did in Y′. By splitting the balls with hyperplanes, we reassign the appropriate amount of volume to each region so that the total volume of each region equals the volume in each region of Y′.

We have just constructed a competitor on $S^n(\sqrt{n})$ to the standard double bubble. The total area difference between the competitor and the standard double

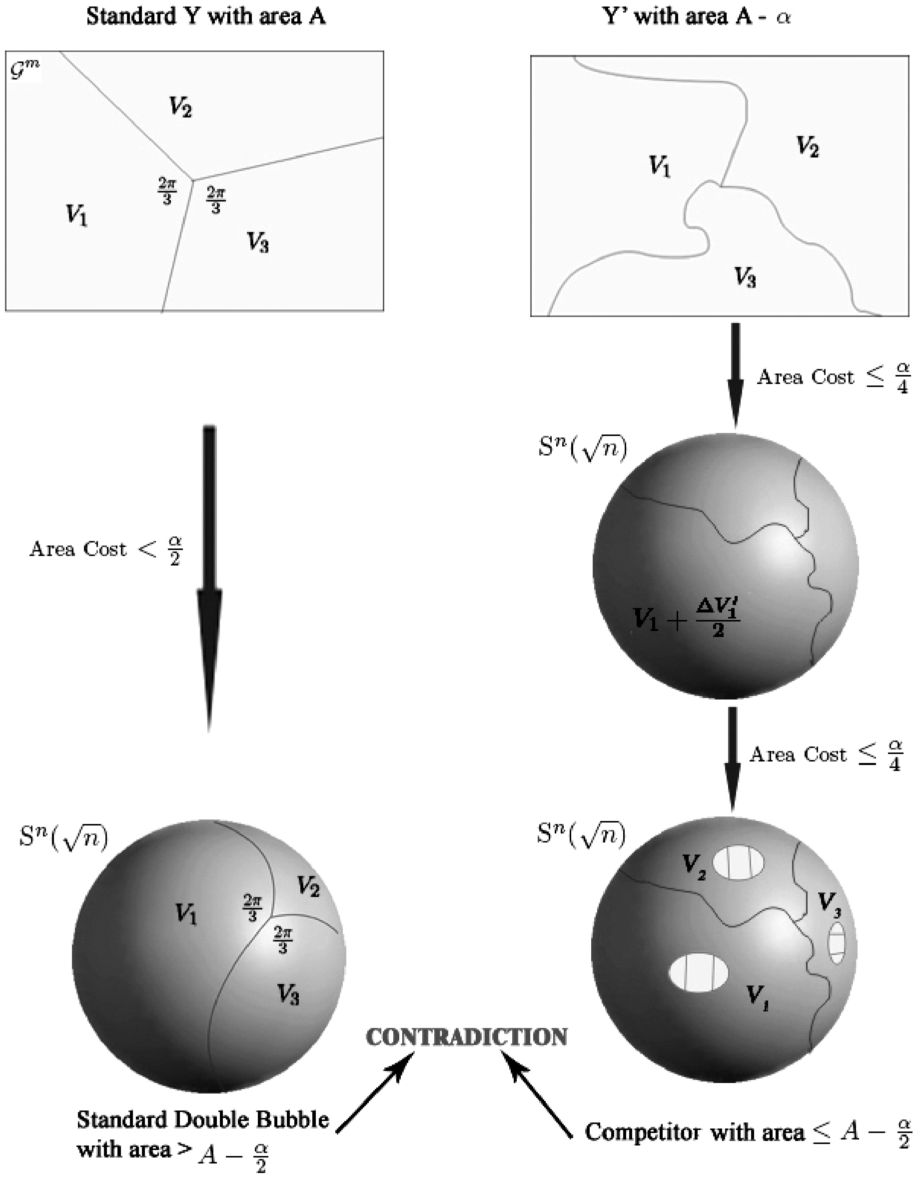

FIGURE 4. Assume there exists a Y′ better than our standard Y. This implies that the preimage of Y′ in $S^n(\sqrt{n})$ beats the standard double bubble, which is a contradiction.

bubble on $S^n(\sqrt{n})$ is less than $\alpha/2$ (area difference between the Y and the standard double bubble) plus $\alpha/4$ (difference between Y′ and preimage on $S^n(\sqrt{n})$) plus $\alpha/4$ (balls to balance volume), and hence less than α. Y′ therefore partitions $S^n(\sqrt{n})$ into the three given volumes and uses less area than the standard double bubble, contradicting our hypothesis. □

Remark. Brakke's Surface Evolver [Br], available online at no cost, is a program expressly designed for the modeling of soap bubbles, foams, liquid solder, capillary shapes, and other liquid surfaces shaped by minimizing energy subject to various constraints. It inspired and helped us to simulate the perimeter-minimizing partition of three areas in G^2. It is a good way to see how the double bubble in Euclidean space turns into a standard Y in Gauss space.

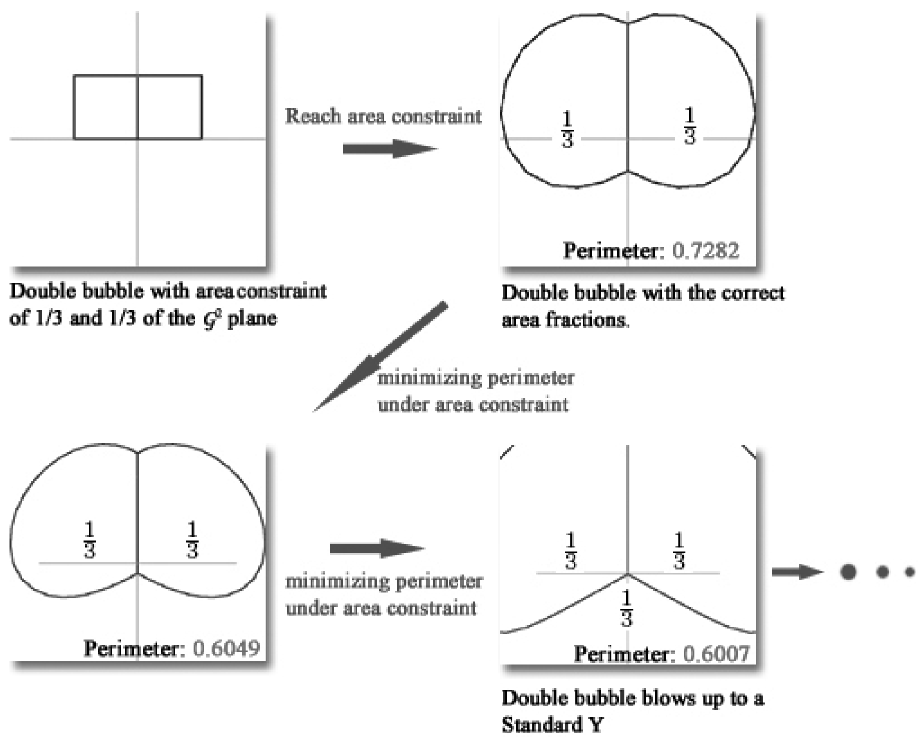

FIGURE 5. The evolution of an equal-area double bubble in G^2 of limiting perimeter 0.5984.

We first evolved the equal-area double bubble with area fractions $(1/3, 1/3, 1/3)$, as in Figure 5. Notice that the double bubble blows up and one vertex moves to the origin with three rays spaced approximately at angles of $2\pi/3$. The evolved shape has perimeter approaching $3/(2\sqrt{2\pi}) = 0.5984$, which is the perimeter of the standard Y at the origin.

We then evolved a double bubble with un-equal area fractions $(1/10, 1/3, 17/30)$. Notice that this bubble also blows up, forming a shifted standard Y (Figure 6).

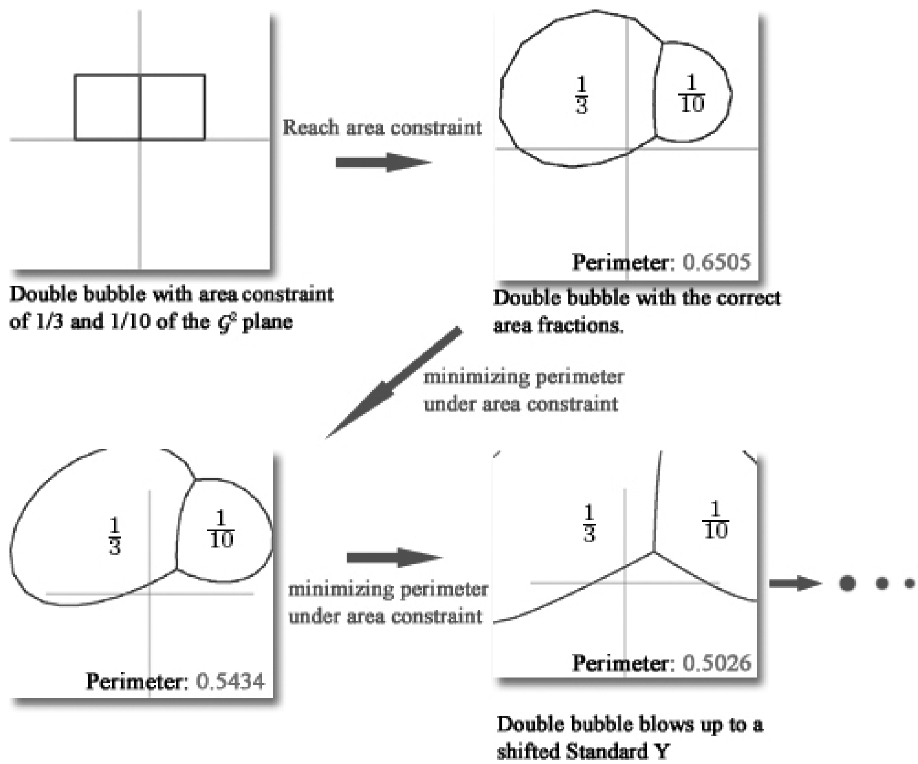

FIGURE 6. The evolution of an unequal-area double bubble in G^2.

Remark. Extending our work we propose the following conjecture.

Conjecture 2.1. *The perimeter-minimizing partition of G^2 into n volumes $V_1 + V_2 + \ldots + V_n = 1$ consists of n halflines and $n-3$ line segments, meeting at* 120 *degrees, as in Figure 7.*

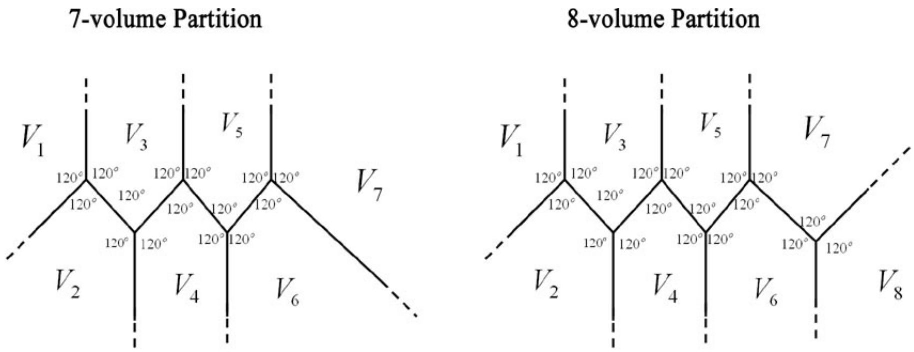

FIGURE 7. Our conjectured area-minimizing partition of G^2 into 7 and 8 volumes.

3. The Double Bubble Conjecture in S^n

Our main Theorem 3.7 shows that the unique area-minimizing partition of the sphere S^n into three nearly equal volumes is a standard double bubble (see Definition 2). The main Lemmas 3.3–3.6 prove the Hutchings function positive in G^m, from which Theorem 3.7 follows. Section 3.3 extends Theorem 3.7 to a larger range of volumes.

A partial solution to the double bubble conjecture in S^3 obtained in [CF] states:

Theorem 1.1. *A least-area enclosure of two equal volumes in S^3 which add up to at most 90 percent of the total volume of S^3 must be the (unique) standard double bubble.*

Theorem 3.7 stems from this result.

3.1. The Hutchings Function in G^m and S^n.

Definition 3. Define T to be the triangular region in R^3 of positive volumes (V_1, V_2, V_3) such that $V_1 + V_2 + V_3 = 1$. Throughout this section, unless otherwise noted, we assume that volumes (V_1, V_2, V_3) are in T. Figures 8 and 9 illustrate this barycentric coordinate system.

Definition 4. The Hutchings function on the sphere S^n is defined as

$$F_{S^n}(V_1, V_2, V_3) = 2A_{S^n}(V_1/2) + A_{S^n}(V_2) + A_{S^n}(V_3) - 2A_{S^n}(V_1, V_2, V_3),$$

where $A_{S^n}(v)$ denotes the surface area of the ball of volume fraction v in $S^n(\sqrt{n})$ and $A_{S^n}(V_1, V_2, V_3)$ denotes the surface area of the standard double bubble which partitions $S^n(\sqrt{n})$ into volume fractions V_1, V_2, V_3. Similarly, the Hutchings function in Gauss space G^m is defined as

$$F_{G^m}(V_1, V_2, V_3) = 2A_{G^m}(V_1/2) + A_{G^m}(V_2) + A_{G^m}(V_3) - 2A_{G^m}(V_1, V_2, V_3),$$

where $A_{G^m}(v)$ denotes the surface area of a halfspace of G^m of volume v and $A_{G^m}(V_1, V_2, V_3)$ denotes the surface area of the standard Y which partitions G^m into volumes V_1, V_2, V_3.

Remark. Hutchings ([Hu], Theorem 4.2) shows that if

$$F_{S^n}(V_1, V_2, V_3) > 0,$$

then the region of volume fraction V_1 is connected in an area-minimizing double bubble enclosing volume fractions V_1, V_2, V_3 in the sphere S^n. Hutchings actually uses a theoretically larger function, taking $A_{S^n}(V_1, V_2, V_3)$ to be the minimum area, so its positivity follows from the positivity of our F_{S^n}.

Note that the positivity of the Hutchings function is independent of the radius of S^n, since changing the radius scales the Hutchings function by a positive factor. The Hutchings theory does not apply directly to G^m, for lack of translational symmetry.

Cotton and Freeman ([CF], Proposition 7.3) show that for a minimizer in S^n, if each region and the exterior is connected, then the minimizer is a standard double bubble.

Lemma 3.1. *As n approaches infinity, $F_{S^n}(V_1, V_2, V_3)$ approaches $F_{G^m}(V_1, V_2, V_3)$.*

Proof. By Remark 2.2, $\lim_{n\to\infty} A_{S^n}(v) = A_{G^1}(v)$. By Proposition 2.12, $\lim_{n\to\infty} A_{S^n}(V_1, V_2, V_3) = A_{G^m}(V_1, V_2, V_3)$. Therefore,

$$F_G(V_1, V_2, V_3) = \lim_{n\to\infty} F_{S^n}(V_1, V_2, V_3).$$

□

Remark. Recall that m-dimensional Gauss space G^m is R^m endowed with density $(2\pi)^{-m/2}e^{-x^2/2}$. Let $(-\infty, x]$ be an interval of weighted length v in G^1. Then

$$A_{G^1}(v) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

where x is defined by

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt = v.$$

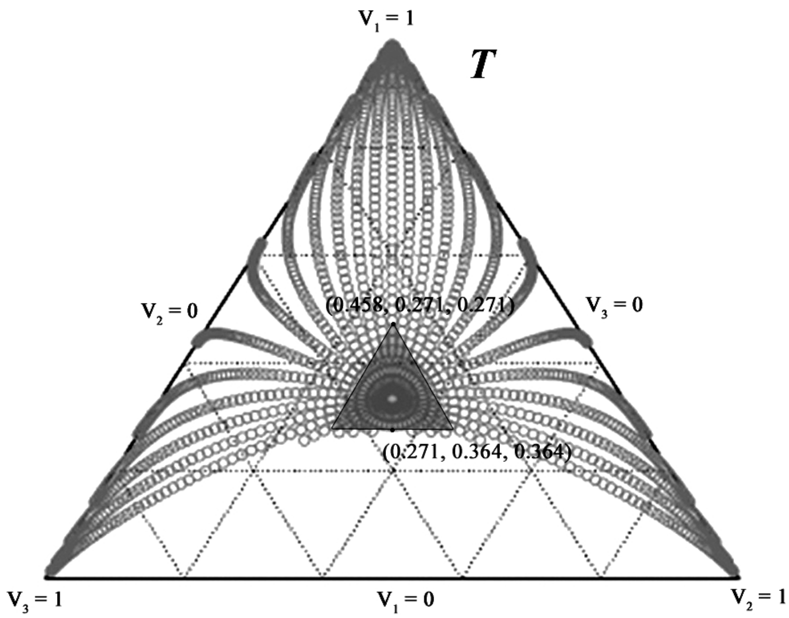

FIGURE 8. Small triangular region in T where all three permutations of the Hutchings function are positive. The larger shaded region is where a particular one of them is positive; the small triangle is the intersection of three such regions.

Writing $A_{G^1}(v)$ as $e^{\psi(x)}$, observe that

$$\frac{dA_{G^1}(v)}{dv} = \frac{dA_{G^1}/dx}{dv/dx} = \psi'(x) = -x. \tag{1}$$

Therefore $A_{G^1}(v)$ is concave with maximum $A_{G^1}(1/2) = 1/\sqrt{2\pi}$.

Definition 5. Define the function $\tilde{F}_{S^n}(V_1, V_2, V_3)$, a lower bound of the Hutchings function $F_{S^n}(V_1, V_2, V_3)$, as

$$\tilde{F}_{S^n}(V_1, V_2, V_3) \quad = \quad 2A_{S^n}(V_1/2) + A_{S^n}(V_2) + A_{S^n}(V_3) - 3A_{S^n}\left(\frac{1}{2}\right).$$

Here $3A_{S^n}(1/2)$ arises as an overestimate for $2A_{S^n}(V_1, V_2, V_3)$ since three great $(n-1)$-hemispheres (each of which has area $A_{S^n}(1/2)/2$) meeting along a great $(n-2)$-sphere partition $S^n(\sqrt{n})$ into volume fractions V_1, V_2, V_3.

Similarly, define the function $\tilde{F}_G(V_1, V_2, V_3)$, a lower bound (for any m) of the Hutchings function $F_{G^m}(V_1, V_2, V_3)$, as

$$\tilde{F}_G(V_1, V_2, V_3) \quad = \quad 2A_{G^1}(V_1/2) + A_{G^1}(V_2) + A_{G^1}(V_3) - \frac{3}{\sqrt{2\pi}}.$$

Here $3/\sqrt{2\pi} = 3A_{G^1}(1/2)$ (as computed in Remark 3.1) arises as an overestimate of $2A_{G^m}(V_1, V_2, V_3)$. Indeed, except for the trivial case $m = 1$ three half-hyperplanes (each of which has area $A_{G^m}(1/2)/2)$ meeting along an (m-2)-dimensional plane passing through the origin can partition G^m into volumes V_1, V_2, V_3. Since the Gauss measure is a spherically symmetric product measure with total measure one, $A_{G^m}(v)$ is independent of m, so we can take $m = 1$.

Lemma 3.2. *As n approaches infinity, $\tilde{F}_{S^n}(V_1, V_2, V_3)$ approaches $\tilde{F}_G(V_1, V_2, V_3)$.*

Proof. By Remark 2.2, $\lim_{n\to\infty} A_{S^n}(v) = A_{G^1}(v)$. Therefore,

$$\tilde{F}_G(V_1, V_2, V_3) = \lim_{n\to\infty} \tilde{F}_{S^n}(V_1, V_2, V_3).$$

□

3.2. **Region of Positivity of the Hutchings Function.** Theorem 3.7 shows that the standard double bubble is an area minimizer for nearly equal volumes in S^n for all n. The proof depends on the positivity of the Hutchings function F_{S^n}, which follows from the positivity of $F_{G^m}(V_1, V_2, V_3)$ and its permutations $F_{G^m}(V_2, V_3, V_1)$ and $F_{G^m}(V_3, V_1, V_2)$.

Lemma 3.3. *For equal volume fractions, $\tilde{F}_G$ is greater than* 0.029.

Proof. Using -100 as an approximation for $-\infty$, we use numerical approximation to find x-values of $-.44$ and $-.97$ for terms of the Hutchings function applied to equal areas. Since

$$\int_{-100}^{-.97} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt < .16666 < \frac{1}{6}$$

and

$$\int_{-100}^{-.44} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt < .33333 < \frac{1}{3},$$

we have that

$$A_{G^1}(\frac{1}{6}) > \frac{1}{\sqrt{2\pi}} e^{-(-.97)^2/2} > .24922$$

and

$$A_{G^1}(\frac{1}{3}) > \frac{1}{\sqrt{2\pi}} e^{-(-.44)^2/2} > .36214.$$

Combining these inequalities yields

$$\begin{aligned}\tilde{F}_G\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) &= 2A_{G^1}\left(\frac{1}{6}\right) + A_{G^1}\left(\frac{1}{3}\right) + A_{G^1}\left(\frac{2}{3}\right) - \frac{3}{\sqrt{2\pi}} \\ &> 2(.24922) + 2(.36214) - \frac{3}{\sqrt{2\pi}} > .029.\end{aligned}$$

□

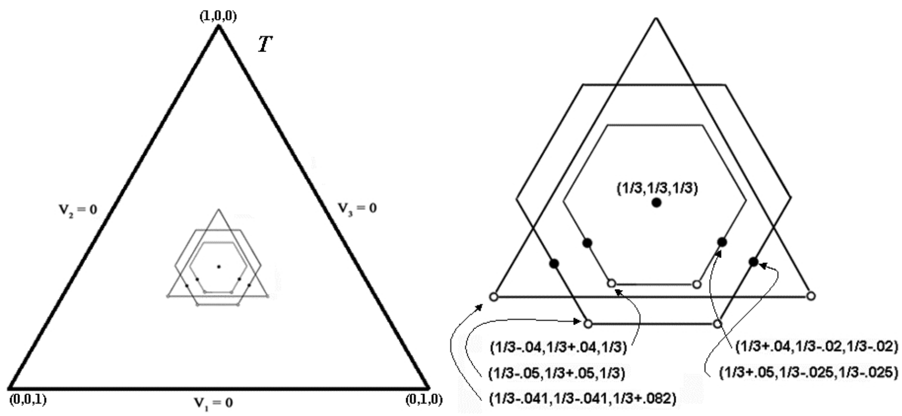

FIGURE 9. Small hexagons and triangle about $(1/3, 1/3, 1/3)$ where we prove the Hutchings function positive.

Lemma 3.4. *For a hexagon* $\{|V_i - 1/3| \leq \delta \leq 1/9\}$, $\tilde{F}_G(V_1, V_2, V_3)$ *is smallest at its two bottom vertices (with minimal* V_1 *as shown with the open circles in Figure 9). Similarly on a triangle* $\{1/3 - V_i \leq \delta \leq 1/18\}$ *the same result holds.*

PROOF. By Remark 3.1, A_{G^1} is concave, hence so is $\tilde{F}_G$. Therefore $\tilde{F}_G$ attains its minimal value on the vertices of the hexagon and triangle.

We temporarily relax the constraint that $(V_1, V_2, V_3) \in T$. Then the partial derivatives of $\tilde{F}_G(V_1, V_2, V_3)$ are given by

$$\begin{aligned}\frac{\partial \tilde{F}_G}{\partial V_1} &= A'\left(\frac{V_1}{2}\right)\\ \frac{\partial \tilde{F}_G}{\partial V_2} &= A'(V_2)\\ \frac{\partial \tilde{F}_G}{\partial V_3} &= A'(V_3).\end{aligned}$$

By (1), $A'(V_1/2) = -x_1$, $A'(V_2) = -x_2$ and $A'(V_3) = -x_3$. Since $V_1/2 \leq 1/2$, x_1 is negative, and since $V_1/2$ is less than or equal to V_2 and V_3, therefore x_1 is less than or equal to x_2 and x_3. Thus for all V_1, V_2, V_3 in the hexagon,

$$\frac{\partial \tilde{F}_G}{\partial V_1} \geq \frac{\partial \tilde{F}_G}{\partial V_2}, \tag{2}$$

$$\frac{\partial \tilde{F}_G}{\partial V_1} \geq \frac{\partial \tilde{F}_G}{\partial V_3}. \tag{3}$$

Returning to T, for motion downward along the upper-righthand edge of the hexagon (as illustrated in Figure 9), V_3 is constant while V_1 decreases at the same rate that V_2 increases. Therefore by (2) $\tilde{F}_G$ is non-increasing. Similar analysis applies to the other non-horizontal edges. Therefore $\tilde{F}_G$ is smallest at the bottom vertices. For the triangle, the same argument applies. $\square$

Lemma 3.5. *In the hexagon* $\{|V_i - 1/3| \leq .04\}$, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ *for large* n.

Proof. By Lemma 3.4, $\tilde{F}_G(V_1, V_2, V_3)$ achieves its minimum at $(1/3, 1/3, 1/3) + .04(-1, 1, 0)$. Computations show that

$$A_{G^1}\left(\frac{1}{6} - .02\right) > .24964, \quad A_{G^1}\left(\tfrac{1}{3} + .04\right) > .37866, \quad A_{G^1}\left(\frac{2}{3}\right) > .36359.$$

Hence,

$$\tilde{F}_G\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .04(-1, 1, 0)\right) =$$

$$= 2A_{G^1}\left(\frac{1}{6} - .02\right) + A_{G^1}\left(\frac{1}{3} + .04\right) + A_{G^1}\left(\frac{2}{3}\right) - \frac{3}{\sqrt{2\pi}} > .004 > 0.$$

Thus in the hexagon $\{|V_i - 1/3| \leq .04\}$, $\tilde{F}_G(V_1, V_2, V_3) > 0$ and by Lemma 3.2, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ for large n. $\square$

Lemma 3.6. *If $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ for infinitely many n, then $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ for all n.*

Proof. By Barthe ([Bar], Proposition 11; see [Ros], Theorem 21),

$$\frac{A_{S^{(n+1)}}(v)}{A_{S^{(n+1)}}(\frac{1}{2})} \leq \frac{A_{S^n}(v)}{A_{S^n}(\frac{1}{2})}. \tag{4}$$

Dividing the modified Hutchings function by $A_{S^n}(1/2)$ yields

$$\frac{\tilde{F}_{S^n}(V_1, V_2, V_3)}{A_{S^n}(\frac{1}{2})} = \frac{A_{S^n}(V_1/2)}{A_{S^n}(\frac{1}{2})} + \frac{A_{S^n}(V_2)}{A_{S^n}(\frac{1}{2})} + \frac{A_{S^n}(V_3)}{A_{S^n}(\frac{1}{2})} - 3$$

By (4), $\tilde{F}_{S^n}(V_1, V_2, V_3)/A_{S^n}(1/2)$ is nonincreasing in n. The result follows. □

Theorem 3.7. *For all n, the $n-$dimensional standard double bubble is the unique area-minimizing partition of S^n into any three volume fractions within $.04$ of $1/3$.*

Proof. By Lemma 3.5, for large n, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$. By Lemma 3.6, for all n, $\tilde{F}_{S^n}(V_1, V_2, V_3) > 0$ and therefore $F_{S^n}(V_1, V_2, V_3) > 0$. By Hutchings (Remark 3.1) and the symmetry of permuting the V_i, each region is connected. By Cotton and Freeman (Remark 3.1), the regions must form a standard double bubble. □

Corollary 3.8. *In G^m the standard Y is an area-minimizing partition for any three volume fractions within $.04$ of $1/3$.*

Proof. This follows immediately from Theorems 3.7 and 2.14. In fact, this result can be reached independent of Lemma 3.6 since it only depends on standard double bubbles with large n. □

3.3. **Extending the known region of positivity for the Hutchings function.** Using computational lower bounds on the Hutchings function we enlarge the known region of positivity for the Hutchings function in Gauss space and hence in high-dimensional spheres from the hexagon $\{|V_i - 1/3| \leq .04\}$ to $\{|V_i - 1/3| \leq .05\}$ and finally to the triangle $\{1/3 - V_i \leq .041\}$. (We do not know how to generalize Lemma 3.6 on lower-dimensional spheres.) Figure 9 shows the inner (.04) and outer (.05) hexagons along with the .041 triangle. An improved overestimate for $A_{G^m}(V_1, V_2, V_3)$ is necessary in order to extend the region of positivity . To this end we prove two lemmas about the concave nature of $A_{G^m}(V_1, V_2, V_3)$.

Remark. The concept of mean curvature can be generalized to hypersurfaces in manifolds with density, such as Gauss space ([Gr], [CHSX], [Mo2]). A first variation formula shows that the curvature κ of a curve in an equilibrium configuration equals minus the rate of change of perimeter or "area" with respect to volume transfer across the curve. A line in Gauss space has constant curvature equal to its signed distance from the origin (Remark 3.1).

Lemma 3.9. *In T, outside of the hexagon $\{|V_i - 1/3| < \delta\}$, $A_{G^m}(V_1, V_2, V_3)$ attains its maximum value at the middle of an edge.*

Proof. Every candidate maximum point lies on the boundary of an expanded hexagon $\{|V_i - 1/3| \leq \delta_1\}$, for some $\delta_1 \geq \delta$. Consider moving along an edge of the hexagon. For example, hold V_1 constant while V_2 changes. By Remark 3.3, $-dA_{G^m}/dV_2$ is a signed distance which is strictly increasing by Lemma 2.9. Thus dA_{G^m}/dV_2 is strictly decreasing and hence A_{G^m} is concave in V_2. By this concavity and symmetry we may assume that the candidate maximum point lies at the middle of an edge. Now consider holding $V_2 = V_3$ and changing V_1. By Remark 3.3, $-dA_{G^m}/dV_1$ is a signed distance which is strictly increasing by Lemma 2.10. Thus dA_{G^m}/dV_1 is strictly decreasing and hence A_{G^m} is concave in V_1. By this concavity and symmetry, we may assume that $\delta_1 = \delta$. □

Theorem 3.10. *In G^m the standard Y is an area-minimizing partition for any three volume fractions within .05 of 1/3.*

Proof. By Corollary 3.8, we need consider only the region R between the outer hexagon (.05) and the inner hexagon (.04) of Figure 9. By Lemma 3.9 and the symmetry of the hexagon, we need consider only two edge midpoints

$$\begin{aligned} p_1 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .04\left(-1, \frac{1}{2}, \frac{1}{2}\right), \\ p_2 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .04\left(1, -\frac{1}{2}, -\frac{1}{2}\right). \end{aligned}$$

By computation $A_{G^m}(p_1)$ is approximately 0.596374, and $A_{G^m}(p_2)$ is approximately 0.596436. Therefore $A_{G^m}(V_1, V_2, V_3) \leq 0.596437$. Using this overestimate for $A_{G^m}(V_1, V_2, V_3)$ in region R, we define a modified $\tilde{F}_G$ (a new lower bound for F_{G^m} in R), changing the constant term from $3/\sqrt{2\pi}$ to 0.596437. By Lemma 3.4, the lower bound for the $\tilde{F}_G$ in the outer hexagon occurs at $(1/3, 1/3, 1/3) + .05(-1, 1, 0)$. By computation this lower bound is approximately 0.0012288, and therefore F_{G^m} is positive in region R.

By Lemma 3.1, $F_{S^n}(V_1, V_2, V_3)$ converges to $F_{G^m}(V_1, V_2, V_3)$. Therefore, for large n $F_{S^n}(V_1, V_2, V_3) > 0$. By symmetry of the hexagon, each $F_{S^n}(V_i, V_j, V_k)$ is positive. By Hutchings (Remark 3.1), each region, as well as its complement, is connected. By Cotton and Freeman (Remark 3.1), the regions must form a standard double bubble in high-dimensional spheres. By Theorem 2.14, the minimizer in Gauss space is a standard Y. □

Theorem 3.11. *In G^m the standard Y is an area-minimizing partition for any three volume fractions greater than or equal to $1/3 - .041$.*

PROOF. By Corollary 3.8, we need consider only the region R between the (.041) triangle and the outer hexagon (.05) of Figure 9. By Lemma 3.9 and the symmetry of the hexagon, we need consider only two edge midpoints

$$\begin{aligned} p_1 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .05\left(-1, \frac{1}{2}, \frac{1}{2}\right), \\ p_2 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + .05\left(1, -\frac{1}{2}, -\frac{1}{2}\right). \end{aligned}$$

By computation $A_{G^m}(p_1)$ is approximately 0.595213, and $A_{G^m}(p_2)$ is approximately 0.595334. Therefore $A_{G^m}(V_1, V_2, V_3) \leq 0.595335$. Using this overestimate for $A_{G^m}(V_1, V_2, V_3)$ in region R, we define a modified $\tilde{F}_G$ (a new lower bound for F_{G^m} in R), changing the constant term from $3/\sqrt{2\pi}$ to 0.595335. By Lemma 3.4, the lower bound for the $\tilde{F}_G$ in the triangle occurs at $(1/3, 1/3, 1/3) + .041(-1, 2, -1)$. By computation this lower bound is approximately 0.0011504, and therefore F_{G^m} is positive in region R.

As in the proof of Theorem 3.10, the minimizer in Gauss space is a standard Y. □

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