Limitations on transversal gates

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University of Michigan[1], Alibaba Group[2]

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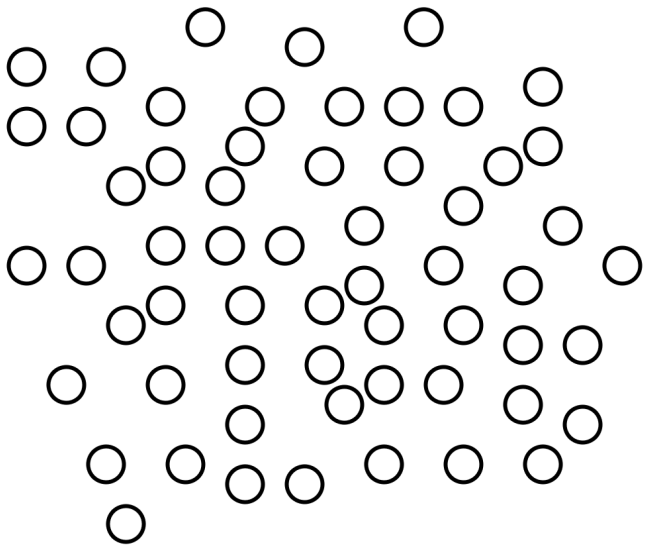
Quantum fault-tolerance



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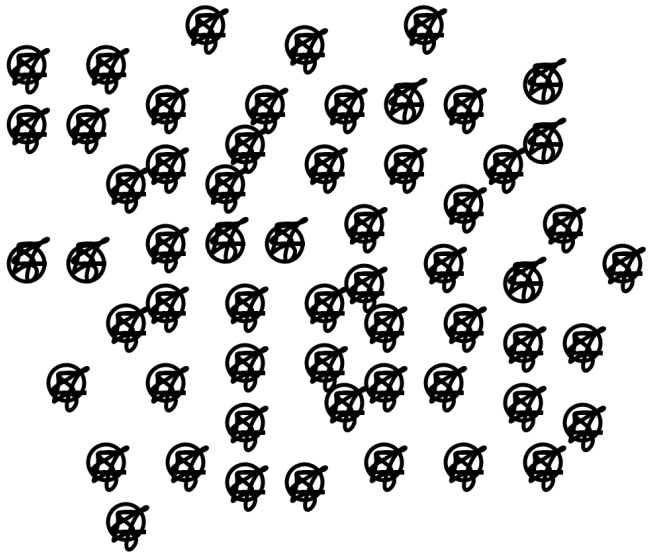

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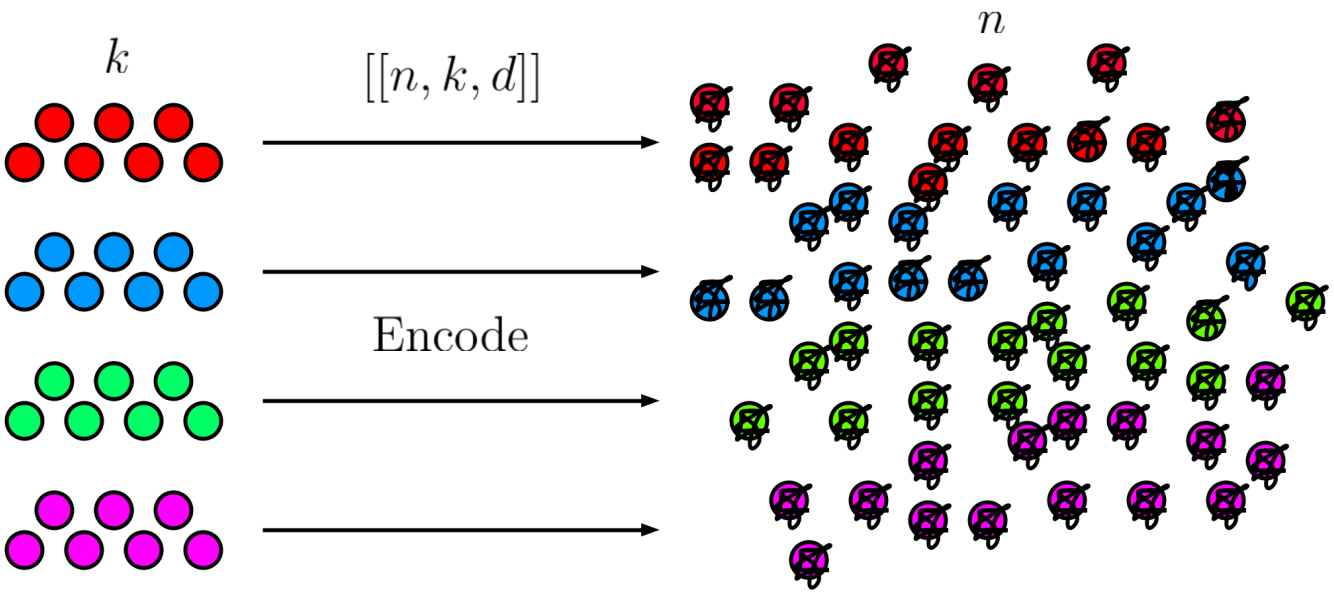

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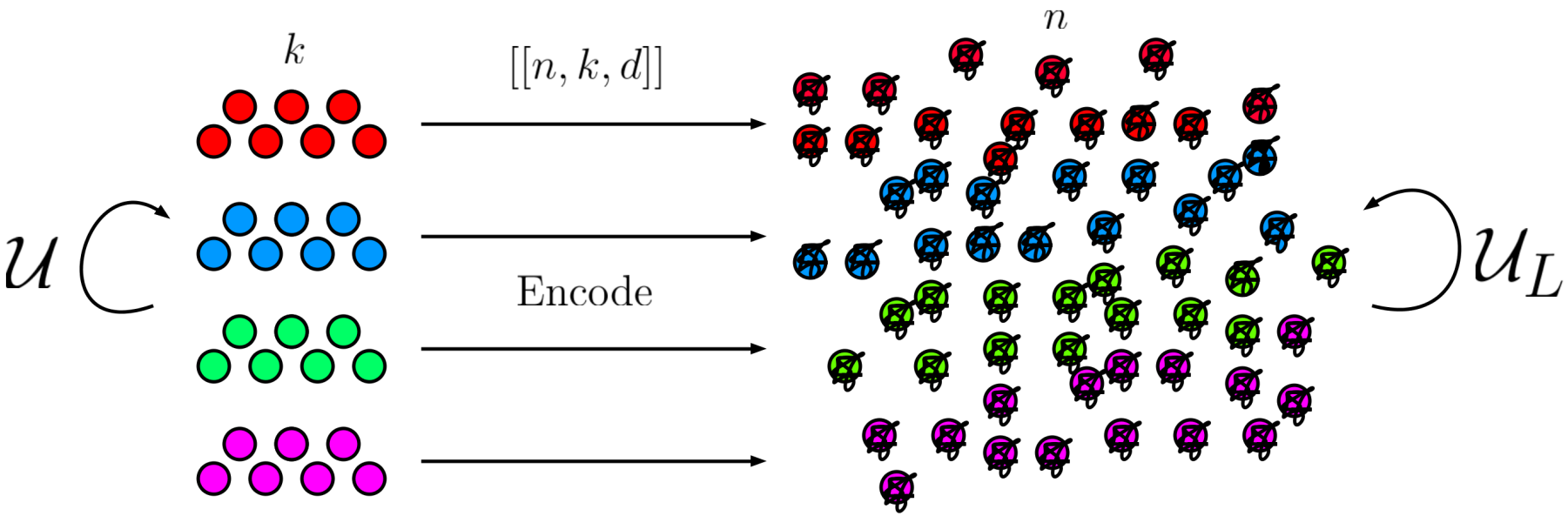
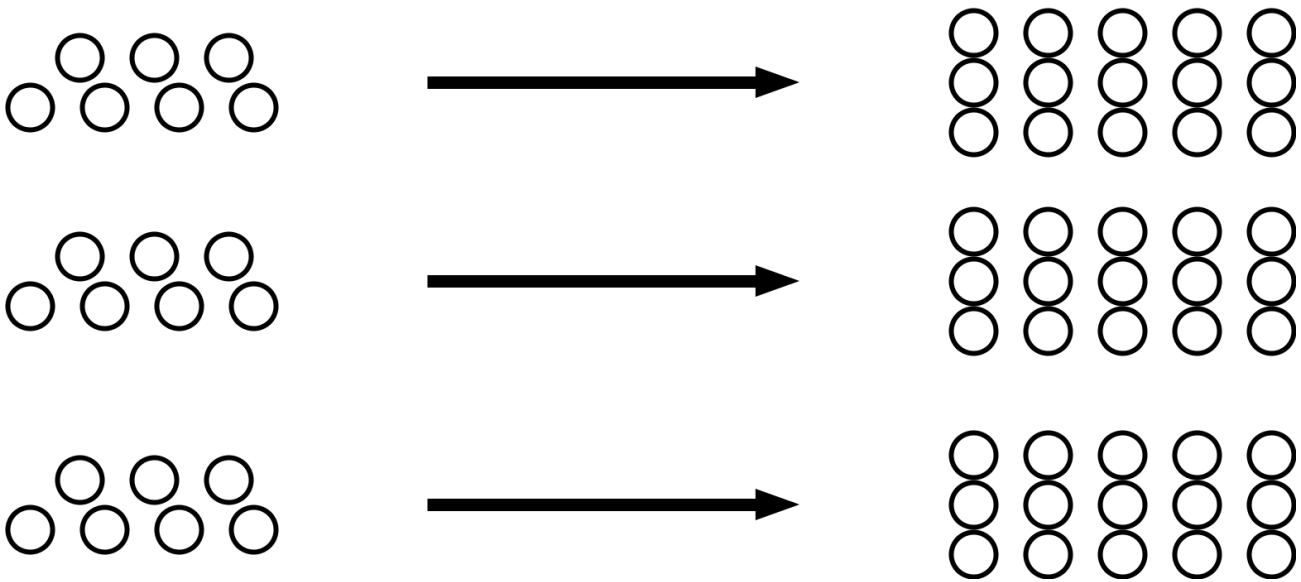


What are transversal gates?

$$[[15, 7, 3]]$$



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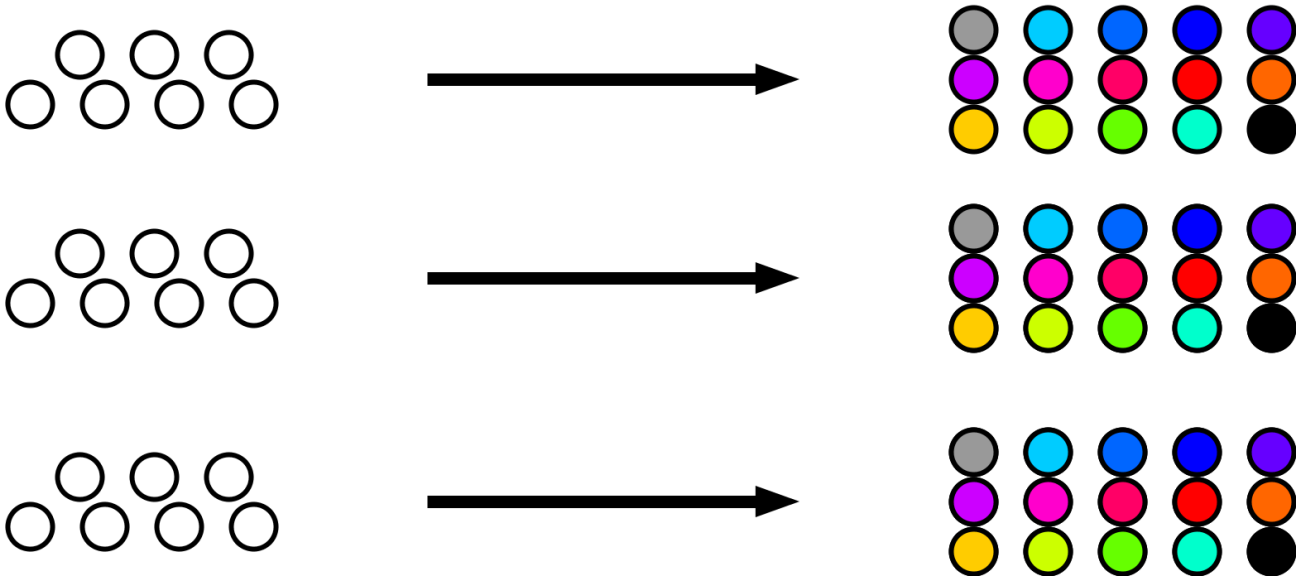

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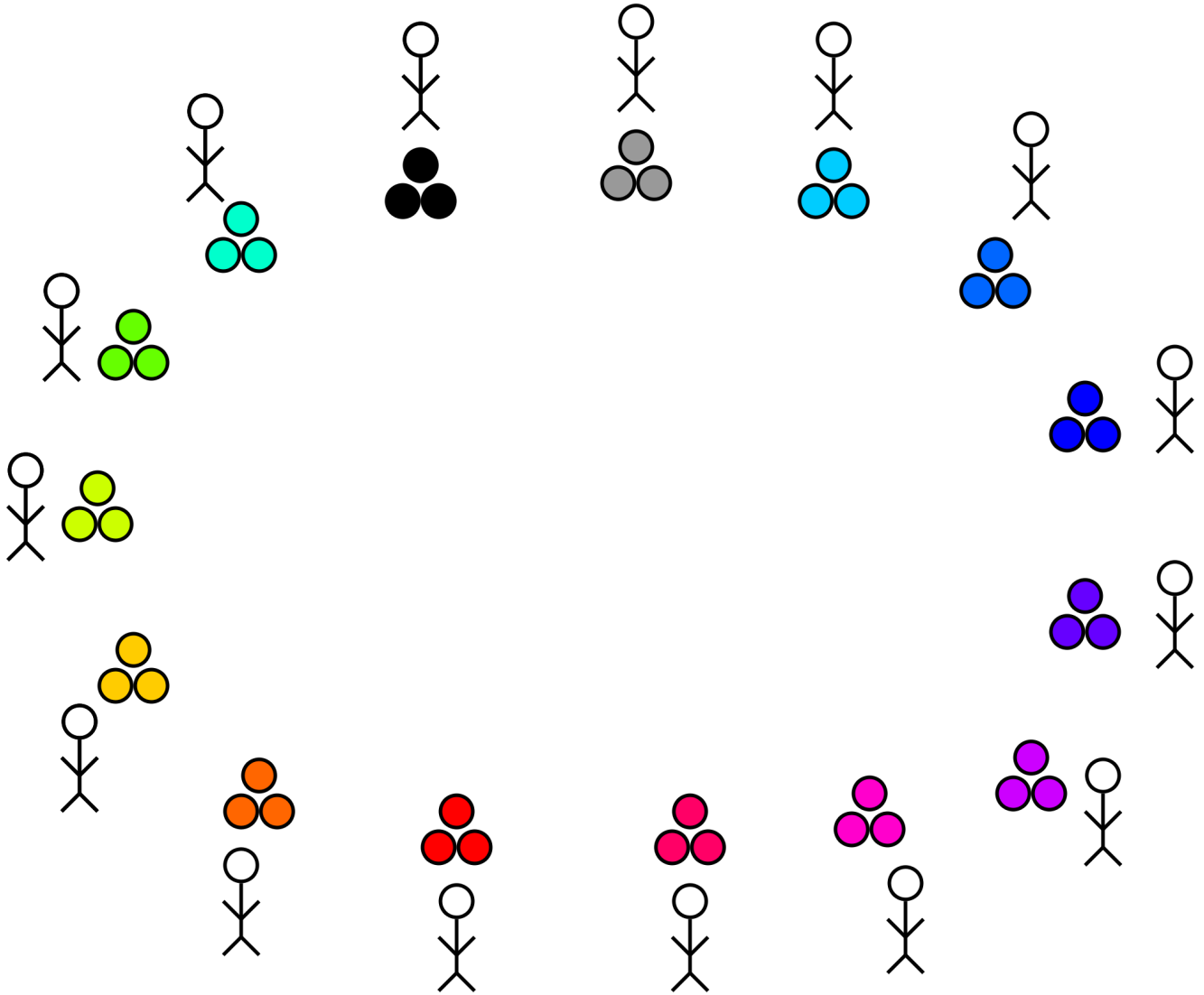

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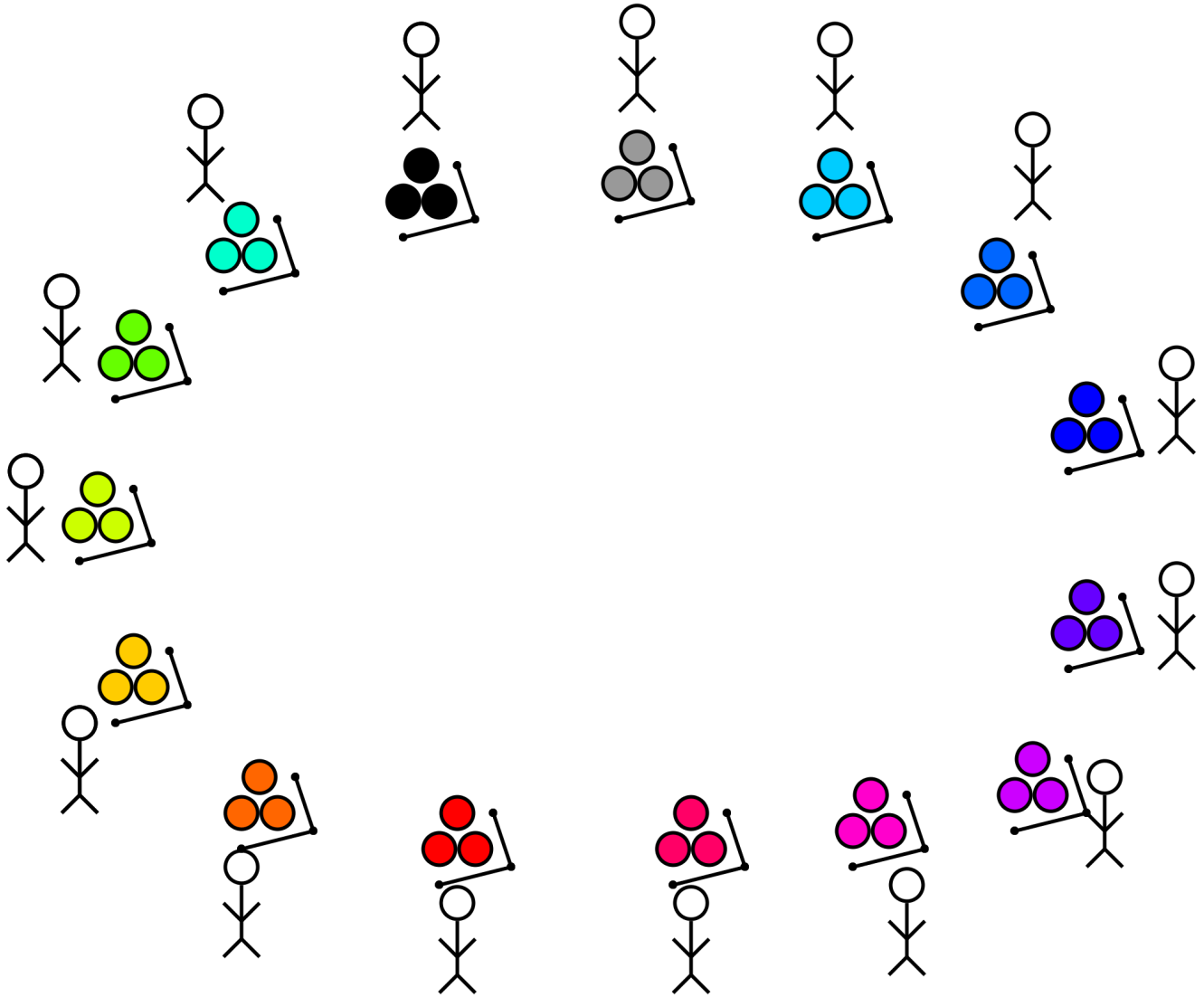

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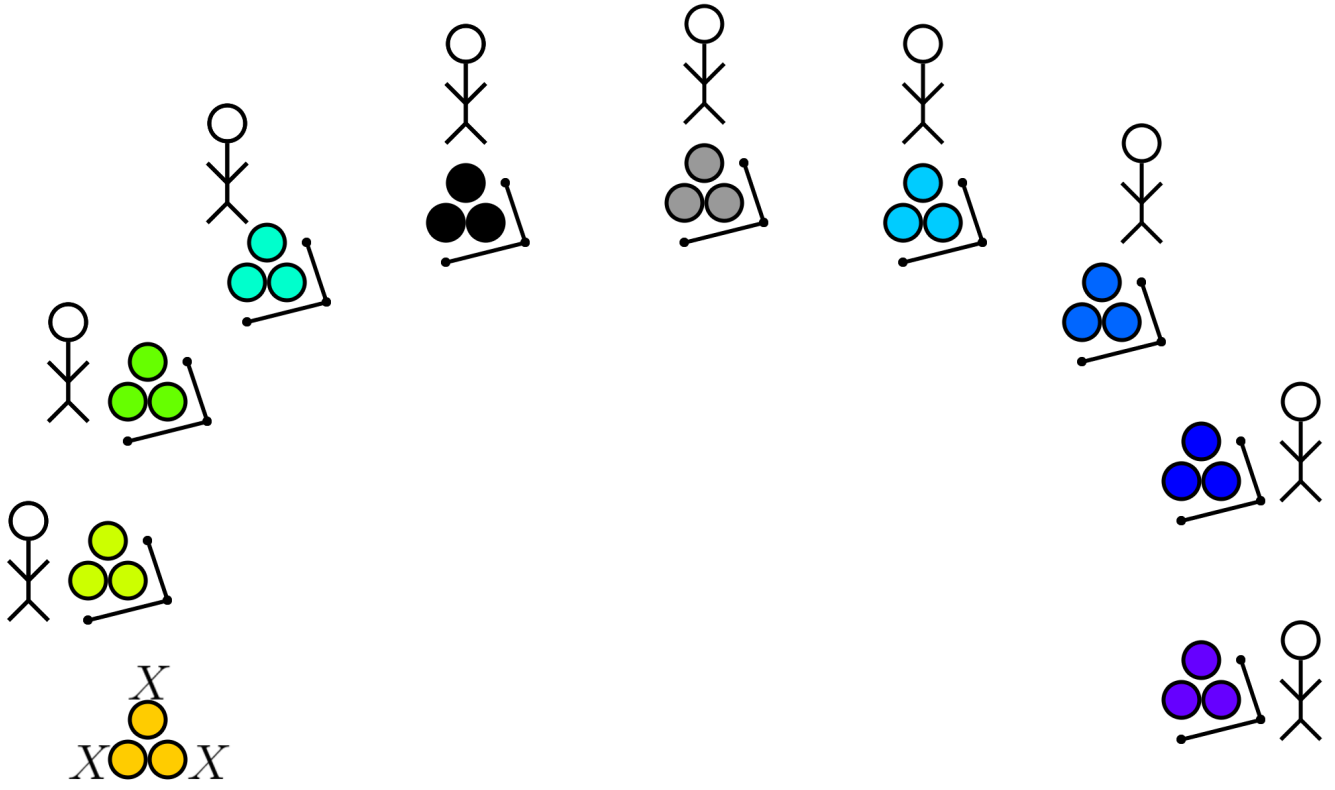

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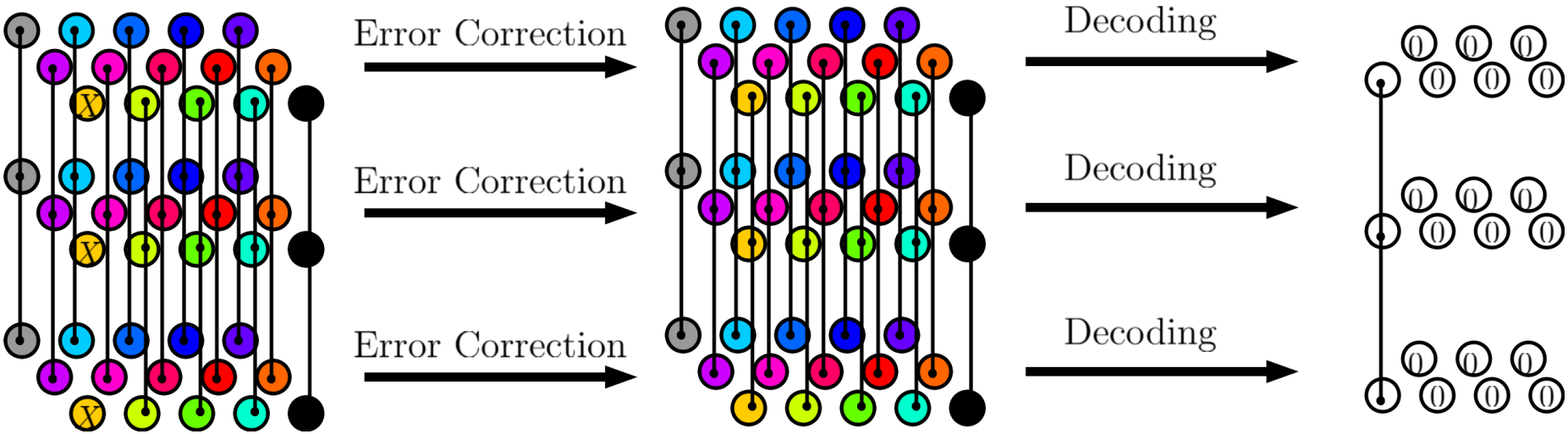


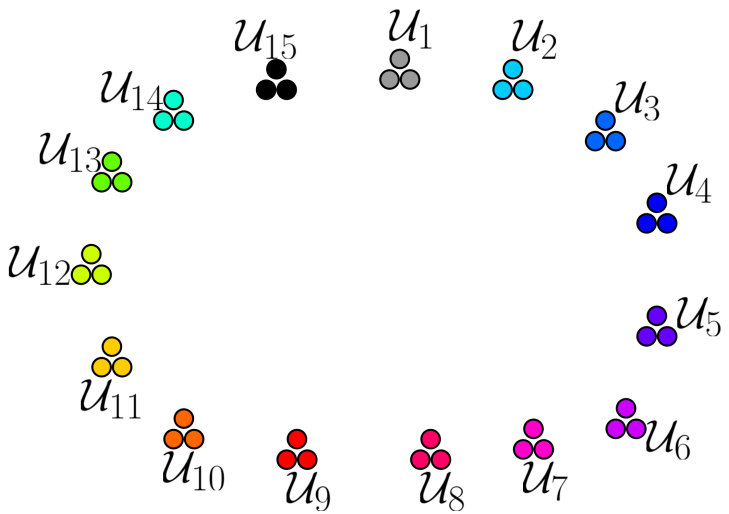
What are transversal gates?

Definition

A transversal p-qubit gate on an n-qubit codespace C is a logical gate $U : U(C^{\otimes p}) = C^{\otimes p}$ that decomposes as

$$U = \bigotimes_{j=1}^{n} U_j$$

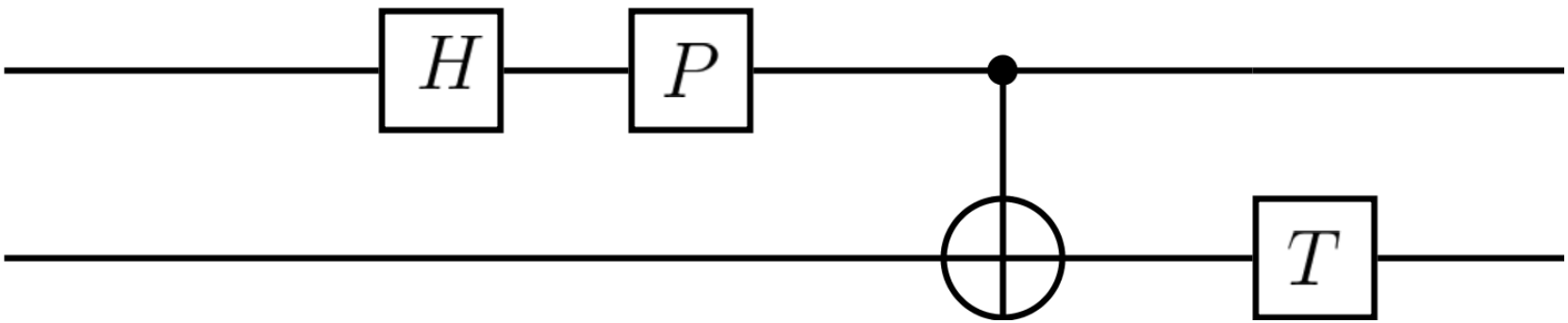
where each U_j acts on a single subsystem of the code.



What are transversal gates?

Logical gates implemented transversally are called $\lfloor \frac{d-1}{2} \rfloor$-*fault-tolerant*: if at most $\lfloor \frac{d-1}{2} \rfloor$ components fail during their application, the resulting errors can be corrected.

What are transversal gates?



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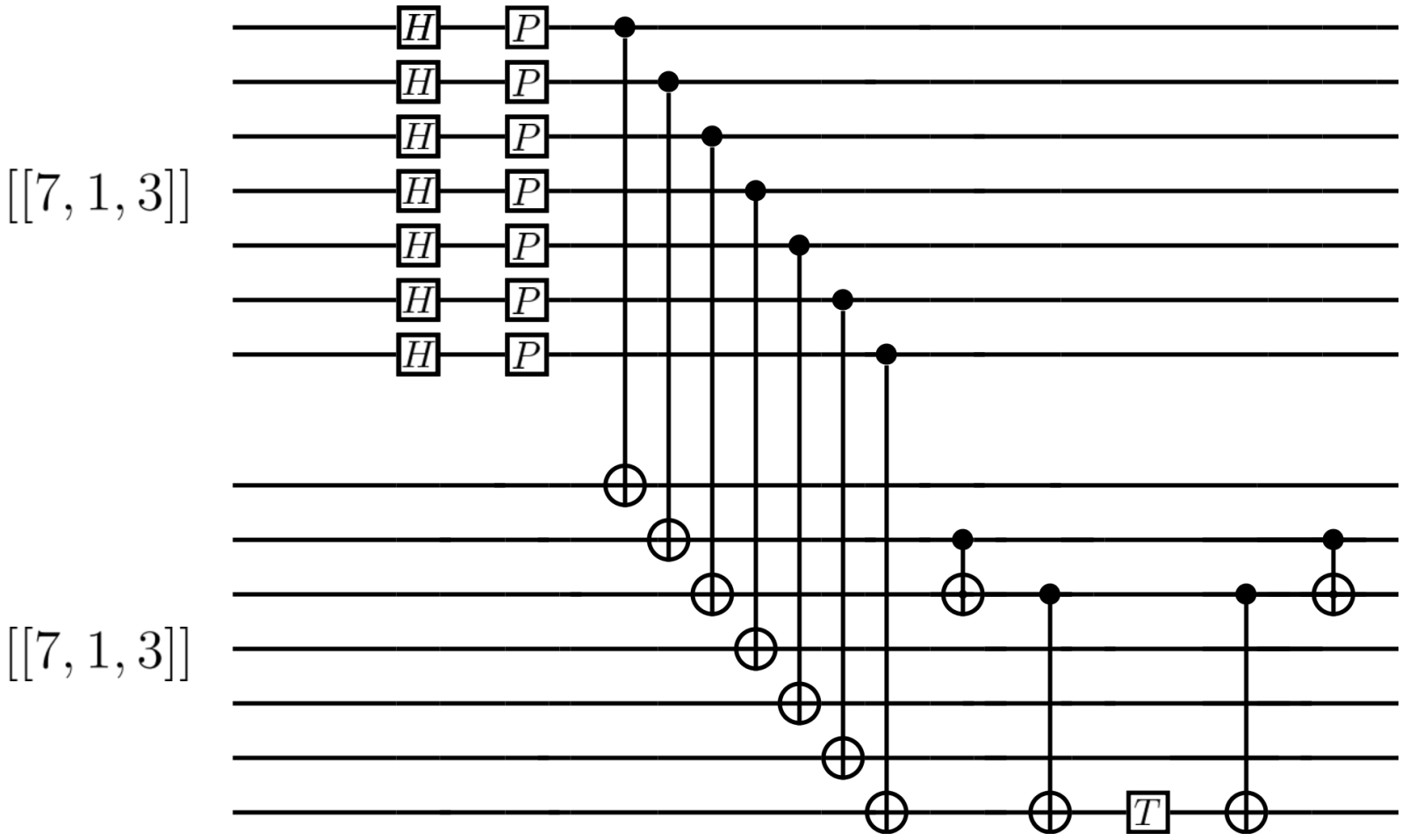

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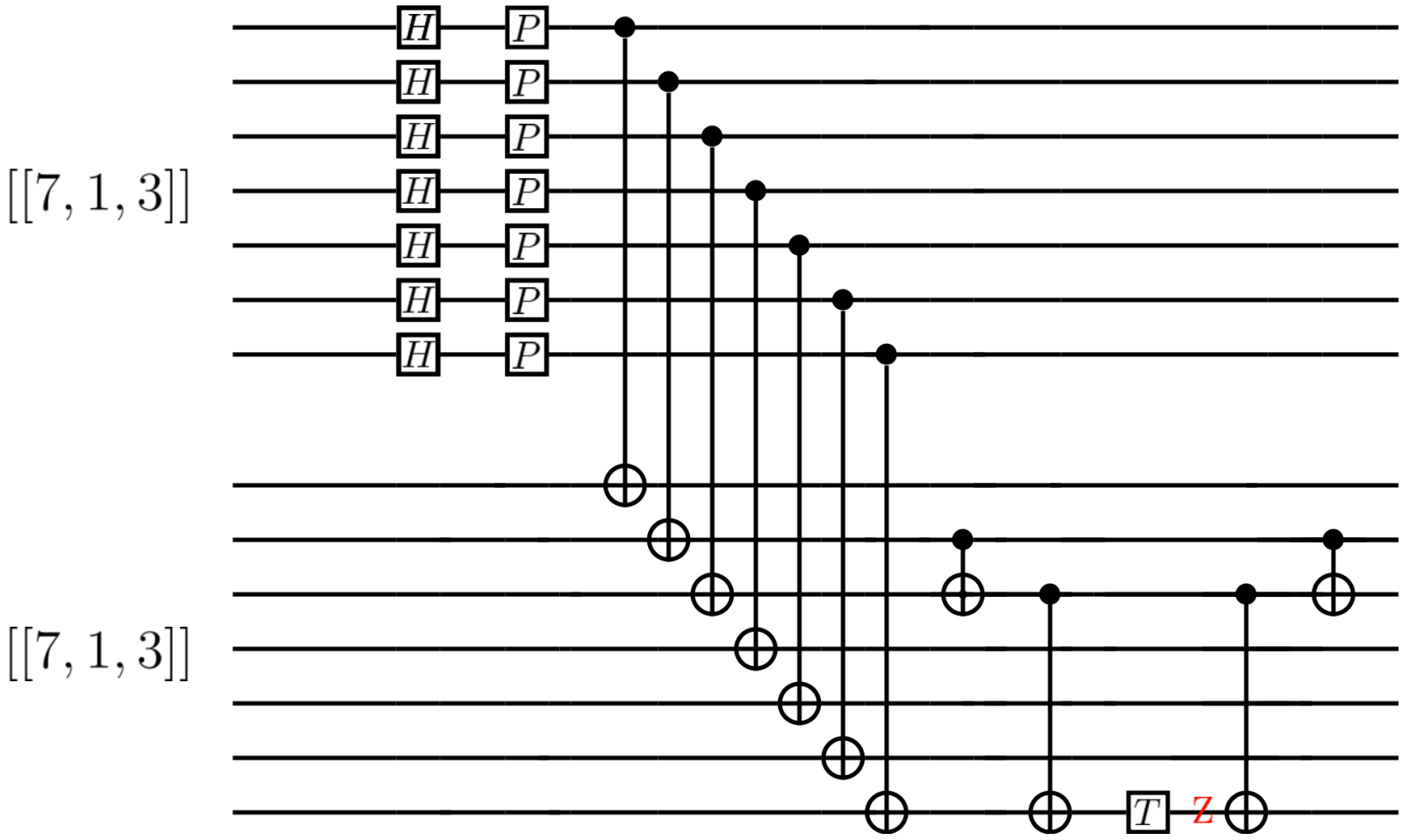

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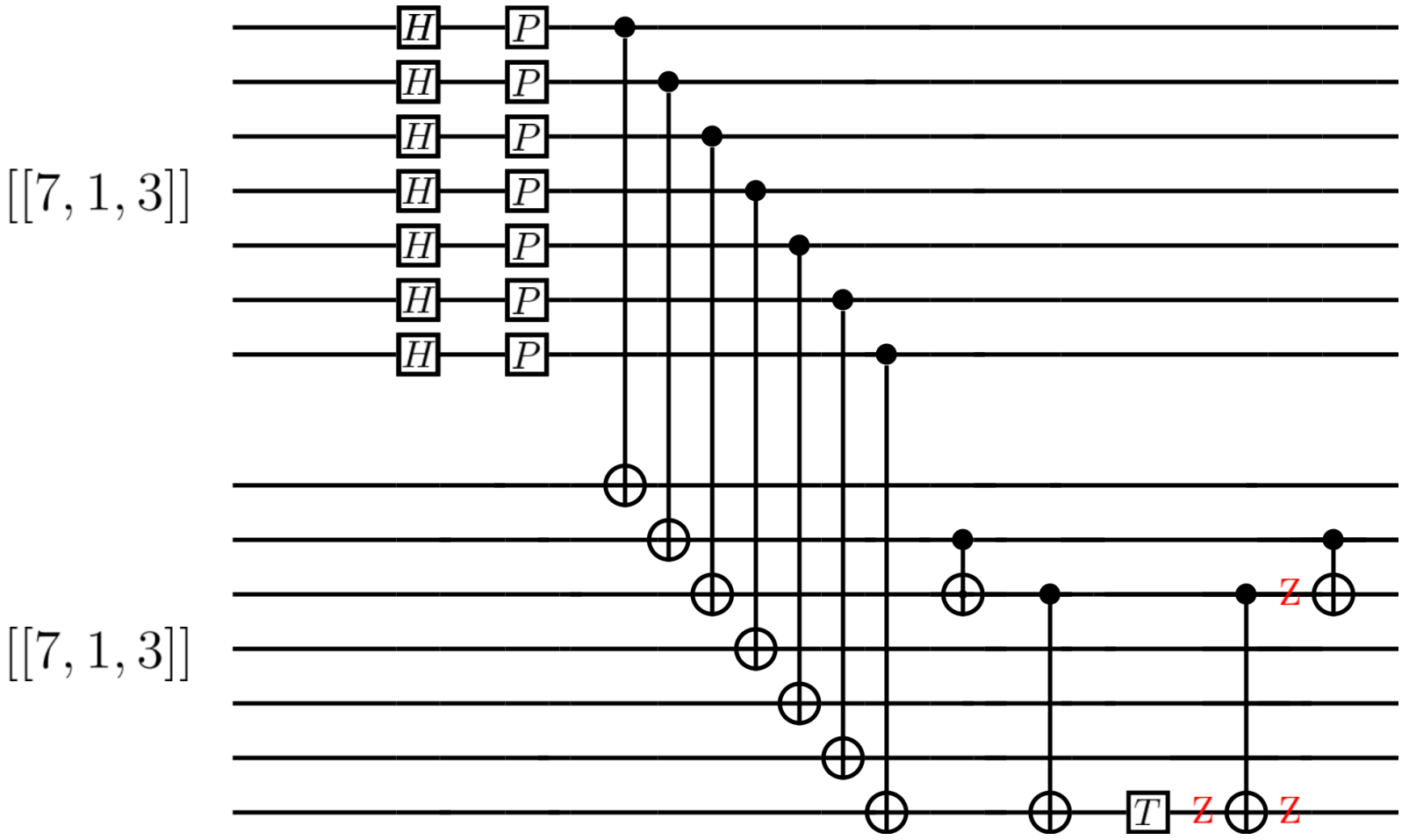

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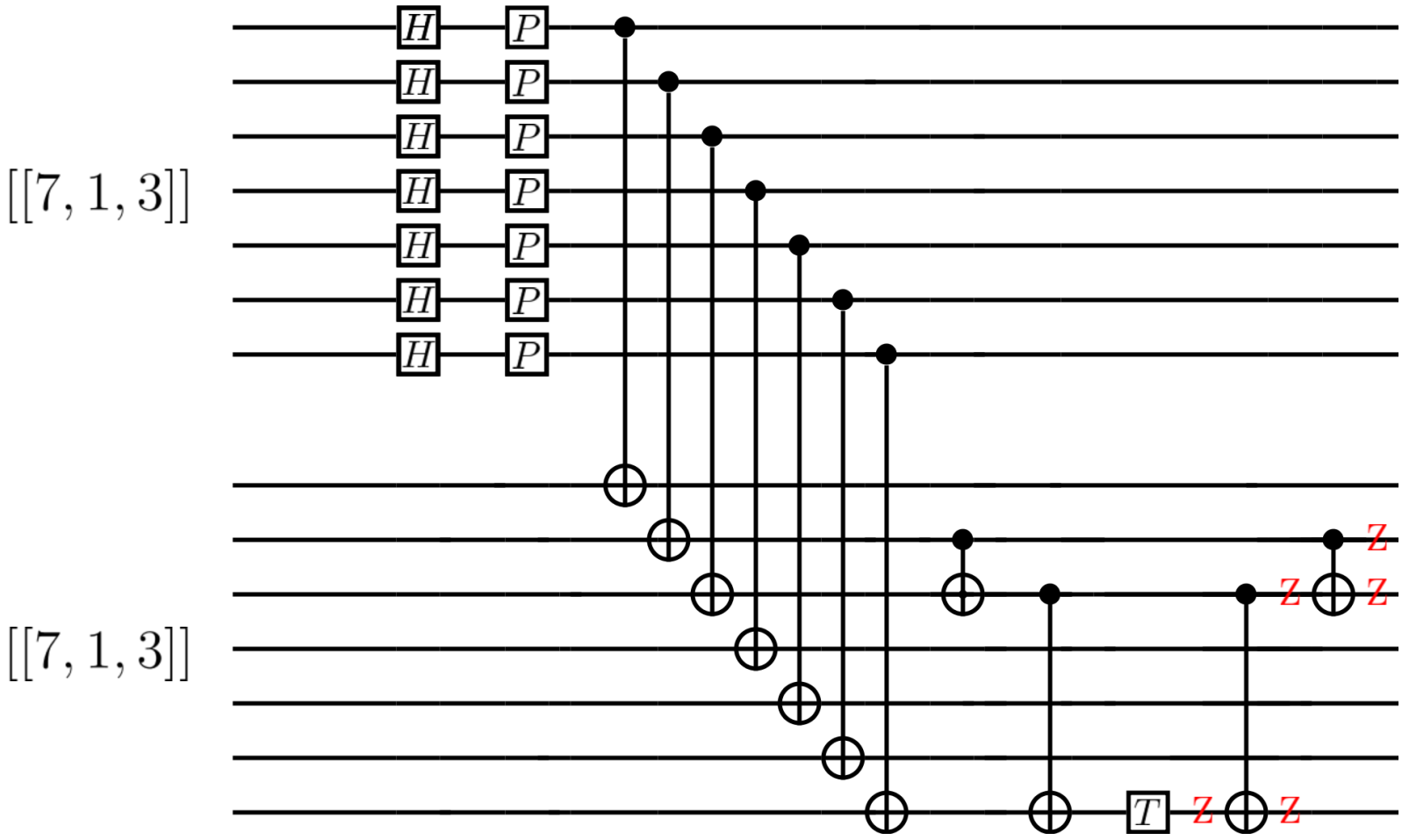


No quantum universal transversal gate set

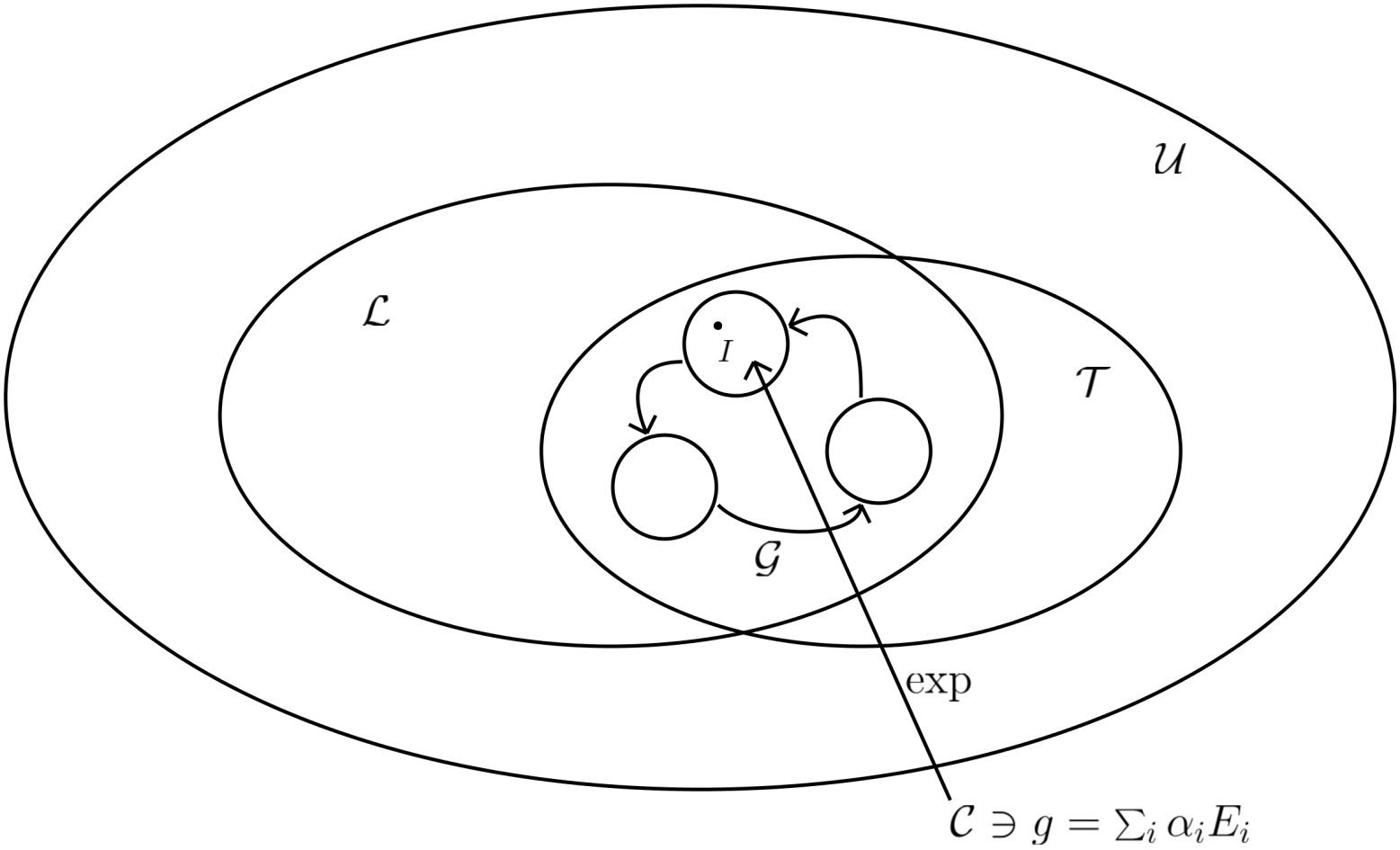
Theorem (Eastin-Knill '09)

For any fixed quantum code that can detect at least 1 *error, the set of logically distinct transversal gates must form a finite group, and so cannot be universal for quantum computing.*

No quantum universal transversal gate set

Proof.

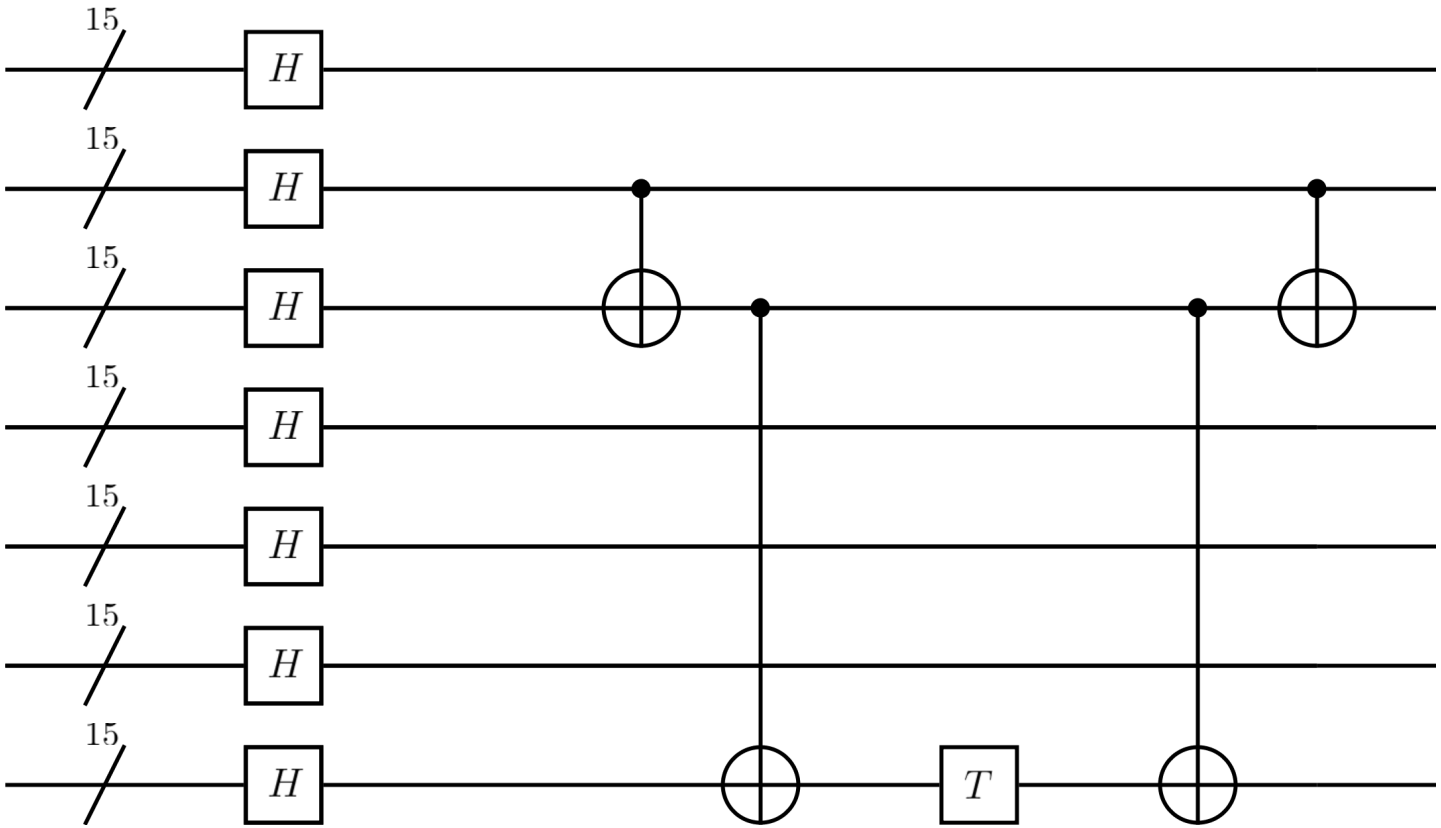
A code C is *error-detecting* if and only if $PEP \propto P$ for all $|E| = 1$.



Importance of a *fixed* partition

$$[[105, 1, 9]] = [[7, 1, 3]]_{outer} \circ [[15, 1, 3]]_{inner}$$

$$\mathcal{G}_{outer} = \langle H, P, CX \rangle \qquad \mathcal{G}_{inner} = \langle T, CX \rangle$$



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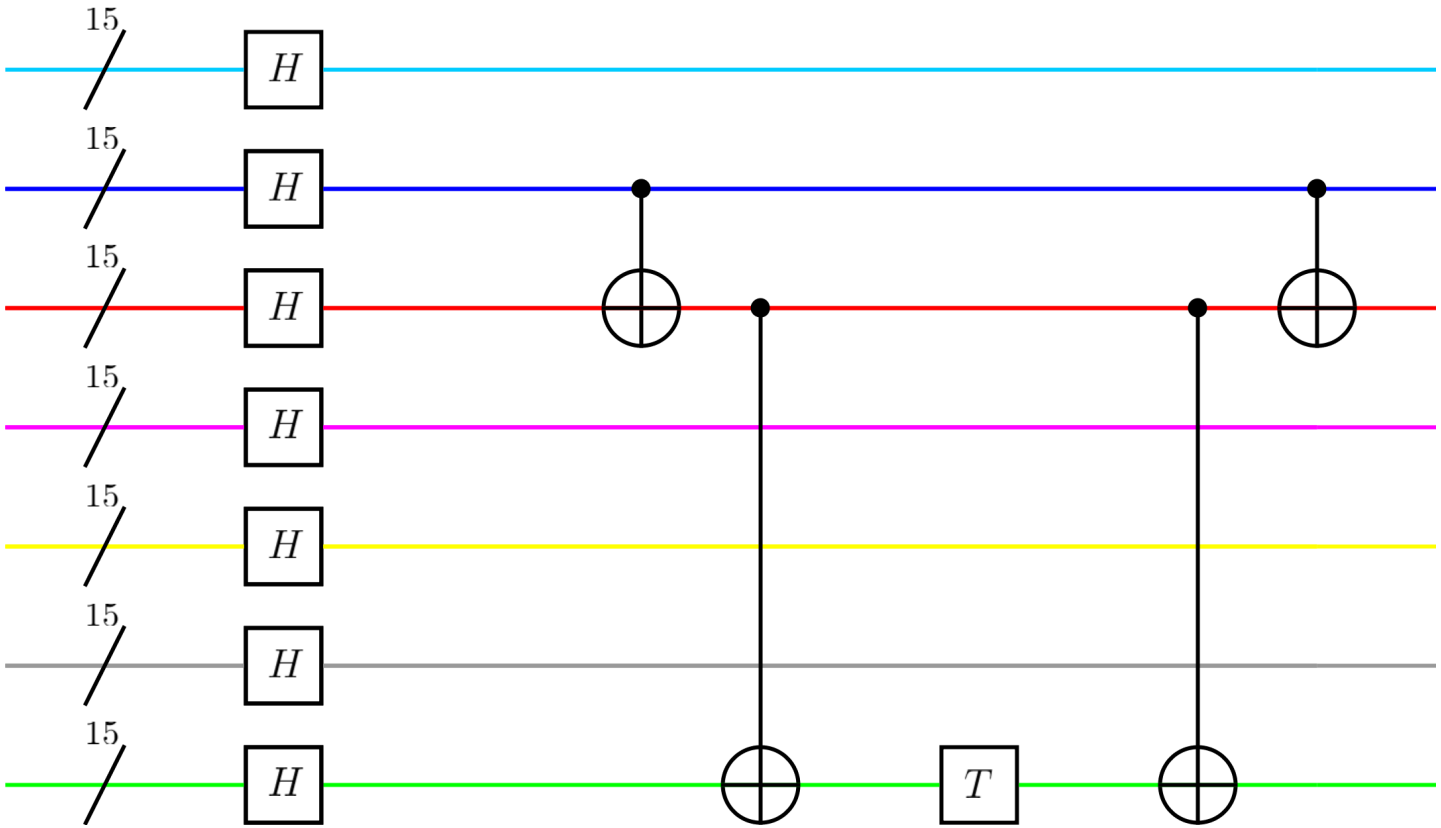

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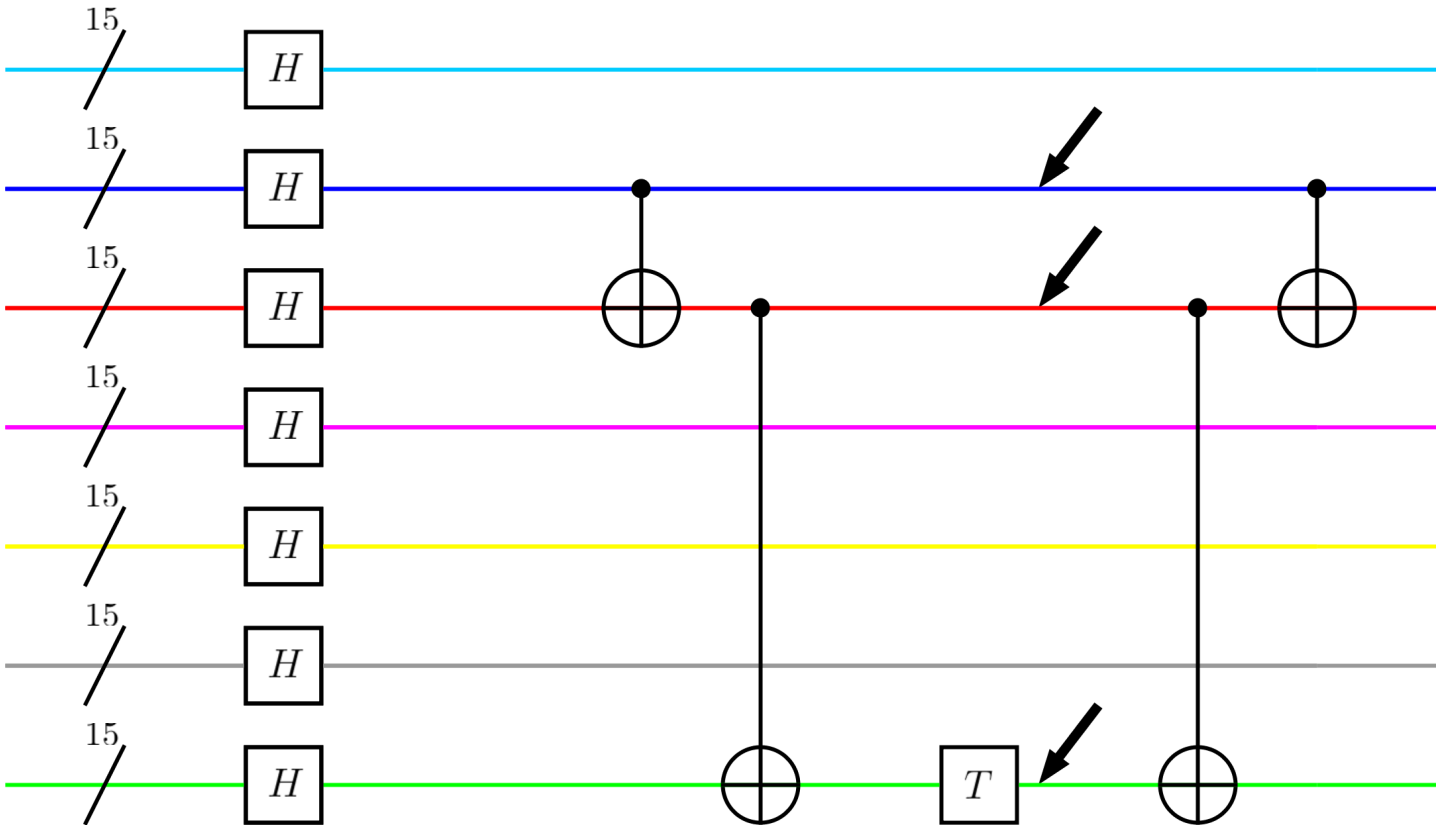

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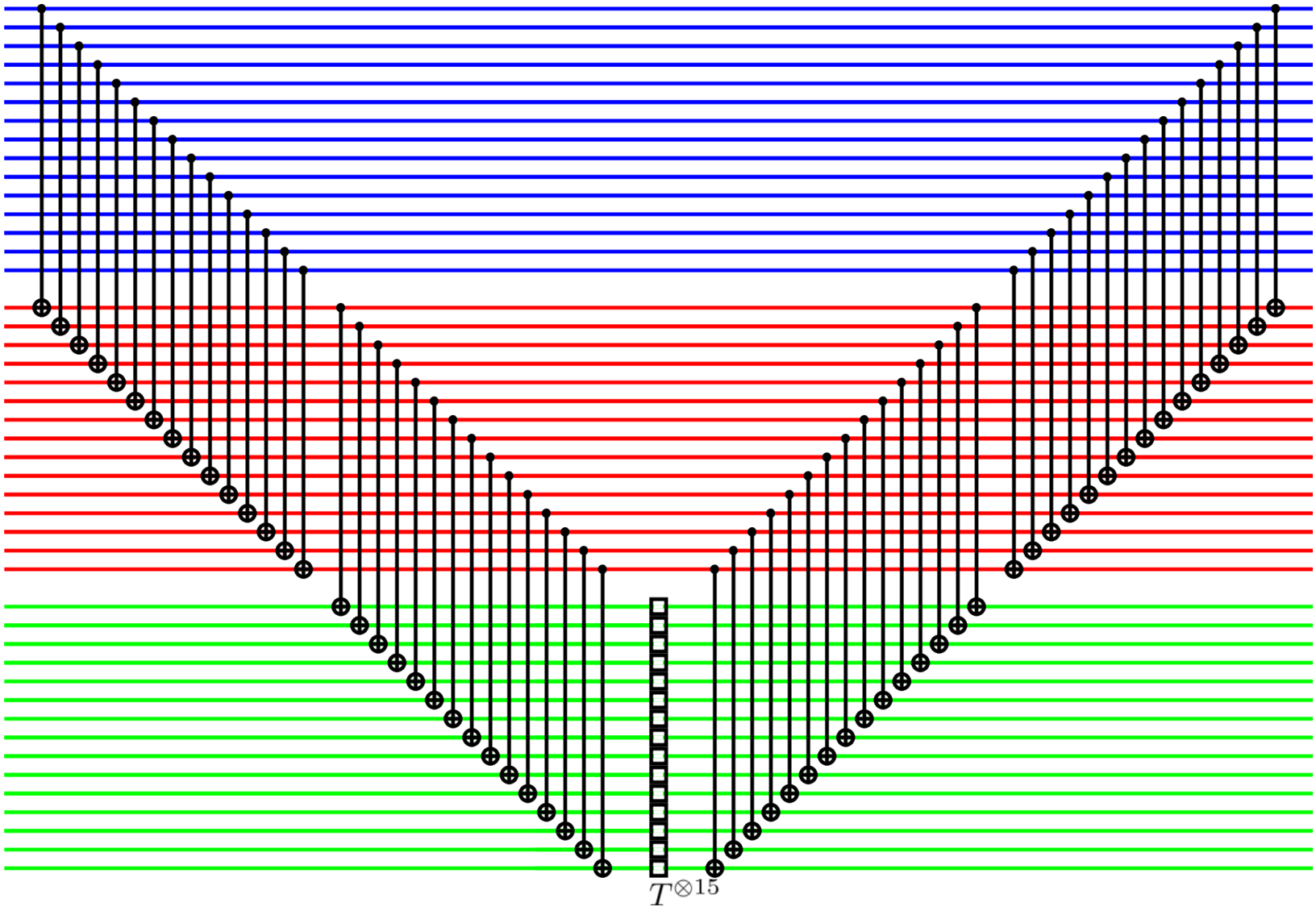

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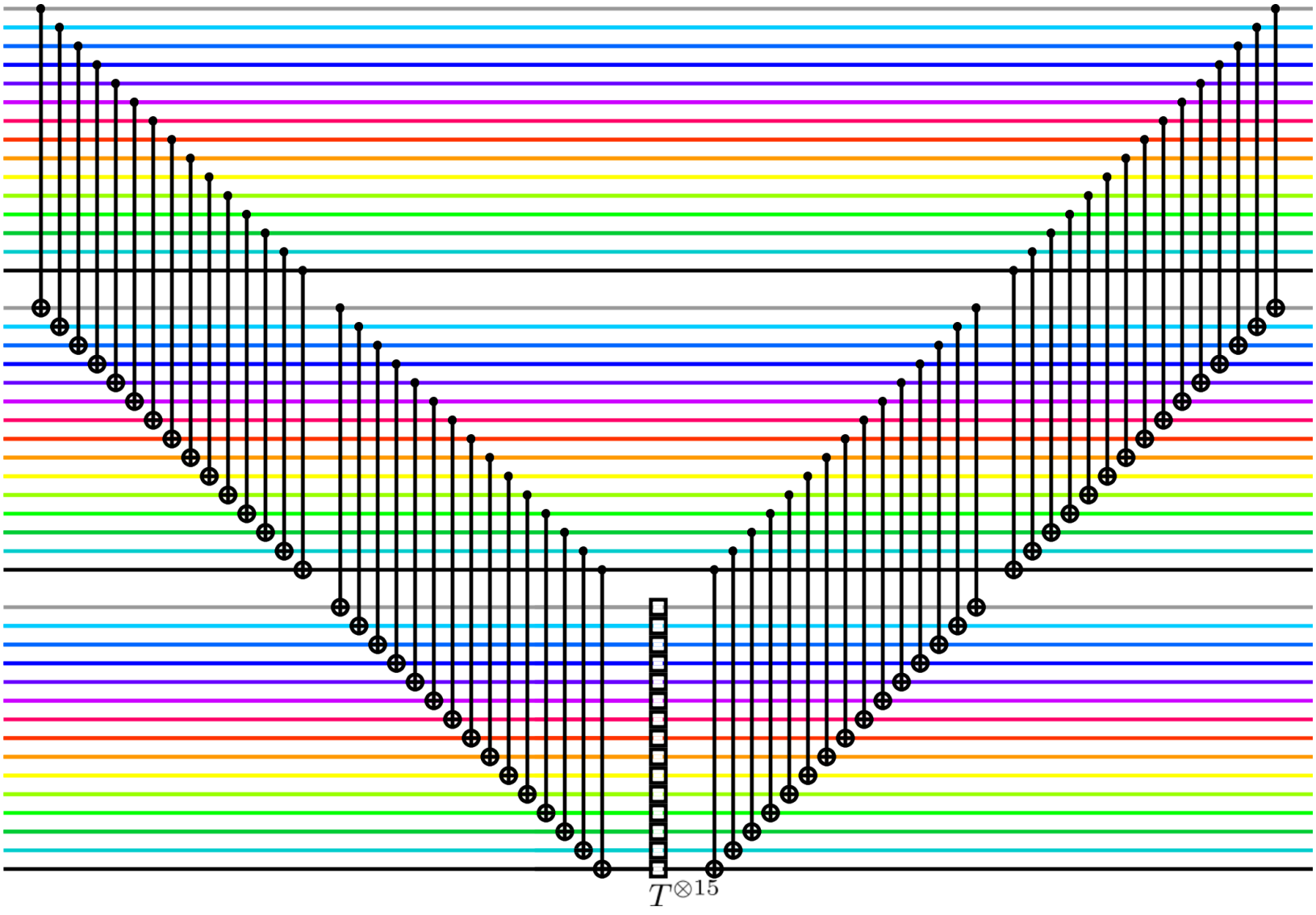

Universal gate sets

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Theorem (Shi '03)

Toffoli and Hadamard together form a universal gate set for quantum computing.

Definition

Toffoli: $|a\rangle|b\rangle|c\rangle \mapsto |a\rangle|b\rangle|c \oplus a \cdot b\rangle$, and is *universal* for classical reversible computing.

A question

Toffoli gates form the dominating portion of many useful quantum algorithms (e.g. Shor's algorithm).

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Question: Can we implement the Toffoli gate transversally on any quantum error-correcting code?

A link to cryptography

(Transversal) logical gates $\leftrightarrow$ (locally) compute on encoded data

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(Transversal) logical gates $\leftrightarrow$ (locally) compute on encoded data

Homomorphic circuits $\leftrightarrow$ compute on encrypted data

(Quantum) Homomorphic Encryption

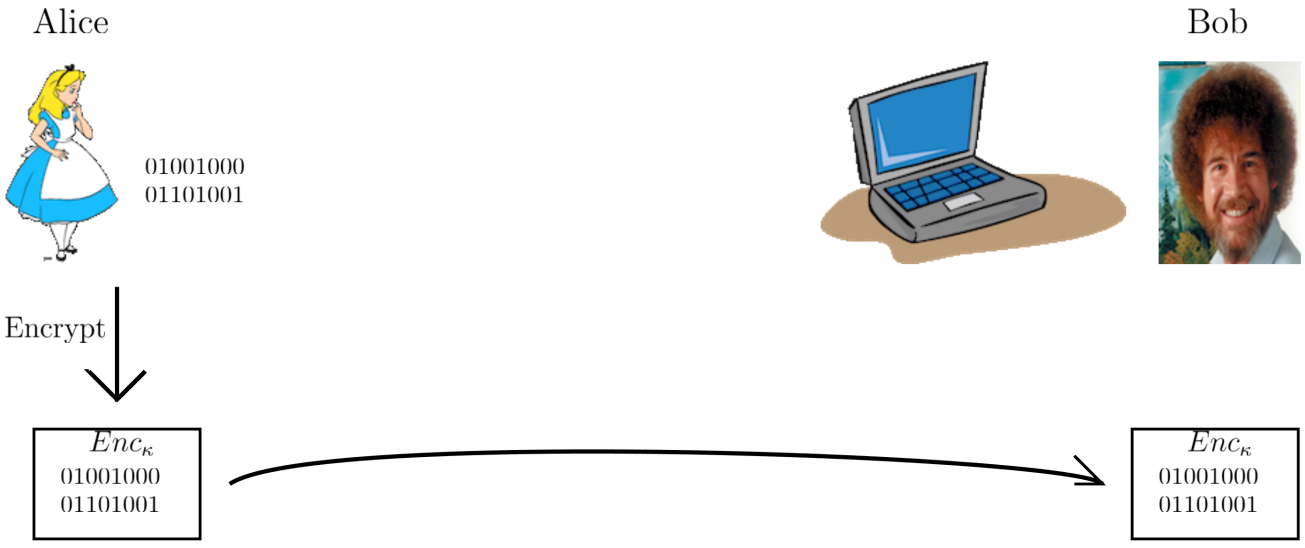
Alice

01001000
01101001

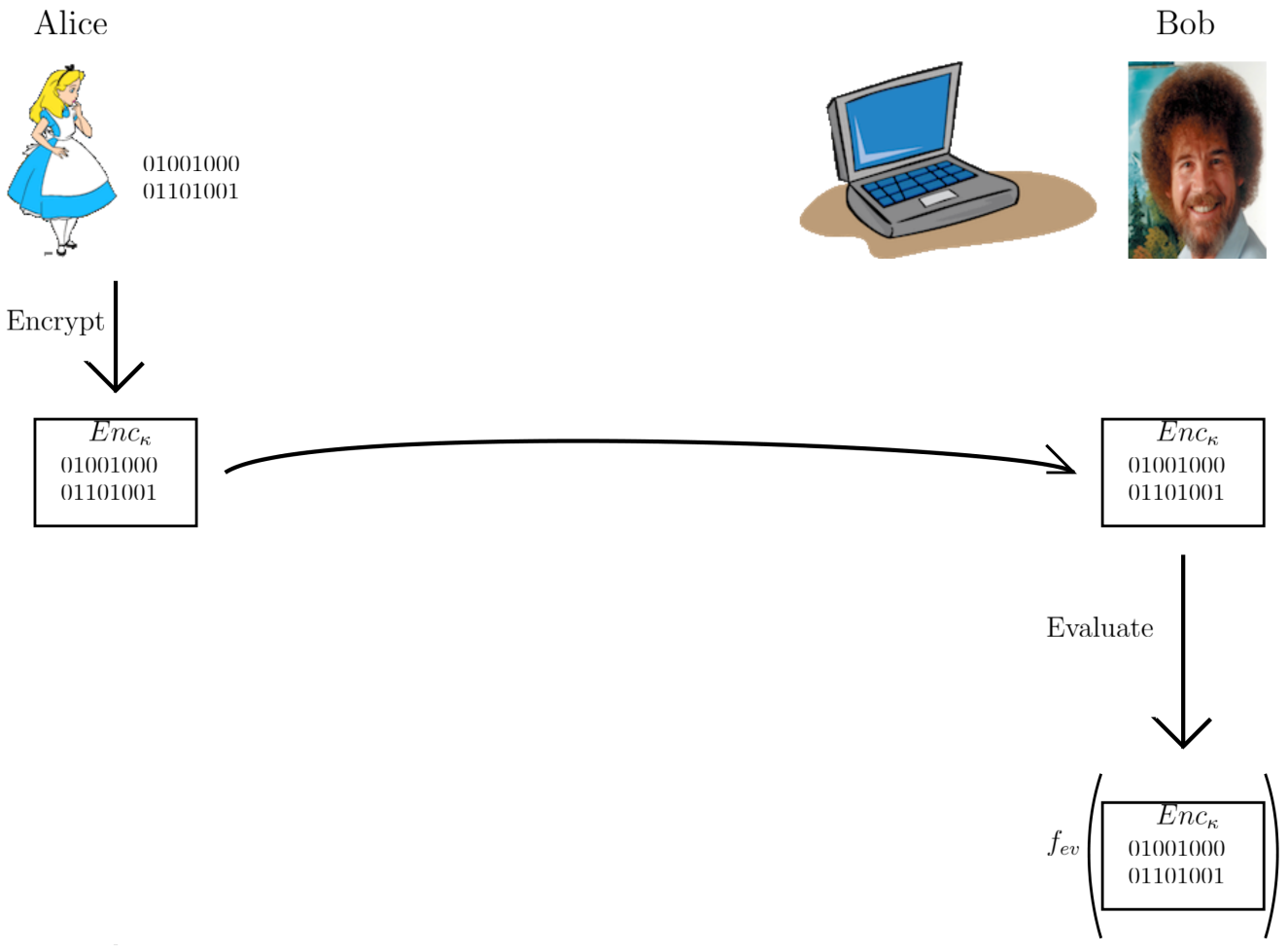
Bob

(Quantum) Homomorphic Encryption

(Quantum) Homomorphic Encryption



(Quantum) Homomorphic Encryption



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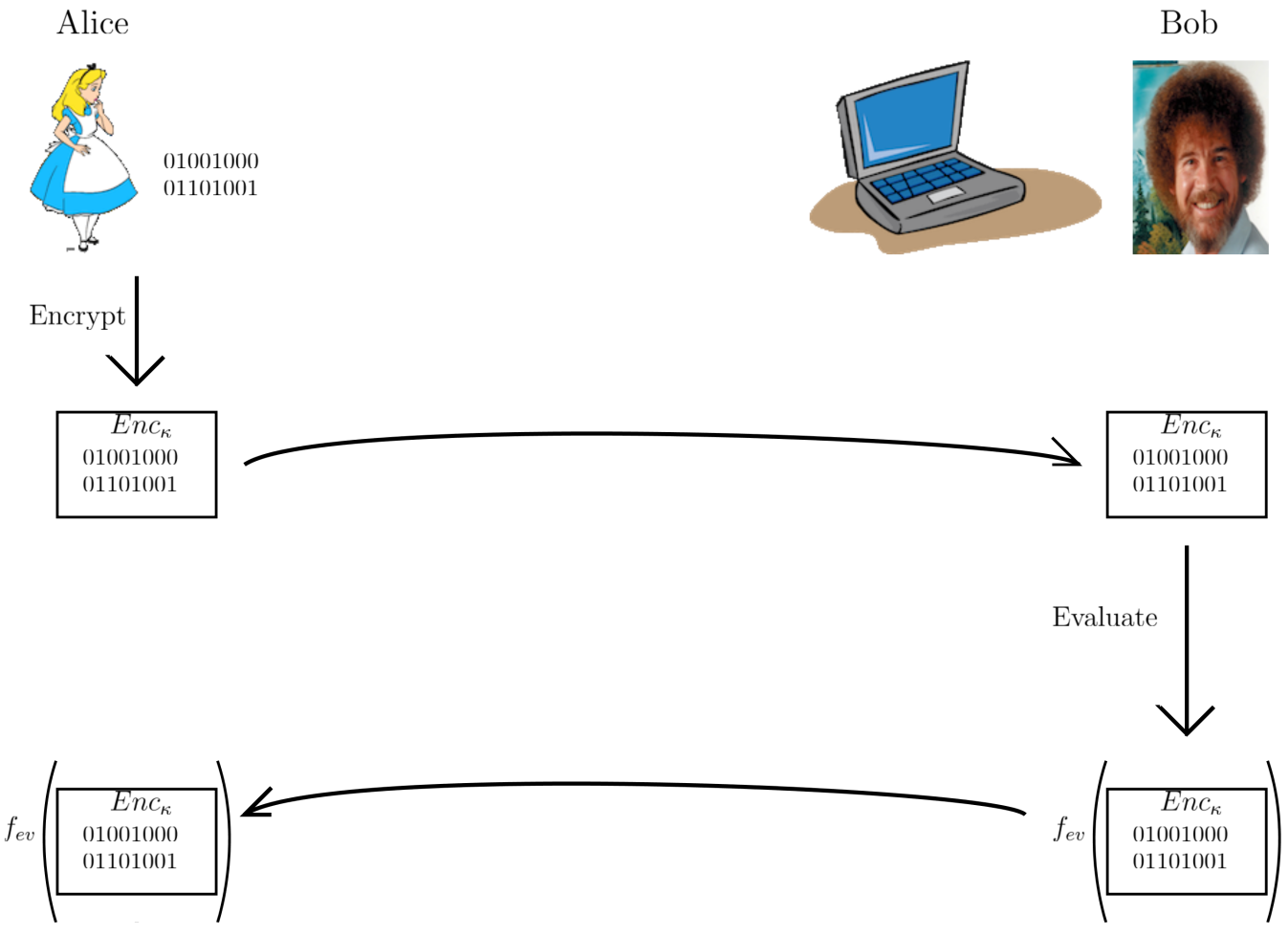

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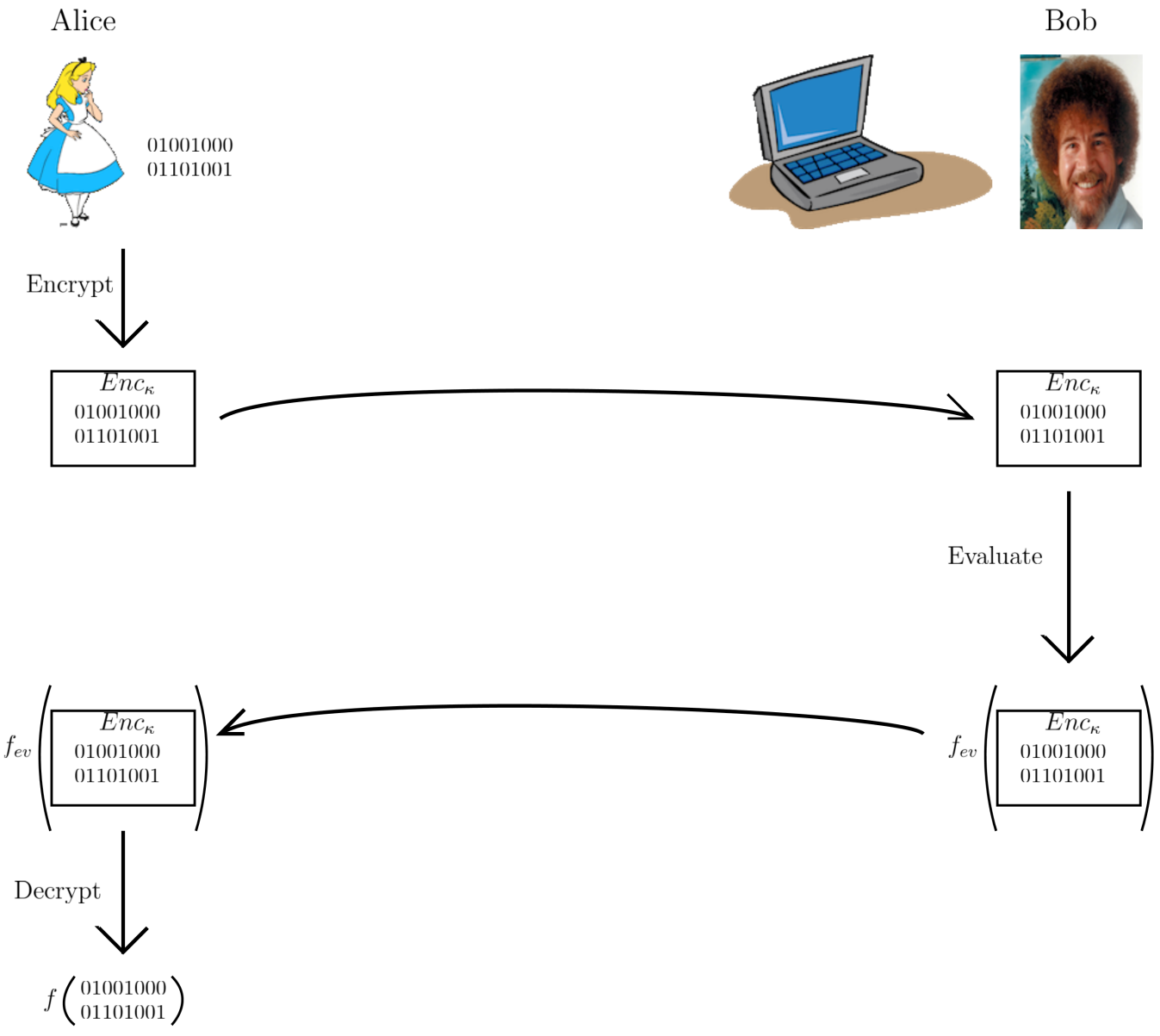

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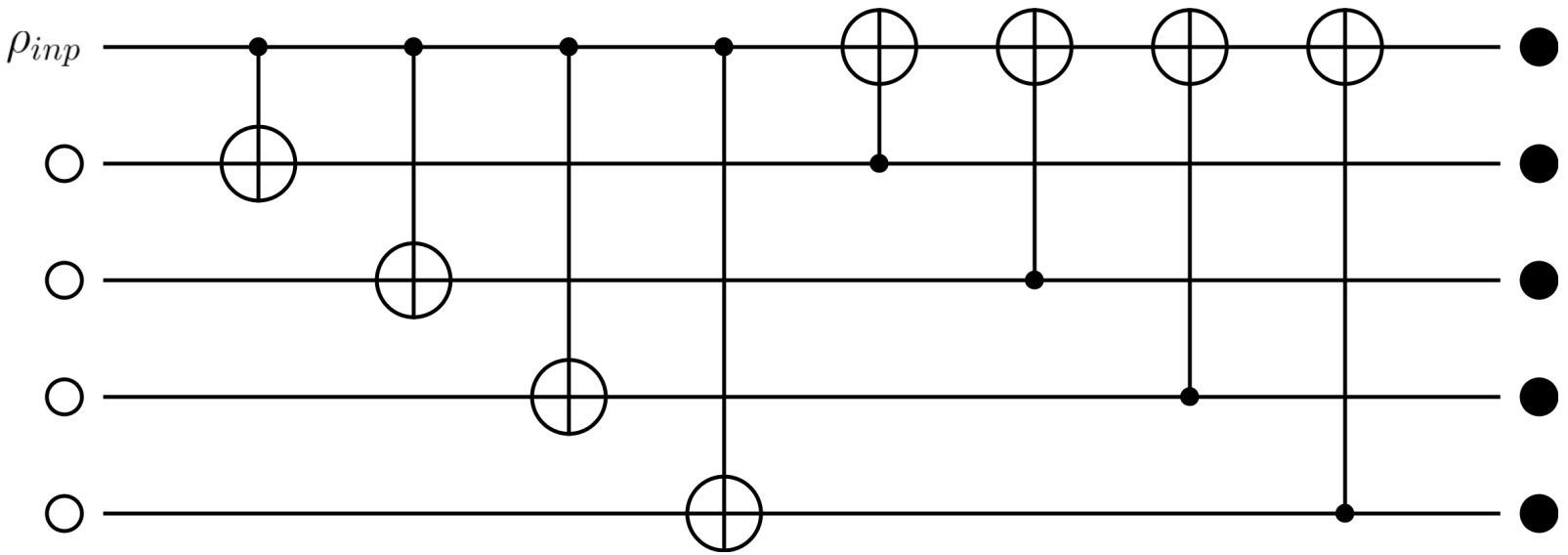
We call such a scheme *compact* and *non-leveled* if the Alice's work is completely independent of the circuit being evaluated.

Information-theoretically secure QHE (OTF'15)



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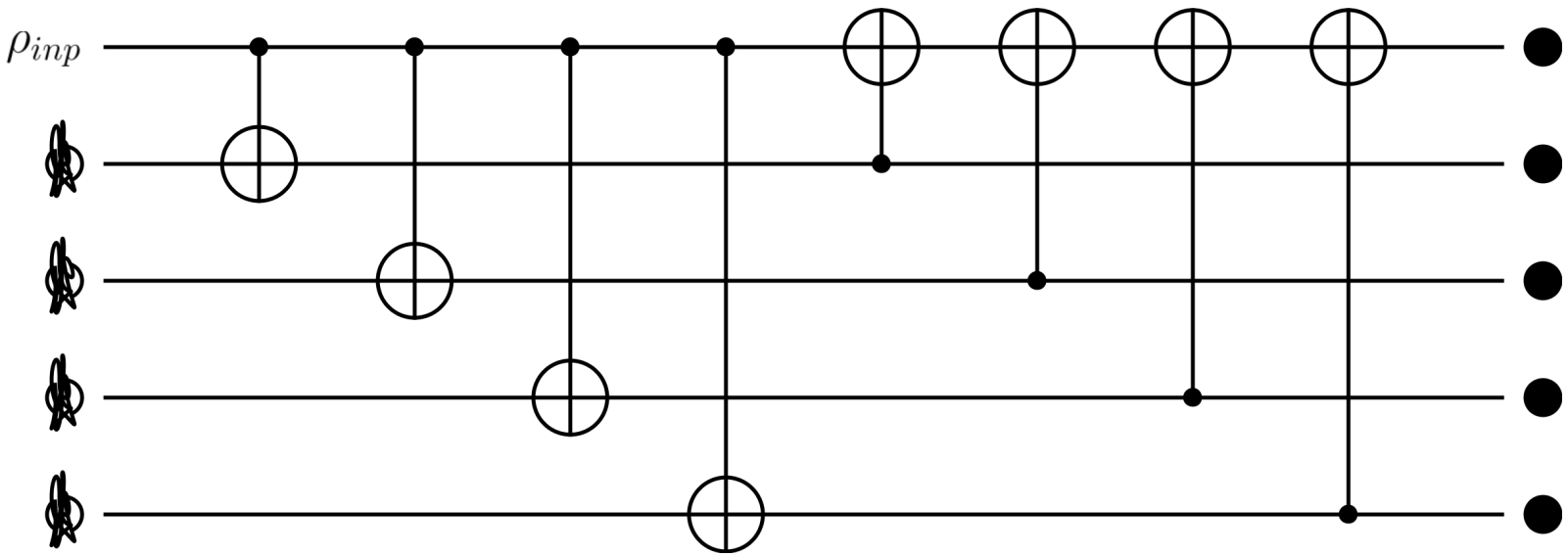

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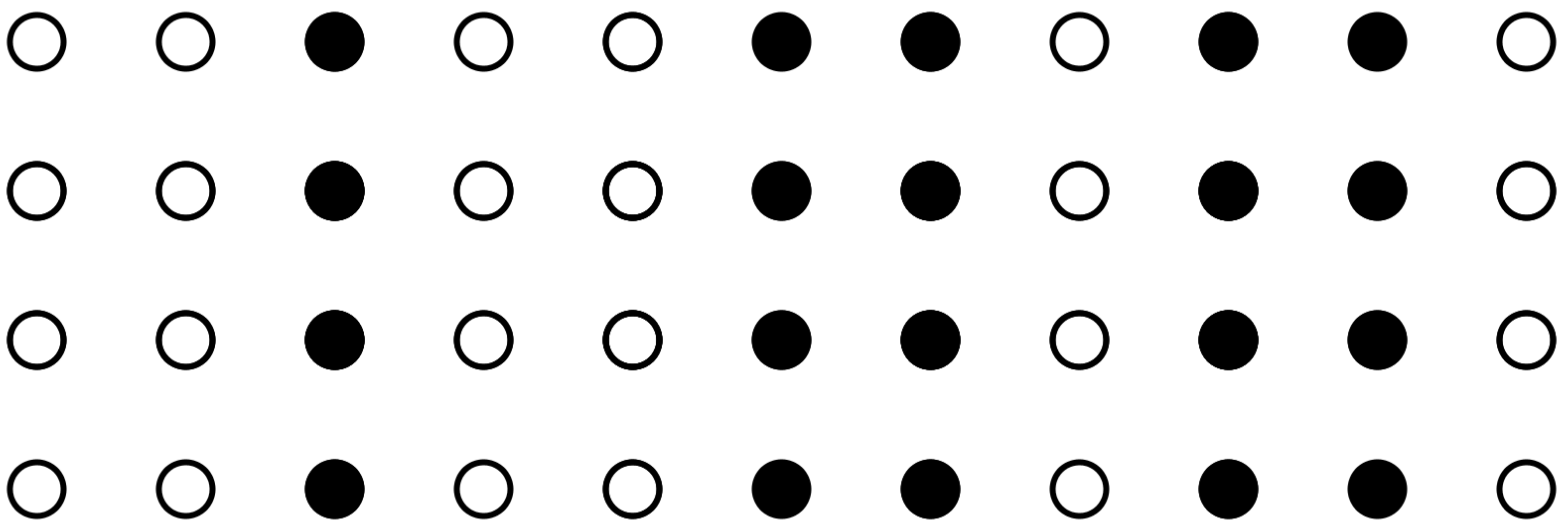

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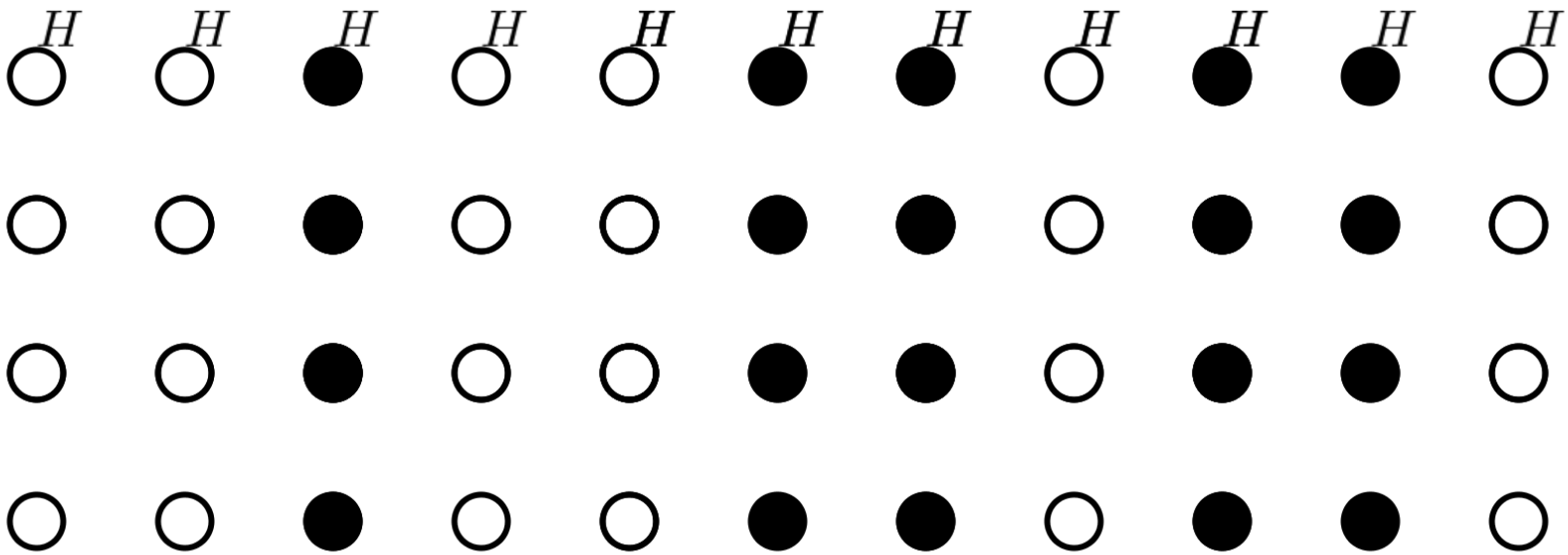

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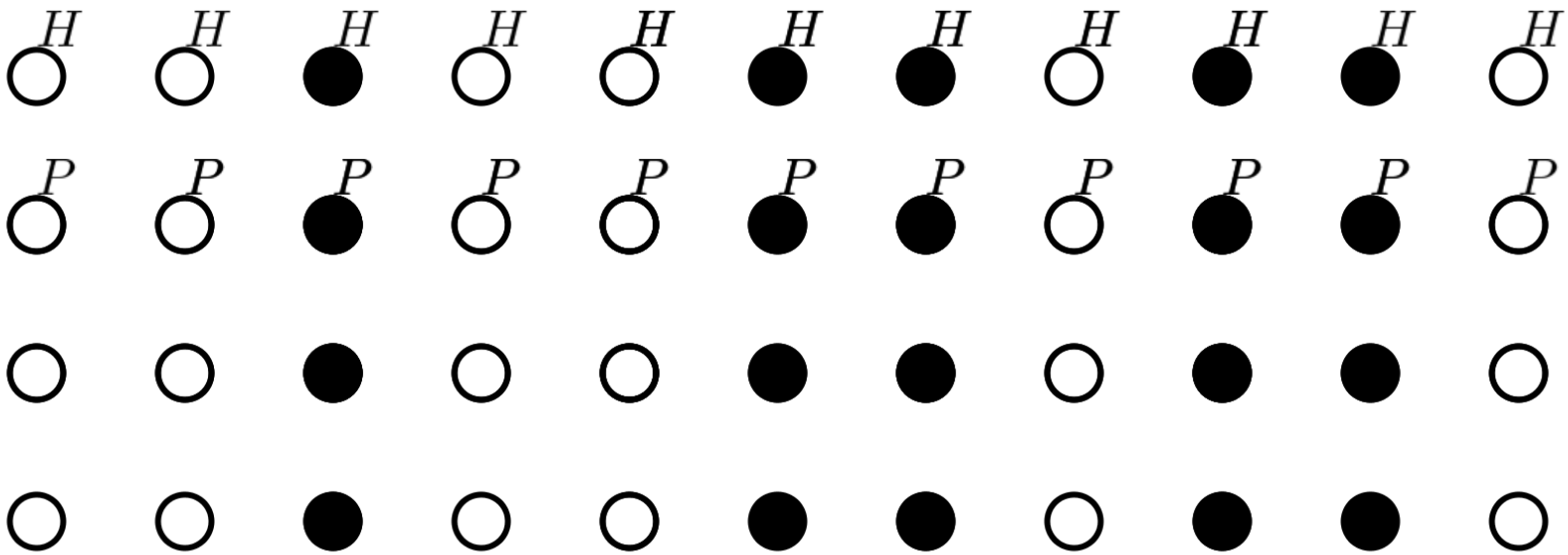

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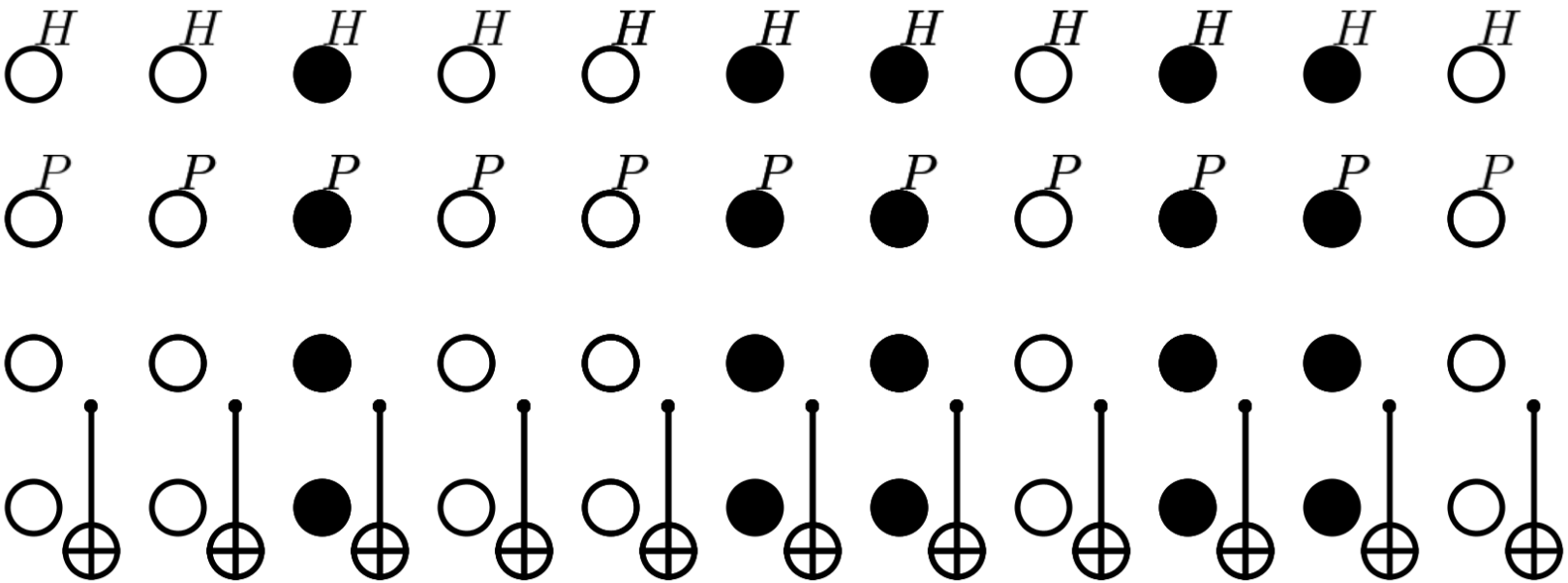


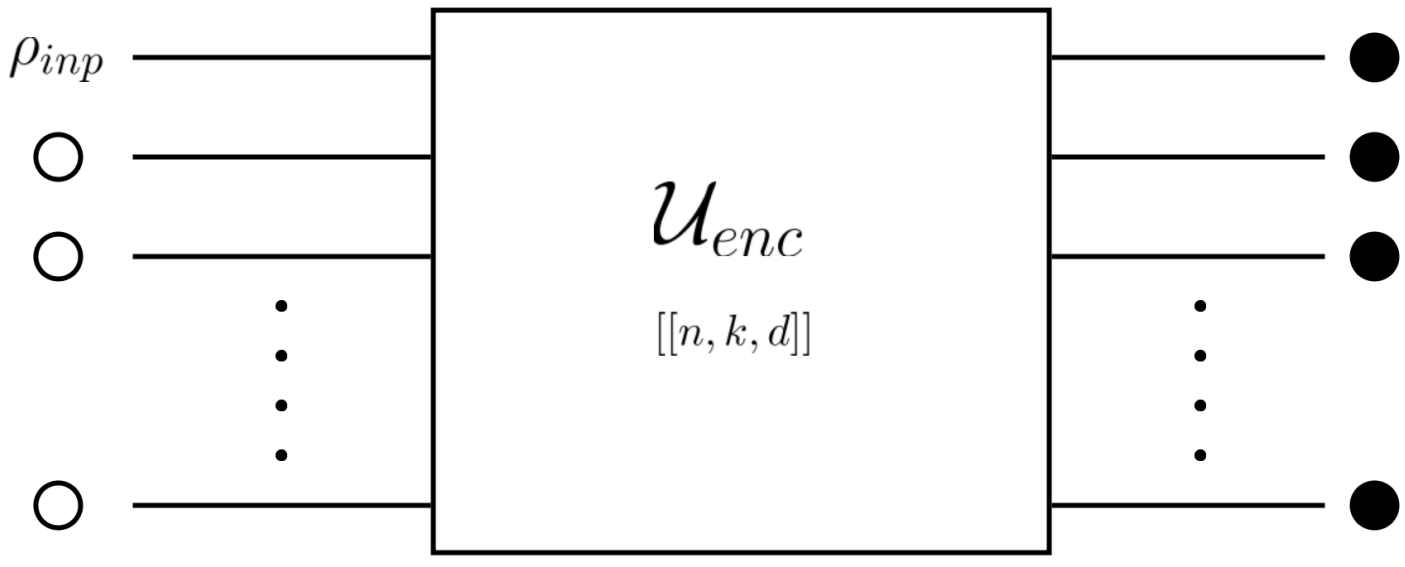
Information-theoretically secure QHE

Let m be the number of noisy qubits that we embed into, and let n be the size of the code. Then for any two p-qubit input ρ, σ with encryptions $\widetilde{\rho}, \widetilde{\sigma}$,

$$\|\widetilde{\rho} - \widetilde{\sigma}\|_1 \leq 2(4^p - 1)\binom{n+m}{n}^{-\frac{1}{2}}.$$

Information-theoretic security for encoding sizes scaling polynomially in the input length.

Information-theoretically secure QHE



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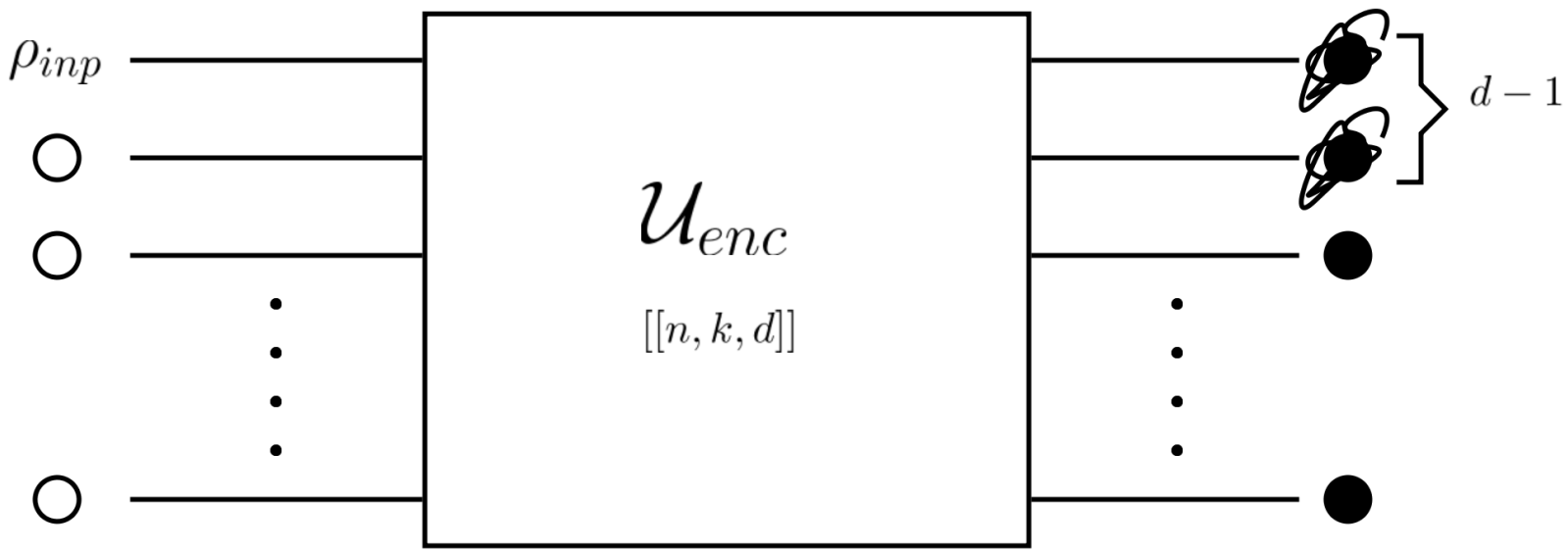

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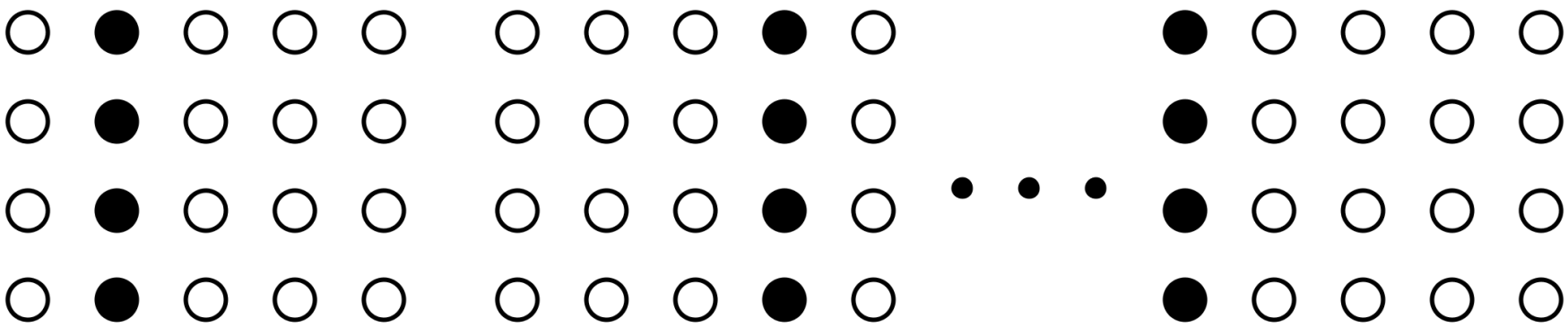

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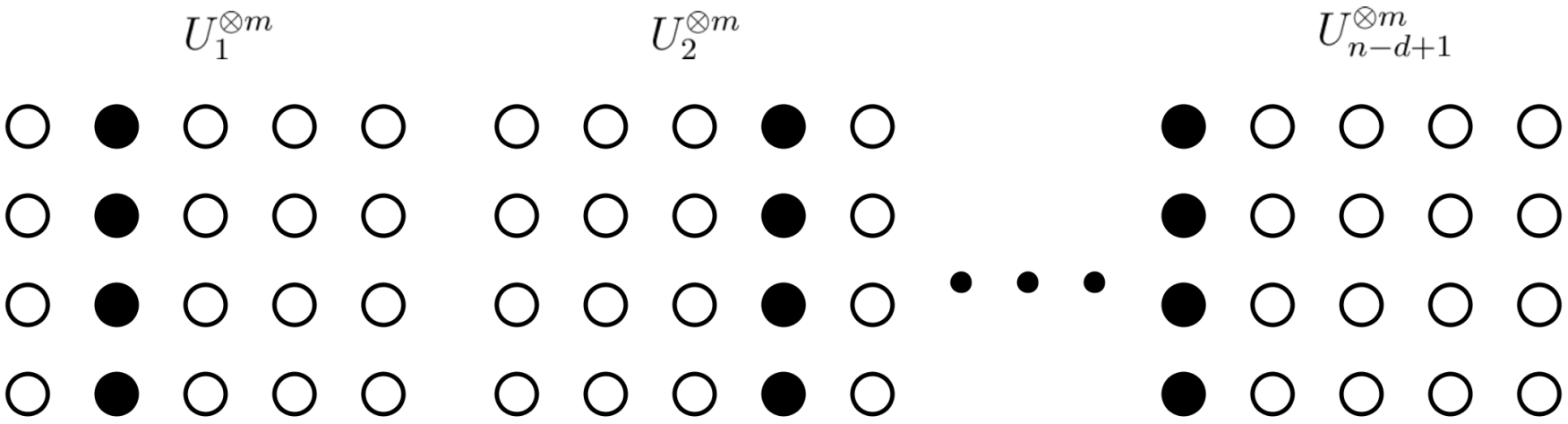

Information-theoretically secure QHE

The transversal gates for the code are the homomorphisms for the encryption scheme, as

$$\mathcal{R}\left(\mathcal{E}_{d-1}\left[(U_1 \otimes U_2 \otimes \ldots \otimes U_n) \cdot |\psi\rangle_L\right]\right) = U_L|\psi\rangle_L.$$

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Let m be the size of the noise we embed into. Under mild assumptions on the code, for any p-qubit inputs ρ, σ with encryptions $\widetilde{\rho}, \widetilde{\sigma}$, there exists a $c > 0$ so that

$$\|\widetilde{\rho} - \widetilde{\sigma}\|_1 \leq \left(\left(\frac{m-1}{m} \right)^{n-d+1} - 1 + 2^{-pc} \left(\frac{2 \cdot 2^p}{m} \right)^{n-d+1} \right)^{\frac{1}{2}}.$$

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In order to be secure, require $m = 2^{pc'}$ for some $c' < 1$ for total encoding size $(n-d+1)p2^{pc'}$.

Information-theoretically secure QHE

Suppose there existed a quantum error-detecting code implementing the Toffoli gate transversally. Then we would obtain a fully-homomorphic encryption scheme with encoding size $\theta(p2^{pc'})$ for some $c' < 1$.

Theorem (Yu, Perez-Delgado, Fitzsimons '14)

Suppose a QHE scheme with perfect information-theoretic security implements a set of homomorphisms S. Then the size of the encoding must grow as $\log_2(|S|)$.

Limitations on ϵ-ITS-QHE

Definition

An (n, m, p)-quantum random access code is an encoding of n bits b into m qubits ρ_b along with a set of n POVM's $\{P_i^0, P_i^1\}_{i=1}^n$ satisfying, for all $b \in \{0,1\}^n$ and $i \in [n]$,

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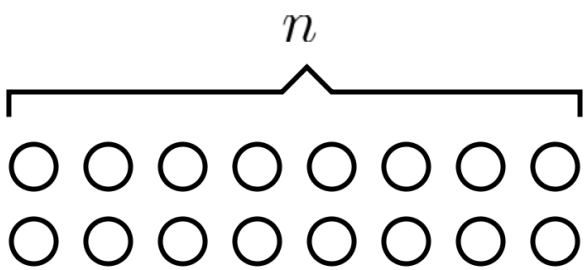

Limitations on ϵ-ITS-QHE

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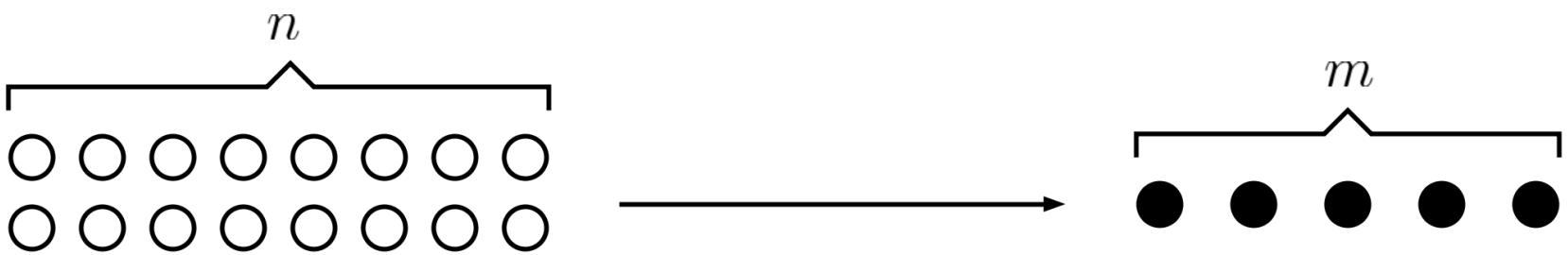

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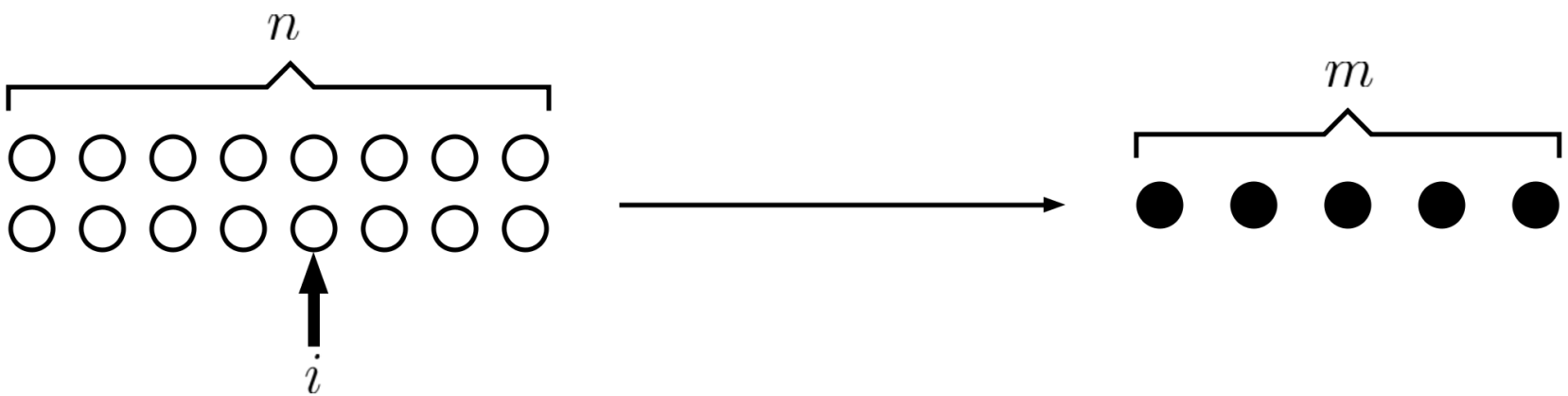

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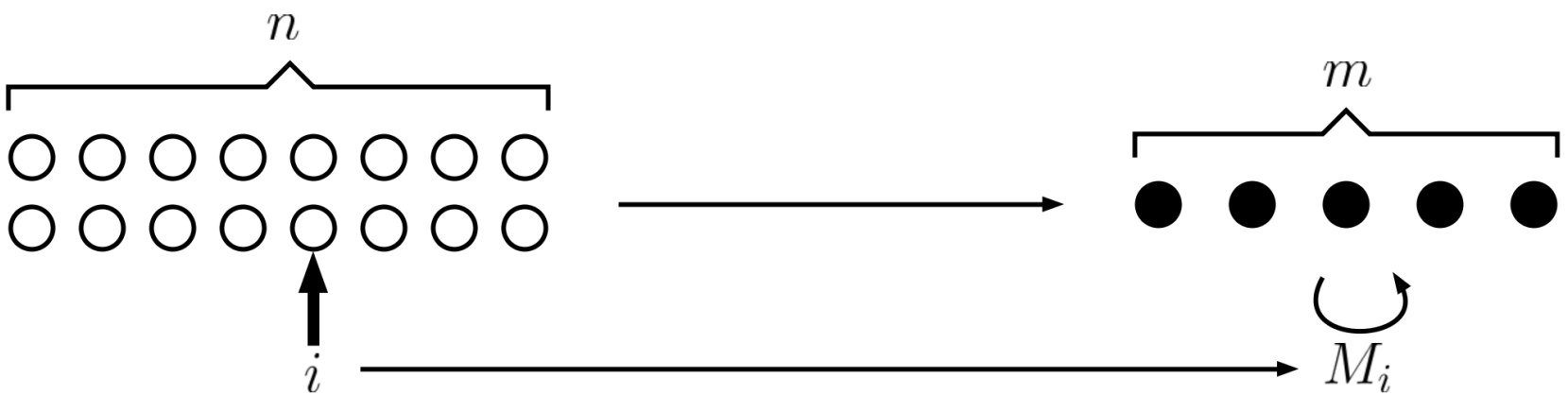

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Theorem (Nayak '99)

For $H(\cdot)$ the binary entropy function, any (n, m, p)-QRAC must satisfy

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Ex. Could have been a $(16, 5, 0.8)$-QRAC but **not** a $(16, 5, 0.9)$-QRAC.

Limitations on ϵ-ITS-QHE

Theorem (N.,Shi)

Consider a QHE scheme that is fully-homomorphic. Let $f(p)$ denote the size of the evaluated ciphertext as a function of the input size p. If, for some $\delta > 0$ and any p-qubit input ciphertexts ρ, ρ',

$$\lim_{p \to \infty} \|\rho - \rho'\|_1 < 1 - \delta,$$

then $f(p) = \Omega(2^p)$.

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By Nayak's bound, $s = \Omega(2^p)$. □

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Theorem (N.,Shi)

Almost no quantum error-detecting code can implement Toffoli (or any classical-universal) gate set transversally.

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Ex. Shor's code: $|0\rangle_L = (|000\rangle + |111\rangle)^{\otimes 3}$; $|1\rangle_L = (|000\rangle - |111\rangle)^{\otimes 3}$.

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No quantum error-detecting code can implement the Toffoli gate strongly transversally.

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- Quantum fault-tolerance schemes aimed at classical computing?

Hey thanks

Thank you!

arXiv:1704.07798