What is an easy IP problem?

(IP) \( \max \{ c^T x : x \in X \} \)

Some possible interpretations (there are others!):

- **Explicit Convex Hull Property** A compact description of the convex hull \( \text{conv}(X) \) is known, which in principle allows up to replace every instance by the LP \( \max \{ c^T x : x \in \text{conv}(X) \} \).

- **Strong Dual Property**: For the given problem class, there exists a strong dual problem

\[
(D) \min \{ w(u) : u \in U \},
\]

allowing us to obtain optimality conditions that can be quickly verified, that is \( x^* \in X \) is optimal if and only if \( \exists u^* \in U : c^T x^* = w(u^*) \).

Note: if the Explicit Convex Hull Property holds, the dual of the LP satisfies the strong dual property!

When LP relaxation is all you need

(IP) \( \max \{ c^T x : Ax \leq b, \ x \in \mathbb{Z}_+^n \} \)

LP relaxation:

(LP) \( \max \{ c^T x : Ax \leq b, \ x \in \mathbb{R}_+^n \} \)

Consider a problem with data \((A, b)\) integral. When does (LP) have an integral optimal solution?
When LP relaxation is all you need

(IP) $\max \{ c^T x : Ax \leq b, \ x \in \mathbb{Z}_+^n \}$

LP relaxation:

(LP) $\max \{ c^T x : Ax \leq b, \ x \in \mathbb{R}_+^n \}$

Consider a problem with data $(A, b)$ integral. When does (LP) have an integral optimal solution?

Recall: a BFS of LP is a feasible solution of the form $x = (x_B, x_N) = (B^{-1}b, 0)$, where $B$ is an $m \times m$ nonsingular submatrix of $[A, I]$, and $I$ is the $m \times m$ identity matrix.

**Observation 3.1: Sufficient condition**

If the optimal basis $B$ has det$(B) = \pm 1$, then the LP relaxation solves (IP).

**Totally Unimodular matrices**

**Totally unimodular matrix**

A matrix $A$ is totally unimodular (TU) if every square submatrix of $A$ has determinant $+1$, $-1$, or $0$. (In particular, $\forall i, j \ a_{ij} \in \{+1, -1, 0\}$.)

**Proposition 3.3’ (equality constraints)**

The linear program $\max \{ c^T x : Ax = b, \ x \in \mathbb{R}_+^n \}$ has an integral optimal solution for all integer vectors $b$ for which it has a finite optimal value if $A$ is totally unimodular.
Some TU (and non-TU) matrices

The following matrices are TU

\[
\begin{pmatrix}
1 & -1 & -1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The following matrices are not TU:

\[
\begin{pmatrix}
1 & -1 \\
1 & 1 \\
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
\end{pmatrix}
\]

Recognizing TU matrices

**Proposition 3.1: necessary and sufficient conditions:**

Let \( A \in \mathbb{R}^{m \times n} \). The following conditions are equivalent:

(i) \( A \) is TU

(ii) The transpose matrix \( A^T \) is TU

(iii) The matrix \([A, I]\) is TU

Other things you can do to rows/columns of \( A \) preserving the TU property: interchanging, deleting, multiplying by \(-1\)

**Proposition 3.2: sufficient conditions:**

A matrix \( A \) is TU if

(i) \( a_{ij} \in \{+1, -1, 0\} \)

(ii) Each column contains at most two nonzero coefficients

(iii) The exists a partition \((M_1, M_2)\) of the set \( M \) of rows such that each column \( j \) containing two nonzero coefficients satisfies

\[
\sum_{i \in M_1} a_{ij} = \sum_{i \in M_2} a_{ij}.
\]
Proposition 3.3’ (equality constraints)
The linear program $\max\{c^T x : Ax = b, x \in \mathbb{R}_+^n\}$ has an integral optimal solution for all integer vectors $b$ for which it has a finite optimal value if $A$ is totally unimodular.

Proposition 3.3 (equality constraints)
The linear program $\max\{c^T x : Ax \leq b, x \in \mathbb{R}_+^n\}$ has an integral optimal solution for all integer vectors $b$ for which it has a finite optimal value if and only if $A$ is totally unimodular.

Assignment problem:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$$

s.t. $$\sum_{j=1}^{n} x_{ij} = 1, \ i = 1, \ldots, n$$
$$\sum_{i=1}^{n} x_{ij} = 1, \ j = 1, \ldots, n$$
$$x \geq 0, \ \text{integer}$$

- Observation: The constraint matrix arising in an assignment problem is TU
  - To see this, apply Proposition 3.2 with $M_1 = \{1, \ldots, n\}$ and $M_2 = \{n + 1, \ldots, 2n\}$
  - Thus, assignment problem can be solved by solving its LP relaxation with, e.g., Simplex method
Minimum cost network flows

Problem: given a directed graph $D = (V, A)$, arc capacities $h_{ij} \forall (i, j) \in A$, supplies $b_i \forall i \in V$, unit flow costs $c_{ij} \forall (i, j) \in A$, find a feasible flow that satisfies all the demands at minimum cost.

Let $V^+(i) = \{ k : (i, k) \in A \}$, $V^-(i) = \{ k : (k, i) \in A \}$

\[
\min \sum_{(i,j) \in A} c_{ij} x_{ij} \\
\sum_{k \in V^+(i)} x_{ik} - \sum_{k \in V^-(i)} x_{ki} = b_i \forall i \in V \\
0 \leq x_{ij} \leq h_{ij} \forall (i, j) \in A
\]

Proposition 3.4

The constraint matrix arising in a minimum cost network flow problem is TU.

Corollary

If the demands and capacities are integral, (i) Each extreme point is integral; (ii) The constraints describe the convex hull of the integral feasible flows.

Maximum $s-t$ flow; minimum $s-t$ cut

We are given a directed graph $D = (V, A)$, two distinguished nodes $s, t \in V$ and capacities $h_{ij} \geq 0, \forall (i, j) \in A$.

Maximum $s-t$ flow problem: find a maximum flow from $s$ to $t$.

Adding uncapacitated arc $(t, s)$, the problem can be formulated as an NFP:

\[
z_{\text{MFP}} = \max x_{st} \\
\sum_{k \in V^+(i)} x_{ik} - \sum_{k \in V^-(i)} x_{ki} = 0 \forall i \in V \\
0 \leq x_{ij} \leq h_{ij} \forall (i, j) \in A
\]

An $s-t$ cut is a partitioning of $V$ into sets $X$ and $\bar{X} = V \setminus X$ such that $s \in X$ and $t \in \bar{X}$. The capacity of an $s-t$ cut is defined as the total capacity of the arcs crossing the cut in the “forward” direction:

\[
\sum_{(i,j) \in A : i \in X, j \notin X} h_{ij}
\]

Minimum $s-t$ cut problem: find a minimum-capacity $s-t$ cut.
Strong duality between max-flow and min-cut problems

**Proof:** Weak duality – easy. 
The dual of the max flow LP formulation:
\[ z_{\text{MFP}} = \min \sum_{(i,j) \in A} h_{ij} w_{ij} \]
\[ u_i - u_j + w_{ij} \geq 0 \quad \forall (i,j) \in A \]
\[ u_s - u_t \geq 1, \quad w_{ij} \geq 0 \quad \forall (i,j) \in A. \]

From total unimodularity, there is an integer opt. solution \((\bar{u}, \bar{v})\). Without loss of generality, \(\bar{s} = 0\).

Let \(X = \{j \in V : \bar{u}_j \leq 0\} \ni s\), \(X = V \setminus X = \{j \in V : \bar{u}_j \geq 1\} \ni t\). Consider another feasible solution to the LP dual:
\[ \hat{u}_j = 0, \quad j \in X, \quad \hat{u}_j = 1, \quad j \in \bar{X}, \quad \hat{w}_{ij} = 1, \quad i \in X, j \in \bar{X}, \quad \hat{w}_{ij} = 0 \text{ otherwise} \]

Then (2nd inequality: \(\hat{w}_{ij} \geq \bar{u}_i - \bar{u}_j \geq 1\) for \(i \in X\) and \(j \in \bar{X}\))
\[ z_{\text{MFP}} \leq \sum_{(i,j) \in A} h_{ij} \hat{w}_{ij} = \sum_{(i,j) \in A, i \in X, j \in \bar{X}} h_{ij} \leq \sum_{(i,j) \in A, i \in X, j \in \bar{X}} h_{ij} \hat{w}_{ij} \leq \sum_{(i,j) \in A} h_{ij} \hat{w}_{ij} = z_{\text{MFP}} \]

Finally, note that \(\sum_{(i,j) \in A, i \in X, j \in \bar{X}} h_{ij}\) is the capacity of an \(s - t\) cut!

Shortest path problem (if time remains)

Problem: given a directed graph \(D = (V, A)\), two distinguished nodes \(s, t \in V\) and arc costs \(c_{ij} \geq 0, \forall (i,j) \in A\), find a minimum cost path from \(s\) to \(t\).

IP formulation:
\[ z = \min \sum_{(i,j) \in A} c_{ij} x_{ij} \]
\[ \sum_{k \in V^+(i)} x_{ik} - \sum_{k \in V^-(i)} x_{ki} = 1 \quad \text{for } i = s \]
\[ \sum_{k \in V^+(i)} x_{ik} - \sum_{k \in V^-(i)} x_{ki} = 0 \quad \forall i \in V \setminus \{s, t\} \]
\[ \sum_{k \in V^+(i)} x_{ik} - \sum_{k \in V^-(i)} x_{ki} = -1 \quad \text{for } i = t \]
\[ x_{ij} \geq 0 \quad \forall (i,j) \in A \]
\[ x \in \mathbb{Z}^{|A|} \]

Here, \(x_{ij} = 1\) if arc \((i,j)\) is on the shortest path, 0 otherwise.

**Theorem 3.5**

\(z\) is the length of a shortest \(s - t\) path if and only if there exist values \(\pi, \ i \in V\) such that \(\pi_s = 0, \ \pi_t = z\), and \(\pi_j - \pi_i \leq c_{ij}\) for \((i,j) \in A\).