What is an easy IP problem?

$$\text{(IP)} \;\; \max\{c^T x : x \in X\}$$

Some possible interpretations (there are others!):

- Explicit Convex Hull Property A compact description of the convex hull conv(X) is known, which in principle allows up to replace every instance by the LP $\max\{c^T x : x \in \text{conv}(X)\}$.
- Strong Dual Property: For the given problem class, there exists a strong dual problem

  $$\text{(D)} \;\; \min\{w(u) : u \in U\},$$

  allowing us to obtain optimality conditions that can be quickly verified, that is $x^\star \in X$ is optimal if and only if $\exists u^\star \in U : \;\; c^T x^\star = w(u^\star)$.

Note: if the Explicit Convex Hull Property holds, the dual of the LP satisfies the strong dual property!

When LP relaxation is all you need

$$\text{(IP)} \;\; \max\{c^T x : Ax \le b, \;\; x \in \mathbb{Z}^n_+\}$$

LP relaxation:

$$\text{(LP)} \;\; \max\{c^T x : Ax \le b, \;\; x \in \mathbb{R}^n_+\}$$

Consider a problem with data (A, b) integral. When does (LP) have an integral optimal solution?

When LP relaxation is all you need

$$\text{(IP)}\ \max\{c^T x : Ax \le b,\ x \in \mathbb{Z}^n_+\}$$

LP relaxation:

$$\text{(LP)}\ \max\{c^T x : Ax \le b,\ x \in \mathbb{R}^n_+\}$$

Consider a problem with data (A, b) integral. When does (LP) have an integral optimal solution?
Recall: a BFS of LP is a feasible solution of the form $x = (x_B, x_N) = (B^{-1}b, 0)$, where B is an $m \times m$ nonsingular submatrix of $[A, I]$, and I is the $m \times m$ identity matrix.

Observation 3.1: Sufficient condition

If the optimal basis B has $\det(B) = \pm 1$, then the LP relaxation solves (IP).

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Totally Unimodular matrices

Totally unimodular matrix

A matrix A is *totally unimodular (TU)* if every square submatrix of A has determinant $+1$, -1, or 0. (In particular, $\forall i, j\ a_{ij} \in \{+1, -1, 0\}$.)

Proposition 3.3′ (equality constraints)

The linear program $\max\{c^T x : Ax = b, x \in \mathbb{R}^n_+\}$ has an integral optimal solution for all integer vectors b for which it has a finite optimal value if A is totally unimodular.

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Some TU (and non-TU) matrices

The following matrices are TU

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following matrices are not TU:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Recognizing TU matrices

Proposition 3.1: necessary and sufficient conditions:

Let $A \in \mathbb{R}^{m \times n}$. The following conditions are equivalent:
(i) A is TU
(ii) The transpose matrix A^T is TU
(iii) The matrix $[A, I]$ is TU

Other things you can do to rows/columns of A preserving the TU property: interchanging, deleting, multiplying by -1

Proposition 3.2: sufficient conditions:

A matrix A is TU if
(i) $a_{ij} \in \{+1, -1, 0\}$
(ii) Each column contains at most two nonzero coefficients
(iii) The exists a partition (M_1, M_2) of the set M of rows such that each column j containing two nonzero coefficients satisfies $\sum_{i \in M_1} a_{ij} = \sum_{i \in M_2} a_{ij}$.

Role of Total Unimodularity in IP

Proposition 3.3′ (equality constraints)

The linear program $\max\{c^T x : Ax = b, x \in \mathbb{R}^n_+\}$ has an integral optimal solution for all integer vectors b for which it has a finite optimal value if A is totally unimodular.

Proposition 3.3 (equality constraints)

The linear program $\max\{c^T x : Ax \leq b, x \in \mathbb{R}^n_+\}$ has an integral optimal solution for all integer vectors b for which it has a finite optimal value if and only if A is totally unimodular.

Assignment problem:

$$\begin{array}{ll} \min & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} & \sum_{j=1}^n x_{ij} = 1, \ i = 1, \ldots, n \\ & \sum_{i=1}^n x_{ij} = 1, \ j = 1, \ldots, n \\ & x \geq 0, \ \text{integer} \end{array}$$

- Observation: The constraint matrix arising in an assignment problem is TU
  - To see this, apply Proposition 3.2 with $M_1 = \{1, \ldots, n\}$ and $M_2 = \{n+1, \ldots, 2n\}$
- Thus, assignment problem can be solved by solving its LP relaxation with, e.g., Simplex method

Minimum cost network flows

Problem: given a directed graph $D = (V, A)$, arc capacities h_{ij} $\forall (i,j) \in A$, supplies b_i $\forall i \in V$, unit flow costs c_{ij} $\forall (i,j) \in A$, find a feasible flow that satisfies all the demands at minimum cost. Let $V^+(i) = \{k : (i,k) \in A\}$, $V^-(i) = \{k : (k,i) \in A\}$

$$\begin{array}{ll} \min & \sum_{(i,j)\in A} c_{ij} x_{ij} \\ & \sum_{k \in V^+(i)} x_{ik} - \sum_{k \in V^-(i)} x_{ki} = b_i \ \forall i \in V \\ & 0 \le x_{ij} \le h_{ij} \ \forall (i,j) \in A \end{array}$$

Proposition 3.4

The constraint matrix arising in a minimum cost network flow problem is TU.

Corollary

If the demands and capacities are integral, *(i)* Each extreme point is integral; *(ii)* The constraints describe the convex hull of the integral feasible flows.

Maximum $s - t$ flow; minimum $s - t$ cut

We are given a directed graph $D = (V, A)$, two distinguished nodes $s, t \in V$ and capacities $h_{ij} \ge 0, \forall (i,j) \in A$.
Maximum $s - t$ flow problem: find a maximum flow from s to t. Adding uncapacitated arc (t, s), the problem can be formulated as an NFP:

$$\begin{array}{lll} z_{\text{MFP}} = & \max & x_{st} \\ & & \sum_{k \in V^+(i)} x_{ik} - \sum_{k \in V^-(i)} x_{ki} = 0 \ \forall i \in V \\ & & 0 \le x_{ij} \le h_{ij} \ \forall (i,j) \in A \end{array}$$

An $s - t$ cut is a partitioning of V into sets X and $\bar{X} = V \setminus X$ such that $s \in X$ and $t \in \bar{X}$. The capacity of an $s - t$ cut is defined as the total capacity of the arcs crossing the cut in the "forward" direction:

$$\sum_{(i,j) \in A: i \in X, j \notin X} h_{ij}.$$

Minimum $s - t$ cut problem: find a minimum-capacity $s - t$ cut.

Strong duality between max-flow and min-cut problems

"Proof:" Weak duality – easy.
The dual of the max flow LP formulation:

$$\begin{array}{rl} z_{\text{MFP}} = \min & \sum_{(i,j)\in A} h_{ij} w_{ij} \\ & u_i - u_j + w_{ij} \geq 0 \ \forall (i,j) \in A \\ & u_t - u_s \geq 1, \ w_{ij} \geq 0 \ \forall (i,j) \in A. \end{array}$$

From total unimodularity, there is an integer opt. solution $(\tilde{u}, \tilde{w})$.
Without loss of generality, $\tilde{u}_s = 0$.
Let $X = \{j \in V : \tilde{u}_j \leq 0\} \ni s$, $\bar{X} = V \setminus X = \{j \in V : \tilde{u}_j \geq 1\} \ni t$.
Consider another *feasible* solution to the LP dual:

$\hat{u}_j = 0,\ j \in X,\ \hat{u}_j = 1,\ j \in \bar{X},\ \hat{w}_{ij} = 1,\ i \in X, j \in \bar{X},\ \hat{w}_{ij} = 0$ otherwise

Then (2nd inequality: $\tilde{w}_{ij} \geq \tilde{u}_j - \tilde{u}_i \geq 1$ for $i \in X$ and $j \in \bar{X}$)

$$z_{\text{MFP}} \leq \sum_{(i,j)\in A} h_{ij}\hat{w}_{ij} = \sum_{(i,j)\in A, i\in X, j\in\bar{X}} h_{ij} \leq \sum_{(i,j)\in A, i\in X, j\in\bar{X}} h_{ij}\tilde{w}_{ij} \leq \sum_{(i,j)\in A} h_{ij}\tilde{w}_{ij} = z_{\text{MFP}}$$

Finally, note that $\sum_{(i,j)\in A, i\in X, j\in\bar{X}} h_{ij}$ is the capacity of an $s-t$ cut!

Shortest path problem (if time remains)

Problem: given a directed graph $D = (V, A)$, two distinguished nodes $s, t \in V$ and arc costs $c_{ij} \geq 0, \forall (i,j) \in A$, find a minimum cost path from s to t.
IP formulation:

$$\begin{array}{rl} z = \min & \sum_{(i,j)\in A} c_{ij} x_{ij} \\ & \sum_{k\in V^+(i)} x_{ik} - \sum_{k\in V^-(i)} x_{ki} = 1 \text{ for } i = s \\ & \sum_{k\in V^+(i)} x_{ik} - \sum_{k\in V^-(i)} x_{ki} = 0 \ \forall i \in V \setminus \{s,t\} \\ & \sum_{k\in V^+(i)} x_{ik} - \sum_{k\in V^-(i)} x_{ki} = -1 \text{ for } i = t \\ & x_{ij} \geq 0 \ \forall (i,j) \in A \\ & x \in \mathbb{Z}^{|A|} \end{array}$$

Here, $x_{ij} = 1$ if arc (i,j) is on the shortest path, 0 otherwise.

Theorem 3.5

z is the length of a shortest $s-t$ path if and only if there exist values $\pi,\ i \in V$ such that $\pi_s = 0,\ \pi_t = z$, and $\pi_j - \pi_i \leq c_{ij}$ for $(i,j) \in A$.