Bounds, relaxations and duality

Given an optimization problem

\[ z = \max \{ c(x) : x \in X \}, \]

how does one find \( z \), or prove that a feasible solution \( x^* \) is optimal or close to optimal?

- Search for a lower and upper bound on the optimal objective value: \( z \leq z \leq \bar{z} \).
- If \( z = \bar{z} \), then the optimal objective value is found.
- If \( x^* \in X \) satisfies \( z \leq c(x^*) \leq \bar{z} \), and \( \bar{z} - z \leq \epsilon \), where \( \epsilon > 0 \) is an “optimality tolerance” parameter, then \( x^* \) is guaranteed to satisfy \( c(x^*) \geq z - \epsilon \).

Primal bounds: “can you beat this?” Lower(upper) bounds on the optimal objective function value of a max(min) problem Any feasible solution can be used to obtain a primal bound.

Dual bounds: can’t get any better than...

Upper(lower) bounds on the optimal objective function value of a max(min) problem are often obtained via relaxations of the problem

(IP) \[ z = \max \{ c(x) : x \in X \subseteq \mathbb{R}^n \} \]

Relaxation

A problem (RP) \[ z^R = \max \{ f(x) : x \in T \subseteq \mathbb{R}^n \} \]

is a relaxation of (IP) if:

(i) \( X \subseteq T \), and (ii) \( f(x) \geq c(x) \) for all \( x \in X \).

Proposition 2.1

If (RP) is a relaxation of (IP), then \( z^R \geq z \).

Proposition 2.3

(i) If a relaxation (RP) is infeasible, the original problem (IP) is infeasible.

(ii) Let \( x^* \) be an optimal solution of (RP). If \( x^* \in X \) and \( f(x^*) = c(x^*) \), then \( x^* \) is an optimal solution of (IP).
**Linear Programming Relaxations**

**LP relaxation**

For the integer program \( \text{max}\{c^T x : x \in P \cap \mathbb{Z}^n\} \) with formulation \( P = \{x \in \mathbb{R}^n : Ax \leq b\} \), the *linear programming relaxation* is the linear program \( z^{LP} = \text{max}\{c^T x : x \in P\} \).

Clearly a relaxation: \( P \cap \mathbb{Z}^n \subseteq P \) and objective function is unchanged.

**Proposition 2.2**

Suppose, \( P_1, P_2 \) are two formulations for the IP \( \text{max}\{c^T x : x \in X \subseteq \mathbb{Z}^n\} \) with \( P_1 \subseteq P_2 \). If \( z_i^{LP} = \text{max}\{c^T x : x \in P^i\} \), \( i = 1, 2 \), then \( z_1^{LP} \leq z_2^{LP} \) for all \( c \).

If \( P \) is the ideal formulation, then \( z^{LP} = z^* \), and any extreme point optimal solution of the LP is an optimal solution of the IP.

**Combinatorial relaxations: TSP**

**TSP**: a TSP tour is an assignment (or permutation) containing no subtours; hence

\[
z^{TSP} = \min_T \left\{ \sum_{(i,j) \in T} c_{ij} : T \text{ is a tour} \right\}
\]

\[
\geq \min_T \left\{ \sum_{(i,j) \in T} c_{ij} : T \text{ is an assignment} \right\} = z^{ASS}
\]
Combinatorial relaxations: symmetric TSP

**STSP:** a TSP with $c_{ij} = c_{ji}$

**Tree and 1-tree**

A *tree* on a set of $k$ nodes is an undirected graph that satisfies any two of the following three properties:

(i) It has $k - 1$ edges (ii) It contains no cycles (iii) It is connected.

A *1-tree* on $n$ nodes is a graph consisting of two edges adjacent to node 1, plus the edges of a tree on nodes $\{2, \ldots, n\}$.

Every tour is a 1-tree, so

$$z_{\text{STSP}} = \min_T \left\{ \sum_{(i,j) \in T} c_{ij} : T \text{ is a tour} \right\}$$

$$\geq \min_T \left\{ \sum_{(i,j) \in T} c_{ij} : T \text{ is a 1-tree} \right\} = z_{1\text{-tree}}$$

**Lagrangian relaxation**

(IP) $z = \max\{c^T x : Ax \leq b, x \in X \subseteq \mathbb{Z}^n\}$.

**Proposition 2.4**

Let $z(u) = \max\{c^T x + u^T (b - Ax) : x \in X\}$. Then $z \leq z(u)$ for all $u \geq 0$.

Above is called the *Lagrangian relaxation* of (IP). It is, indeed a relaxation: larger feasible region, and for any $x$ feasible for (IP),

$$z(u) \geq c^T x + u^T (b - Ax) \geq c^T x.$$
Duality: a dual problem

**Duality**

The two problems (IP) \( z = \max \{c^T x : x \in X \subseteq \mathbb{Z}^n \} \), and (D) \( w = \min \{w(u) : u \in U \} \) form a (weak)-dual pair if \( c(x) \leq w(u) \) for all \( x \in X \) and all \( u \in U \). When \( z = w \), they form a strong-dual pair.

Advantage: any feasible solution of the dual provides a dual bound (whereas a relaxation needs to be solved to optimality).

**Proposition 2.6**

(i) If (D) is unbounded, the original problem (IP) is infeasible.

(ii) If \( x^* \in X \) and \( u^* \in U \) satisfy \( c(x^*) = w(u^*) \), the \( x^* \) is optimal for (IP) and \( w^* \) is optimal for (D).

**Proposition 2.5**

The integer program \( z = \max \{c^T x : Ax \leq b, \ x \in \mathbb{Z}^n_+ \} \) and the linear program \( w^{LP} = \min \{u^T b : u^T A \geq c, \ u \in \mathbb{R}^m_+ \} \) form a weak dual pair.

A dual pair — combinatorial perspective

Given a graph \( G = (V, E) \), a **matching** \( M \subseteq E \) is a set of disjoint edges; a **covering by nodes** is a set \( R \subseteq V \) of nodes such that every edge has at least one endpoint in \( R \).

**Proposition 2.7**

The problem of finding a maximum cardinality matching and the problem of finding a minimum cardinality covering by nodes form a weak-dual pair.

**Proof:** If \( M = \{(i_1, j_1), \ldots, (i_k, j_k)\} \) then the nodes \( i_1, j_1, \ldots, i_k, j_k \) are distinct, and any covering by nodes \( R \) must contain at least one node from each pair. Hence \( |R| \geq k = |M| \).
A dual pair — LP perspective

**Node-edge incidence matrix**

Let \( n = |V| \) and \( m = |E| \). Define an \( n \) by \( m \) node-edge incidence matrix \( A \) as follows: \( a_{j,e} = 1 \) if node \( j \) is an endpoint of edge \( e \), and 0 otherwise.

Maximum cardinality matching: \( z = \max \{ e^T x : Ax \leq e, \ x \in \mathbb{Z}^m_+ \} \)

Minimum cardinality cover by nodes:

\( w = \min \{ e^T y : A^T y \geq e, \ y \in \mathbb{Z}^n_+ \} \)

Let \( z^{\text{LP}} \) and \( w^{\text{LP}} \) be the values of the corresponding LP relaxations. Then

\[ z \leq z^{\text{LP}} = w^{\text{LP}} \leq w, \]

establishing weak-duality.

Not a strong-dual pair! Consider \( A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \)