Simplex method: outline

- The Simplex Method is a family of algorithms for solving LPs in standard form (and their duals)
- Goal: identify an optimal basis, as in Definition 3.3
- Versions we will consider:
  - Primal
  - Dual (along with LP sensitivity analysis)
  - Parametric primal-dual

Simplex method(s) for solving LPs in standard form

- Idea behind the Simplex method(s) for LPs in standard form:
  - Start at a basis $B$
  - Move to an adjacent basis
  - Repeat until
    - an optimal basis (Definition 3.3) is reached, or
    - lack of optimal solution is proven
- Questions:
  - What initial basis do we choose?
  - How do we move between adjacent bases?
  - How can we guarantee that the process terminates with a correct conclusion?
  - How long will it take?
Primal Simplex Method

- Assumptions:
  - The primal LP is feasible
  - A basis associated with a BFS is available

- The Primal Simplex Algorithm
  - Start at a BFS $x$ and associated basis $B$
  - Calculate $\bar{c}^T = c^T - c^T_B B^{-1} A$
  - If $\bar{c} \geq 0$, stop — current basis is optimal. Otherwise, pick some $j \in N$ with $\bar{c}_j < 0$.
  - Move to an adjacent feasible basis in which $A_j$ is a basic column.
  - Repeat until an optimal basis is reached, or unboundedness (of the LP) is proven.

Feasible directions and basic directions

(LP) $\min c^T x \text{ s.t. } x \in P; \quad P = \{x : Ax = b, \ x \geq 0\}$

**Definition 3.1**

Let $x \in P$. $d \in \mathbb{R}^n$ is a **feasible direction** at $x$ if $\exists \theta > 0$: $x + \theta d \in P$.

- **Special case:** $x$ is a BFS with basis $B$; $j$ — non-basic index
- $d$ is the $j$th **basic direction** at $x$ if
  $$d_i = 0, \ i \in N, \ i \neq j, \ d_j = 1, \ d_B = (d_B(1), \ldots, d_B(m))^T = -B^{-1} A_j$$

- **Note:**
  - If $x$ is non-degenerate, $d$ is a feasible direction
  - If $x$ is degenerate, $d$ may not be a feasible direction!

- **Effect on the cost function of moving in $j$th basic direction:** $j$th **reduced cost**
  $$\bar{c}_j = c^T d = c^T_B d_B + c_j = c_j - c^T_B B^{-1} A_j$$
An iteration of the (primal) simplex method: a pivot

1. Let \( x = (x_B; x_N) = (B^{-1}b; 0) \) be a BFS, and \( B \) — the associated basis (assume for now \( x \) is non-degenerate)
2. Compute \( \bar{c}_j = c_j - c_B^T B^{-1} A_j \) \( \forall j \in N \). If \( \bar{c} \geq 0 \) — terminate; optimal solution found. Else, choose \( j : \bar{c}_j < 0 \).
3. Let \( d \) be the \( j \) basic direction at \( x \):
   \[
d_i = 0, \ i \in N, \ i \neq j, \ d_j = 1, \ d_B = (d_{B(1)}, \ldots, d_{B(m)})^T = -B^{-1}A_j
   \]
   If \( d_B \geq 0 \), terminate; LP is unbounded.
4. Else, let
   \[
   \theta^* = \min_{i: d_B(i) < 0} \frac{x_B(i)}{-d_B(i)}, \ I = \arg\min_{i: d_B(i) < 0} \frac{x_B(i)}{-d_B(i)}
   \]
5. Let \( l \) be as above, \( \theta^* = \frac{x_B(l)}{-d_B(l)} \). Form a new basis matrix by replacing \( A_B(l) \) with \( A_j \) (\( x_l \) leaves, and \( x_j \) enters, the basis). The values of the new basic variables:
   \[
y_j = \theta^*, \ y_B(i) = x_B(i) + \theta^* d_B(i), \ i \neq l
   \]

“Sanity checks”

**Theorem 3.2**

(a) The columns \( A_B(i), \ i \neq l \) and \( A_j \) are linearly independent and therefore, the matrix constructed in step 5 is a basis.

(b) The vector \( y = x + \theta^* d \) is a basic feasible solution associated with the above matrix.

**Theorem 3.3**

Assume that the feasible set is nonempty and that every basic feasible solution is nondegenerate. Then the simplex method terminates after a finite number of iterations. At termination, there are the following two possibilities:

(a) We have an optimal basis \( B \) and an associated basic feasible solution which is optimal.

(b) We have found a vector \( d \) satisfying \( Ad = 0, \ d \geq 0, \ c^T d < 0 \), and the problem is unbounded.
What if there are degenerate BFSs?

- If current BFS $x$ is degenerate, then it is possible that $\theta^* = 0$.
- It is still possible, however, to perform a change of basis, although we will stay at the same solution, and see no improvement in the objective.
- It is possible that a series of iterations as above will cause us to “loop” through a series of bases, returning to the original one: *cycling*.
- Can be avoided by appropriate *pivoting rules*.

Pivot selection

How to choose $j$ and $l$ (if there are ties) in steps 2 and 5 of the algorithm? Some possible *pivoting rules* (out of many others):

- Choose $j = \text{argmin } \bar{c}_j$ — fastest rate of decrease of the cost.
- Choose $j = \text{argmax } \theta^*_j |\bar{c}_j|$ — largest cost decrease.
- Choose the smallest $j$ with $\bar{c}_j < 0$ — less computation. If the *smallest subscript* rule is also used to break ties in choosing $l$, no cycling occurs in degenerate problems
  - Thus, with appropriate pivoting rules, simplex method will find an optimal basis matrix in any standard form LP with finite optimal value!
- Lexicographic pivoting rule (See Section 3.4) — another anti cycling pivoting rule.
Unanswered (for now) questions

- How do we find a starting feasible basis, or find out/prove that the LP is infeasible?
- What is the computational efficiency of the (Primal) Simplex Method:
  - Time to execute 1 iteration?
  - Number of iterations required?
- Let’s postpone these for now, and explore
  - other properties of optimal bases.
  - other variants of the simplex method.

Sensitivity analysis: changes in $c$

$$(P) \min c^T x \quad \begin{cases} \text{s.t.} & Ax = b \quad (D) \max b^T p \\ x \geq 0 & \text{s.t.} & p^T A \leq c^T \end{cases}$$

- Suppose optimal basis $B$ is known, but $c$ becomes $c + t\Delta c$, for some given $\Delta c \in \mathbb{R}^n$
- For what values of $t$ is $B$ still optimal?
  - $x_B = B^{-1}b \geq 0$ does not change
  - Need: $(c_j + t\Delta c_j) - (c_B + t\Delta c_B)^T B^{-1} A_j \geq 0$ for all $j \in N$
- Let $\Delta \bar{c}_j = \Delta c_j - \Delta c_B^T B^{-1} A_j$
- $B$ stays dual-feasible (and hence optimal) for
  $$\max_{j: \Delta \bar{c}_j > 0} \frac{\bar{c}_j}{\Delta \bar{c}_j} \leq t \leq \min_{j: \Delta \bar{c}_j < 0} \frac{\bar{c}_j}{\Delta \bar{c}_j}$$

- If change in $c$ is such that $B$ is no longer dual-feasible, re-solve with Primal Simplex (starting at current basis $B$)
Sensitivity analysis: changes in \( b \)

- Suppose optimal basis \( B \) is known, but \( b \) becomes \( b + t\Delta b \), for some given \( \Delta b \in \mathbb{R}^m \)
- For what values of \( t \) is \( B \) still optimal?
  - \( s_N = c_N - c_B^T B^{-1} A_N = \bar{c}_N \geq 0 \) does not change
  - Need: \( B^{-1}(b + t\Delta b) \geq 0 \)
- Let \( \Delta x_B = B^{-1} \Delta b \)
- \( B \) stays primal-feasible (and hence optimal) for

\[
\max_{i: \Delta x_B(i) > 0} -\frac{x_B(i)}{\Delta x_B(i)} \leq t \leq \min_{i: \Delta x_B(i) < 0} -\frac{x_B(i)}{\Delta x_B(i)}
\]

- If change in \( b \) is such that \( B \) is no longer primal-feasible, re-solve?
- Note: current basis \( B \) is dual-feasible, but not primal feasible...

Primal vs. Dual simplex

- “Primal” Simplex:
  - Starts with a primal-feasible basis
  - Goes from a basis to an (adjacent) basis
  - Strives to achieve dual feasibility (hence selection of entering variable)
  - Maintains primal feasibility throughout (hence selection of exiting variable)
- “Dual” Simplex?
  - Starts with a dual-feasible basis
  - Goes from a basis to an (adjacent) basis
  - Strives to achieve primal feasibility
  - Maintains dual feasibility throughout
- Development of Dual Simplex algorithm:
  - Given a dual-feasible basis,
  - Determine which variable should exit to work towards primal feasibility
  - Then determine which variable should enter to maintain dual feasibility
An iteration of the dual simplex method: a pivot

1. Let \((p; s) = (p; s_B; s_N) = ((c_B^T B^{-1})^T; 0; \bar{c}_N)\) with \(s_N \geq 0\) be a dual BFS, and \(B\) — the associated basis.
2. Compute \(x_B = B^{-1}b\). If \(x_B \geq 0\) — terminate; solution found. Else, choose \(l : x_B(l) < 0\). \((x_B(l)\) leaves the basis).
3. Let \(v\) be the \(l\)th row of \(B^{-1}A\):
   \[v^T = e_i^T B^{-1}A\]
   If \(v \geq 0\), terminate; (primal) LP is infeasible.
4. Else, let \(j = \arg\min_{i : v_i < 0} \frac{\bar{c}_i}{-v_i}\) \((x_j\) enters the basis).
5. Let \(l\) and \(j\) be as above. Form a new basis matrix by replacing \(A_B(l)\) with \(A_j\).

Analysis of the dual simplex pivot

Entering variable: \(x_j\); exiting variable: \(x_B(l)\)

- New basic \((p^{\text{new}})^T = p^T + t^*\Delta p = c_B^T B^{-1} + \frac{\bar{c}_j}{v_j} e_i^T B^{-1}\)
- For \(i\) in the (new) basis, \(c_i^{\text{new}} = 0\) by construction.
- For \(i\) nonbasic (after the basis change),
  \[c_i^{\text{new}} = c_i - (p^{\text{new}})^T A_i = \bar{c}_i - \frac{\bar{c}_j}{v_j} e_i^T B^{-1} A_i = \bar{c}_i - \frac{\bar{c}_j}{v_j} \cdot v_i\]
  - For \(i = B(l)\): \(c_B^{\text{new}}(l) = x_B(l) - \frac{\bar{c}_j}{v_j} \cdot v_B(l) = 0 - \frac{\bar{c}_j}{v_j} \cdot 1 \geq 0\)
  - For \(i \neq B(l)\): \(\bar{c}_i^{\text{new}} = \bar{c}_i - \frac{\bar{c}_j}{v_j} \cdot v_i \geq 0\) by choice of \(j\)
- New objective value:
  \[(p^{\text{new}})^T b = c_B^T B^{-1}b + \frac{\bar{c}_j}{v_j} e_i^T B^{-1}b = c_B^T B^{-1}b + \frac{\bar{c}_j}{v_j} x_B(l) \geq c_B^T B^{-1}b\]
  - If \(\bar{c}_j > 0\), then the pivot is non-degenerate and new dual BFS has a better (dual) objective function value
    - If all pivots are non-degenerate, bases are never repeated, proving convergence
    - Usual anti-cycling pivoting rules can be used if needed
Sensitivity analysis summary

Given a primal and dual feasible basis $B$,

- If $c$ changes to $c + t \Delta c$, find range of $t$ for which $B$ remains dual-feasible
- If $b$ changes to $b + t \Delta b$, find range of $t$ for which $B$ remains primal-feasible
- If both changes happen simultaneously, combined analyses finds range of $t$ for which $B$ remains optimal

Simplex method summary so far

✓ Described:
  - Primal simplex method, to use when a starting primal-feasible $B$ is available
  - Dual simplex method, to use when a starting dual-feasible $B$ is available

✓ Argued that both versions of simplex terminate after a finite number of iterations
  - Either finds an optimal basis,
  - or proves that the problem is unbounded

Easy argument with no degeneracy; references to anti-cycling pivoting rules with degeneracy

✓ Still unanswered:
  - How many iterations?
  - How much time per iteration?
  - How do we get a primal- or dual- feasible basis to start with?
    - or re-design simplex method to be able to start at any basis!