To start: some more AMPL

A simple AMPL model of a production problem

Variables: \( x_B \), \( x_C \) — number of tons of bands, coils, resp., to make

\[
\begin{align*}
\text{max} & \quad 25x_B + 30x_C \\
\text{s.t.} & \quad (1/200)x_B + (1/140)x_C \leq 40 \\
& \quad 0 \leq x_B \leq 6000 \\
& \quad 0 \leq x_C \leq 4000
\end{align*}
\]

File “prod0.mod”:

```ampl
var XB;
var XC;

maximize profit: 25*XB+30*XC;

subject to Time: (1/200)*xB+(1/140)*xC<=40;
subject to B_limit: 0<=xB<=6000;
subject to C_limit: 0<=xC<=4000;
```

Modeling: Components of a general production model

- **Sets**, e.g.,
  - of possible products: bands, coils;
  - of required resources: Time

- **Parameters**, e.g., resource consumption rates and total availability, unit profits, bounds, etc.

- **Variables**

- **Objective function** to be minimized (or maximized)

- **Constraints** that the solution must satisfy

Observations:

- The first two bullets refer to data; the last three — to components of the model

  - Recall the distinction between an optimization problem and an instance of that problem

- AMPL allows us to separate the description of the problem from the problem instance (model from data)

  - See files “prod.mod” and “prod.dat” (AMPL book, 2nd ed.)

```ampl
ampl: model prod.mod; data prod.dat; solve;
```
Modeling: Further refinements of the production model

- Lower bounds on production amounts (commitments)
- Multi-stage production (multiple resources utilized)

**Given:**
- $P$: a set of products,
- $S$: a set of production stages,
- $a_{p,s}$: tons per hour of product $p$ produced in stage $s$
- $b_s$: hours available in each stage
- $c_p$: profit per ton for each product
- $u_p, l_p$: upper (market) and lower (commitment or 0) bounds on production of each product, respectively

**Model:**

$$\text{max} \sum_{p \in P} c_p x_p$$

$$\text{s.t.} \sum_{p \in P} (1/a_{p,s}) x_p \leq b_s, \quad \forall s \in S$$

$$l_p \leq x_p \leq u_p, \quad \forall p \in P$$

**AMPL files:** “steel4.mod” and “steel4.dat”

Recap, and outline of Lecture 5

**Previously:**
- Graphical solution of 2-variable LPs
- Mathematical background:
  - Convex sets, Convex combinations, Convex hulls
  - Hyperplanes, Half-spaces
  - Polyhedra: $\{x \in \mathbb{R}^n \mid Ax \geq b\}$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$
    - Feasible region of any LP is a polyhedron
  - Two geometry-based definitions of a corner point $x$ of polyhedron $P$
    - **Extreme point:** “$x$ is not a convex combination of any two other points in $P$”
    - **Vertex:** “There exists a vector $c$ such that $x$ is the unique solution of $\min \{c' y \mid \text{s.t. } y \in P\}$

**Today:**
- Algebraic definition of a corner point of $P$
- Are these three definitions equivalent?
**Active constraints**

Consider a polyhedron \( P \subset \mathbb{R}^n \) defined as

\[
P = \{ x \in \mathbb{R}^n \mid a_i'x \geq b_i, \ i \in M_1, \ a_i'x \leq b_i, \ i \in M_2, \ a_i'x = b_i, \ i \in M_3 \}
\]

where \( M_1, M_2, M_3 \) are finite index sets.

**Definition 2.8**

If a vector \( x^* \) satisfies \( a_i'x^* = b_i \) for some \( i \in M_1 \cup M_2 \cup M_3 \), we say that the corresponding constraint is **active**, or **binding**, at \( x^* \).

- Note: can apply the definition whether or not \( x^* \) is feasible
- Idea of the algebraic definition of a corner point: a point that
  - is feasible,
  - is uniquely identified by the system of equations obtained from active constraints
- To fully develop this idea, need linear algebra tools (esp. Thms. 1.2 and 1.3)
- Along the way, will introduce some concepts useful in developing algorithms for solving LPs

**Linear algebra tools**

- **Linear combination** of vectors \( x^1, \ldots, x^K \): a vector
  \[
y = \sum_{k=1}^{K} a_k x^k \text{ for some numbers } a_k \in \mathbb{R}, \ k = 1, \ldots, K
  \]
- Vectors \( x^1, \ldots, x^K \in \mathbb{R}^n \) are **linearly dependent** if some linear combination of them (with some of \( a_k \)'s non-zero) evaluates to \( 0 \)
  - i.e., vectors are **linearly independent** if system of equations
    \[
    \sum_{k=1}^{K} a_k x^k = 0
    \]
    has a unique solution: \( a_1 = \cdots = a_K = 0 \)
Linear algebra tools

**Theorem 1.2**

Let $D$ be a square matrix. The following statements are equivalent:

(a) The matrix $D$ is invertible
   - i.e., exists square matrix $D^{-1}$ such that $D \cdot D^{-1} = I$, where $I$ is the identity matrix

(b) The matrix $D'$ is invertible
(c) The determinant of $D$ is nonzero
(d) The rows of $D$ are linearly independent
(e) The columns of $D$ are linearly independent
(f) $\forall d$, the linear system $Dx = d$ has a unique solution
(g) $\exists d$ such that the linear system $Dx = d$ has a unique solution

- “$\forall$” is shorthand for “for all”, or “for any”
- “$\exists$” is shorthand for “there exists”, or “we can find”

Linear algebra tools

- A non-empty subset $S$ of $\mathbb{R}^n$ is a **subspace** if, for any $x_1, \ldots, x^K \in S$, any linear combination of these vectors is also contained in $S$
  - Every subspace $S$ contains $0$
  - Subspaces of $\mathbb{R}^1$: $\{0\}$ and $\mathbb{R}^1$
  - Subspaces of $\mathbb{R}^2$: $\{0\}$, any line passing through $0$, and $\mathbb{R}^2$
  - Subspaces of $\mathbb{R}^3$: $\{0\}$, any line passing through $0$, any 2-dimensional plane passing through $0$, and $\mathbb{R}^3$
  - etc.

- The **span** of vectors $x_1, \ldots, x^K$ is the set of all their linear combinations.
  - By construction, a subspace

- A **basis** of a subspace $S$ is a collection of linearly independent vectors whose span is equal to $S$
  - Every basis of a given subspace $S$ has the same number of vectors, called the **dimension** of $S$. 