Recap, and outline of Lecture 4

Previously:
- Convex functions
- Optimization with piecewise linear convex functions — equivalent LP formulations
- Using AMPL software to represent an LP model
  - Using a solver to solve the LP

Today:
- Geometry of LP
  - Illustration with a 2-variable example
  - Convex sets
  - Hyperplanes, halfspaces and polyhedra
- Modeling:
  - Separating the model (problem) from the data (instance) in AMPL

Production problem from Lecture 3
- A steel company takes slabs of steel as input and produces two products: bands and coils.
- The products come off the production line at two different rates: bands — at 200 tons per hour, coils — at 140 tons per hour
- A total of 40 hours of production time is available.
- The profits are: bands — $25 per ton, coils — $30 per ton.
- Finally, due to limited market demand, the management instructs us not to exceed 6,000 tons of bands and 4,000 tons of coils in a week (demands are independent of each other).
- What is the production plan that maximizes the profit?

Variables: $x_B$, $x_C$ — number of tons of bands, coils, resp., to make

$$\begin{align*}
\text{max} & \quad 25x_B + 30x_C \\
\text{s.t.} & \quad \frac{1}{200}x_B + \frac{1}{140}x_C \leq 40 \\
& \quad 0 \leq x_B \leq 6000 \\
& \quad 0 \leq x_C \leq 4000
\end{align*}$$
Geometry of LPs in $\mathbb{R}^2$

Production model: feasible region

$$\begin{align*}
\text{min} & \quad -25x_B - 30x_C \\
\text{s.t.} & \quad (1/200)x_B + (1/140)x_C \leq 40 \\
& \quad 0 \leq x_B \leq 6000 \\
& \quad 0 \leq x_C \leq 4000
\end{align*}$$

Geometry of LPs in $\mathbb{R}^2$

Production model: finding optimal solutions

$$\text{min} \quad -25x_B - 30x_C$$
Geometric study of LPs

**Conjecture**

Every LP is either
- Infeasible, or
- Unbounded (with optimal cost $-\infty$ for minimization problems), or
- Has an optimal solution (or multiple optimal solutions),
  - and, if the feasible set has at least one corner, an optimal solution can be found among the corners of the feasible set.

What about, e.g., $\min x_1 \text{ s.t. } -1 \leq x_1 \leq 1, x_2$ unrestricted

Inequality representation of LP constraints

**Claim**

Feasible set of any LP with $n$ variables can be expressed as:

$$\{ x \in \mathbb{R}^n \mid a_i'x \geq b_i, \ i = 1, \ldots, m \}$$

for some $m, a_i \in \mathbb{R}^n, \ i = 1, \ldots, m,$ and $b_i \in \mathbb{R}, \ i = 1, \ldots, m$.

- $a_i'x \leq b_i$ can be re-written as $-a_i'x \geq -b_i$
  - Vectors $x$ that satisfy the first constraint will satisfy the second constraint, and vice versa
- $a_i'x = b_i$ can be re-written as $a_i'x \geq b_i, \ -a_i'x \geq -b_i$
- $x_i \geq 0$ can be re-written as $e_i'x \geq 0$
- $x_i \leq 0$ can be re-written as $-e_i'x \geq 0$
  - $e_i \in \mathbb{R}^n$ is the $i$th unit vector: all zeros, 1 in the $i$th place
Compact inequality representation

- Let
  \[ A = \begin{bmatrix} -a_1' & - \\ \vdots \\ -a_m' & - \end{bmatrix} \in \mathbb{R}^{m \times n}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m. \]

- Constraints \( a_i'x \geq b_i \), \( i = 1, \ldots, m \) can be compactly written as
  \[ Ax \geq b \]

- \( Ax = \begin{bmatrix} a_1'x \\ \vdots \\ a_m'x \end{bmatrix} \in \mathbb{R}^m \)

- Inequality interpreted componentwise: \((Ax)_i \geq b_i\) for every \( i \)

- We will study geometry of LPs expressed in the form
  \[
  \begin{align*}
  \text{minimize} & \quad c'x \\
  \text{s.t.} & \quad Ax \geq b
  \end{align*}
  \]

Polyhedra; Bounded (and unbounded) sets

**Definition 2.1**

A **polyhedron** is a set that can be described in the form \( \{x \in \mathbb{R}^n \mid Ax \geq b\} \), where \( A \) is an \( m \times n \) matrix and \( b \) is a vector in \( \mathbb{R}^m \).

- Feasible set of any LP is a polyhedron
- A feasible set can either be unbounded, or be confined to a finite region, i.e., be bounded:

**Definition 2.2**

A set \( S \subset \mathbb{R}^n \) is **bounded** if there exists a constant \( K \) such that for any \( x \in S \), the absolute value of every component of \( x \) is bounded above by \( K \).
Hyperplanes and halfspaces

Definition 2.3
Let $\mathbf{a}$ be a non-zero vector in $\mathbb{R}^n$, and let $b \in \mathbb{R}$.
(a) The set $\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}' \mathbf{x} = b \}$ is called a hyperplane.
(b) The set $\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}' \mathbf{x} \geq b \}$ is called a halfspace.

Observations:
- A hyperplane is the boundary of corresponding halfspace
- Vector $\mathbf{a}$ is perpendicular to the hyperplane
- A polyhedron is the intersection of a finite number of hyperplanes.

Convex sets

Definition 2.4
A set $S \subset \mathbb{R}^n$ is convex if for any $\mathbf{x}, \mathbf{y} \in S$, and any $\lambda \in [0, 1]$, we have $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in S$.

- If $\lambda \in [0, 1]$, then the point $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}$ belongs to the line segment joining $\mathbf{x}$ and $\mathbf{y}$.
- Thus, a set is convex if and only if the segment joining any two of its elements belongs to the set.

Theorem 2.1, Part II
(a) The intersection of convex sets is convex.
(b) Every polyhedron is a convex set.

- Let $S_i, \ i \in I$ be a collection of sets, where $I$ is some index set. The intersection of these sets is
  $S = \bigcap_{i \in I} S_i = \{ \mathbf{x} \mid \mathbf{x} \in S_i, \ i \in I \}$
Convex combinations and the convex hull

**Definition 2.5**
Let \( \mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^n \) and \( \lambda_1, \ldots, \lambda_k \in \mathbb{R} \) be nonnegative with \( \sum_{i=1}^{k} \lambda_i = 1 \).

(a) The vector \( \sum_{i=1}^{k} \lambda_i \mathbf{x}^i \) is a **convex combination** of the vectors \( \mathbf{x}^1, \ldots, \mathbf{x}^k \).

(b) The **convex hull** of the vectors \( \mathbf{x}^1, \ldots, \mathbf{x}^k \) is the set of all convex combinations of these vectors.

**Theorem 2.1, Part II**
(c) A convex combination of a finite number of elements of a convex set also belongs to that set.

(d) The convex hull of a finite number of vectors is a convex set.

“Corner points” of polyhedra: 2 geometric points of view

**Definition 2.6**
Let \( P \) be a polyhedron. A vector \( \mathbf{x} \in P \) is an **extreme point** of \( P \) if one cannot represent \( \mathbf{x} \) as a convex combination of two other elements of \( P \).

**Definition 2.7**
Let \( P \) be a polyhedron. A vector \( \mathbf{x} \in P \) is a **vertex** of \( P \) if there exists some \( \mathbf{c} \) such that \( \mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y} \) for all \( \mathbf{y} \in P, \mathbf{y} \neq \mathbf{x} \).