Recap, and outline of Lecture 2

Last time:
- Optimization Problems
  - Set of decisions to make; A single quantitative criterion to compare decisions; Rules governing interactions of decisions
- Mathematical programming problems
  - Decision variables; Objective function (to minimize or maximize); Constraints
- Linear Programming problems (LPs)
  - Continuous variables; Linear objective function; Constraints are linear inequalities and equations
- Examples
  - The Diet Problem
    - Distinction between a problem and problem instance
    - Basic model of the radiation treatment planning problem

Today
- Formal definitions of LP components and solutions
- More modeling examples
- AMPL modeling language and solving LPs

Vector notation in LP

$$\begin{align*}
\text{minimize}_{x_1, x_2, x_3, x_4} & \quad 2x_1 - x_2 + 4x_3 \\
\text{subject to} & \quad x_1 + x_2 + x_4 \leq 2 \\
& \quad 3x_2 - x_3 = 5 \\
& \quad x_3 + x_4 \geq 3 \\
& \quad x_1 \geq 0 \\
& \quad x_3 \leq 0
\end{align*}$$

We write:
- \(x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\)
- \(c = \begin{pmatrix} c_1 \\ \vdots \\ c_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \\ 0 \end{pmatrix}\)
- Objective function: \(c'x = \sum_{j=1}^{4} c_j x_j\)
- Inner product of \(c\) and \(x\)
- \(a_1 = (a_1, \ldots, a_4)' = (1, 1, 0, 1)', \quad b_1 = 2\)
- First constraint: \(a_1'x \leq b_1\)
- All vectors are columns, but in the text we write \(x = (x_1, \ldots, x_4)\) for short
A general LP

Given

- $n$ (# of variables);
- Subsets $N_1$ and $N_2$ of $\{1, \ldots, n\}$ indicating which variables are restricted in sign;
- Cost vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$;
- Finite index sets $M_1, M_2, M_3$ and, for each $i \in M_1 \cup M_2 \cup M_3$, a vector $a_i \in \mathbb{R}^n$ and a scalar $b_i$

the corresponding LP is

$$\begin{align*}
\min_x & \quad c'x \\
\text{s.t.} & \quad a'_i x \geq b_i, \quad i \in M_1 \\
& \quad a'_i x \leq b_i, \quad i \in M_2 \\
& \quad a'_i x = b_i, \quad i \in M_3 \\
& \quad x_j \geq 0, \quad j \in N_1 \\
& \quad x_j \leq 0, \quad j \in N_2
\end{align*}$$

Note: maximizing $c'x$ is equivalent to minimizing $(-c)'x$.

A few names:

- $x_1, \ldots, x_n$ — decision variables
- If $j \notin N_1 \cup N_2$, $x_j$ is unrestricted, or free
- Vector $x = (x_1, \ldots, x_n)$ satisfying all the constraints — a feasible solution, or feasible vector, or feasible point
- Set of all feasible solutions — feasible region, or feasible set
- The function $c'x$ — objective, or cost, function
- A feasible solution $x^*$ that minimizes the objective function over the feasible region — an optimal solution
  - i.e., $c'x^* \leq c'x$ for any feasible $x$
- The value $c'x^*$ — optimal cost, or optimal (objective) value

“Pathological” cases:

- If no feasible solutions exist, we have an infeasible problem
- If for every $K \in \mathbb{R}$ there exists a feasible $x$ such that $c'x < K$, the optimal cost is $-\infty$, or unbounded below (or the problem is unbounded)

We will show that a feasible LP that is not unbounded has an optimal solution
Recall: Idealized model of radiation treatment planning

Given the set $S$ of pixels partitioned into tumor ($T$), critical structure ($C$), and normal tissue ($N$), determine the beamlet intensities $w \in \mathbb{R}^P$, such that the resulting dose delivered to the tumor is at least $\gamma_T$, to the critical structure — at most $\gamma_C$, and to the normal tissue — at most $\gamma_N$, and the total dose is minimized.

$$\min_{w,D} \sum_{j \in S} D_j$$

s.t.

$$D_j = \sum_{p=1}^P A_{j,p} w_p, \quad j \in S$$

$$w_p \geq 0, \quad p = 1, \ldots, P$$

$$D_j \geq \gamma_T, \quad j \in T$$

$$D_j \leq \gamma_C, \quad j \in C$$

$$D_j \leq \gamma_N, \quad j \in N$$

Problem: this LP will be infeasible

Radiation Treatment Planning: modified model I

- Idea: impose bounds only on dose to tumor pixels; penalize dose to other pixels
- $\gamma_L^T, \gamma_U^T$ — lower and upper bounds on dose to the tumor pixels
- $\theta_C, \theta_N > 0$ — pre-specified parameters, a.k.a. weights;

$$\min_{w,D} \theta_C \sum_{j \in C} D_j + \theta_N \sum_{j \in N} D_j$$

s.t.

$$D_j = \sum_{p=1}^P A_{j,p} w_p, \quad j \in S$$

$$w_p \geq 0, \quad p = 1, \ldots, P$$

$$\gamma_L^T \leq D_j \leq \gamma_U^T, \quad j \in T$$

$$w_p \leq 2 \cdot \frac{1}{P} \sum_{p=1}^P w_p, \quad p = 1, \ldots, P$$

Problem: sometimes, it is necessary to “underdose” a part of the tumor to avoid serious side effects
Modified model II

- Idea: for each pixel \( j \in S \) specify desired, or target, dose \( t_j \), and attempt to meet these targets
- Penalty for both under- and over-dosing (i.e., going below or over \( t_j \))

\[
\min_{w,D} \max_{j \in S} |D_j - t_j|
\]

s.t. \( D_j = \sum_{p=1}^{P} D_{j,p} w_p, \quad j \in S \)

\( w_p \geq 0, \quad p = 1, \ldots, P \)

Objective is not a linear function! It is, however, a piecewise linear convex function.

Convex functions

Definition 1.1

(a) A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called convex if \( \forall x, y \in \mathbb{R}^n \), and \( \forall \lambda \in [0, 1] \),

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

(b) A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called concave if \( \forall x, y \in \mathbb{R}^n \), and \( \forall \lambda \in [0, 1] \),

\[
f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)
\]

Theorem 1.1

Let \( f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R} \) be convex functions. Then, the function \( f \) defined by \( f(x) = \max_{i=1,\ldots,m} f_i(x) \) is convex.
Using LP to solve problems with PWLC cost functions

**Definition: PWLC function**

Let $c_1, \ldots, c_m \in \mathbb{R}^n$ and $d_1, \ldots, d_m \in \mathbb{R}$. 

$$f(x) = \max_{i=1,\ldots,m}(c_i^T x + d_i)$$

is a *piecewise linear convex function*

$$\min_x \ \max_{i=1,\ldots,m}(c_i^T x + d_i)$$

s.t. \ linear constraints on $x$

is equivalent to:

$$\min_{x,z} \ \ z$$

s.t. \ \ $z \geq c_i^T x + d_i, \ i = 1, \ldots, m$

linear constraints on $x$

Two mathematical programs, $(P1)$ and $(P2)$ are *equivalent* if

- They are either both infeasible, or have the same optimal cost, and
- Given an optimal solution of $(P1)$, we can (easily) calculate an optimal solution of $(P2)$, and vice versa