Recap for Lecture 11: LPs in standard form

\[
\begin{align*}
\min & \quad c'x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Assumptions and notation:
- \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \)
- \( Ax = b \) has at least one solution
- \( A \) has full row rank (and so \( m \leq n \))
- Let \( A_j \) denote the \( j \)th columns of \( A \)
- Let \( P = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \} \)

From studying LP geometry, we know...
- Assuming the above LP is feasible, \( P \) has one (or several) extreme points
- If the above LP is not unbounded, there is at least one optimal solution found at an extreme point (a.k.a. Basic Feasible Solution) of \( P \)

Idea for an LP-solving algorithm

- First idea: enumerate all BFSs and pick the one with the best objective value
- Issues:
  - A lot of potential BSs ("n-choose-m"); a priori cannot tell which ones are BFSs
  - What if the problem is unbounded?!
- Need: a method that
  - Looks at BFSs in a "smart" order (i.e., improving)
  - Has a smart way of detecting
    - When the current BFS is optimal, or
    - When the LP is unbounded
Feasible directions

**Definition 3.1**

Let $x$ be an element of a polyhedron $P$. A vector $d \in \mathbb{R}^n$ is said to be a **feasible direction** at $x$, if there exists a positive scalar $\theta$ for which $x + \theta d \in P$.

**Intuition:** (see Exercise 3.2)

- If $x \in P$ is not an optimal solution, there should be an “improving” feasible direction at $x$.
- If $x \in P$ is an optimal solution, there are no directions that are both improving and feasible at $x$.

What can we say about feasible directions at $x$ which is a BFS?

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Review: Terminology of BSs for standard form systems

- In a standard form LP, BSs are associated with **bases**.
- $B(1), \ldots, B(m)$ — indices characterizing the basis.
- Columns $A_{B(1)}, \ldots, A_{B(m)}$ — **basic columns**; they form a basis of $\mathbb{R}^m$.

$$
B = \begin{bmatrix}
A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)}
\end{bmatrix} \in \mathbb{R}^{m \times m} \text{ is the basis matrix}
$$

- Variables $x_B = (x_{B(1)}, \ldots, x_{B(m)})'$ — **basic variables**; the remaining variables are **nonbasic**.
- At a BS associated with this basis:
  - Values of non-basic variables $x_j$, $j \neq B(1), \ldots, B(m)$ are 0.
  - Values of basic variables are determined by the unique solution of $Bx_B = b$: $x_B = B^{-1}b$
    - If $x_{B(i)} \neq 0$ for all $i$, this BS is non-degenerate (and has a unique basis associated with it).
    - If $x_{B(i)} = 0$ for some $i$, this BS is degenerate, and may have several different bases associated with it.
    - If $x_B = B^{-1}b \geq 0$, this BS is a BFS.
Basic directions

- Let \( x \) be a BFS, with \( B(1), \ldots, B(m) \) being the indices of the basic variables
- Idea: move to an adjacent BFS (more specifically, to an adjacent basis)
  - Recall: adjacent basis is one which has exactly one different index/column of \( A \)

The \( j \)th basic direction:

- Let \( j \neq B(1), \ldots, B(m) \)
- Move from \( x \) to \( x + \theta d \), where
  - \( d_j = 1 \)
  - \( d_i = 0 \) for all \( i \neq j, B(1), \ldots, B(m) \)
  - Need \( A(x + \theta d) = b \), i.e., \( Ad = 0 \)
    - \( 0 = Ad = \sum_{i=1}^{n} A_i d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j = Bd_B + A_j \)
    - \( d_B = -B^{-1}A_j \)

Properties of the \( j \)th basic direction at \( x \)

\[
d_j = 1, \quad d_i = 0 \text{ for } i \neq j, B(1), \ldots, B(m), \quad d_B = -B^{-1}A_j
\]

Is it a feasible direction?

- Need: \( x + \theta d \geq 0 \) for some positive \( \theta \)
  - Only need to consider the changes in the basic variables: \( x_B + \theta d_B \)
- If \( x \) is a nondegenerate BFS, \( x_B > 0 \), and \( d \) is always a feasible direction
- If \( x \) is a degenerate BFS, \( d \) may not be a feasible direction
  - If \( x_{B(i)} = 0 \) and \( d_{B(i)} < 0 \)

Is it an improving direction?

- Change in cost as we move from \( x \) to \( x + \theta d \):
  \[
c'(x + \theta d) - c'x = \theta c'd = \theta(c_j + c'_B d_B) = \theta(c_j - c'_B B^{-1}A_j)
\]
- Denote \( \bar{c}_j = c_j - c'_B B^{-1}A_j \)
- If \( \bar{c}_j < 0 \), \( d \) is an improving direction
Reduced costs

**Definition 3.2**
Let \( x \) be a basic solution, let \( B \) be an associated basis matrix, and let \( c_B = (c_{B(1)}, \ldots, c_{B(m)})' \) be the vector of costs of the basic variables. For each \( j \), we define the **reduced cost** \( \bar{c}_j \) of the variable \( x_j \) according to the formula 
\[
\bar{c}_j = c_j - c_B' B^{-1} A_j.
\]

Note:
- Reduced costs \( \bar{c} \) are associated with a particular basis \( B \).

Interpretation:
- For a non-basic index \( j \), \( \bar{c}_j \) represents the rate of cost change along the \( j \) basic direction.
- For a basic index \( B(i) \), \( \bar{c}_{B(i)} = c_{B(i)} - c_B' B^{-1} A_{B(i)} = 0 \)

Optimality conditions

**Theorem 3.1**
Consider a basic feasible solution \( x \) associated with a basis matrix \( B \), and let \( \bar{c} \) be the corresponding vector of reduced costs.
(a) If \( \bar{c} \geq 0 \), then \( x \) is optimal.
(b) If \( x \) is optimal and nondegenerate, then \( \bar{c} \geq 0 \).

Proof of (a):
- Let \( y \) be any feasible solution, and \( d = y - x \) (feasible, but not necessarily basic, direction).
- \( Ad = 0 \) implies (\( N \) — set of non-basic indices)
  \[
  Bd_B + \sum_{i \in N} A_i d_i = 0 \Rightarrow d_B = -\sum_{i \in N} (B^{-1} A_i) d_i
  \]
- \( c'd = c_B' d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c_B' B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i \)
- For any \( i \in N \), \( x_i = 0 \), and hence \( d_i \geq 0 \)
- Conclusion: \( c'y \geq c'x \) for any feasible \( y \).
Optimality conditions

**Theorem 3.1**
Consider a basic feasible solution \( x \) associated with a basis matrix \( B \), and let \( \bar{c} \) be the corresponding vector of reduced costs.

(a) If \( \bar{c} \geq 0 \), then \( x \) is optimal
(b) If \( x \) is optimal and nondegenerate, then \( \bar{c} \geq 0 \).

**Proof of (b):**
- Sup. \( x \) is optimal and non-deg., but \( \bar{c}_j < 0 \) for some \( j \in N \)
- \( j \)th basic direction is a feasible direction of cost decrease — contradiction!

**Definition 3.3**
A basis matrix \( B \) is said to be **optimal** if:

(a) \( B^{-1}b \geq 0 \), and
(b) \( \bar{c}' = c' - c'_B B^{-1}A \geq 0' \).

Note: An optimal \( B \) corresponds to an optimal BFS, but an optimal degenerate BFS may be associated with a non-optimal \( B \).

Simplex method: Idea

Assume for now
- LP (in standard form) is feasible
- All its BFSs are non-degenerate
- We have found a starting basis and corresponding BFS

**Idea of the Simplex Method**
- Calculate \( \bar{c} \) at the current BFS
  - If \( \bar{c} \geq 0 \), stop — current basis is optimal (and hence current BFS is an optimal solution)
  - O/w, pick a \( j \) with \( \bar{c}_j < 0 \); move along the \( j \)th basic direction until an adjacent BFS with a better cost is reached; repeat.

**Questions:**
- How far do we move along a basic direction? Are we guaranteed to arrive at an adjacent BFS?
- Will this process terminate?
- What if the LP is unbounded?
- What about possible degeneracy?
- How do we find the initial BFS? (And what if the LP is not feasible?)