IOE 202: Lecture 4 Outline

- Announcements
- Last time...
- More examples of optimization problems and linear programming models

Last time

- Optimization problems and (mathematical) optimization models
  - Optimization model components: decision variables, objective function, constraints
  - Terminology: solution, feasible solution, optimal solution
- Linear Programming (LP) models — special kind of optimization model
  - Easy to represent and solve with computer packages, such as Excel solver
  - If a problem can be modeled with an LP model, usually this is the model of choice!
- Modeled and solved as LP models:
  - Pet food supplier’s problem
  - Monet picture frame manufacturer problem
Example: problem of optimal resource allocation

- The Monet company produces four types of picture frames, labeled A, B, C, and D. The table below lists the unit selling price Monet charges for each type of frame.
- Each type requires a certain amount of skilled labor, metal, and glass, as shown in the table. For production during the coming week, Monet can purchase up to 4000 hours of skilled labor, 6000 ounces of metal, and 10,000 ounces of glass. The unit costs are also indicated in the table.
- Also, market constraints are such that it is impossible to sell more than 1000 type-A frames, 2000 type-B frames, 500 type-C frames, and 1000 type-D frames.
- How many frames of each type should Monet produce during the coming week to maximize its profit?

Data (inputs) for the Monet production problem

<table>
<thead>
<tr>
<th>Frame type</th>
<th>Skilled labor</th>
<th>Metal</th>
<th>Glass</th>
<th>Selling price</th>
<th>Maximal production</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frame A</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>$28.50</td>
<td>1000</td>
</tr>
<tr>
<td>Frame B</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>$12.50</td>
<td>2000</td>
</tr>
<tr>
<td>Frame C</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>$29.25</td>
<td>500</td>
</tr>
<tr>
<td>Frame D</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>$21.50</td>
<td>1000</td>
</tr>
</tbody>
</table>

Max. amount of resource: 4000 hours, 6000 oz, 10,000 oz.
Resource Unit prices: $8.00 per hour, $0.50 per 1 oz, $0.75 per 1 oz.
Mathematical model for Monet:

Decision variables: \( x_A, x_B, x_C, x_D \) denote the number of frames A, B, C, and D to produce, respectively.

Mathematical model:

maximize \( 6x_A + 2x_B + 4x_C + 3x_D \)  
subject to \( 2x_A + x_B + 3x_C + 2x_D \leq 4000 \)  
\( 4x_A + 2x_B + x_C + 2x_D \leq 6000 \)  
\( 6x_A + 2x_B + x_C + 2x_D \leq 10000 \)  
\( x_A \leq 1000 \)  
\( x_B \leq 2000 \)  
\( x_C \leq 500 \)  
\( x_D \leq 1000 \)  
\( x_A, x_B, x_C, x_D \geq 0 \)

This model is also a linear programming model, since all constraints are linear, and the objective function is a linear function.

Some comments

- An alternative (but equivalent) formulation can be constructed by also including variables to represent the amount of labor, metal, and glass purchased. Objective function and constraints could be expressed in terms of this variables.
- Both integer and non-integer levels of production were allowed in our formulation. (Is it a reasonable assumption?)
- Next time, we will discuss the changes in the models/solution methods when the last assumption is not reasonable; for now, let us allow variables to be non-integer.
Optimal solution of the Monet problem

- Optimal solution:
- Optimal profit:
- Which inequality constraints are “tight” (i.e., hold as equalities) at the optimal solution?

An alternative way to characterize the optimal solution: “Do not make any frames of type D. Produce as many frames of type A as you can sell. Use up all available labor and metal.”

Perturbation Theorem

Above, we characterized the optimal solution to the Monet problem by identifying which constraints were tight, and which were slack at that optimal solution. This characterization comes in handy if the data of the problem turns out to be slightly different than we assumed when we formulated the problem.

Perturbation “Theorem”
If the data of the linear program are perturbed by small amounts, its optimal solution can change, but its tight constraints stay tight, and its slack constraints stay slack.
Solving LPs graphically

max \quad 3x_1 + 5x_2
\quad \leq 4
\quad 2x_2 \leq 12
\quad 3x_1 + 2x_2 \leq 18.
\quad x_1, \quad x_2 \geq 0

Demystifying the perturbation theorem

max \quad 3x_1 + 5x_2
\quad \leq 4
\quad 2x_2 \leq 12
\quad 3x_1 + 2x_2 \leq 21
\quad x_1, \quad x_2 \geq 0
Shipping model

A forest production company manufactures plywood in three plants located in different timber zones, and ships it to four depots. Each plant has a monthly production capacity, and each depot has a monthly demand (these demands must be satisfied exactly). The table below specifies the capacities, demands, and the unit shipping costs from each plant to each depot. The cost of producing plywood has been omitted since it is the same in each plant. What is the cheapest way to ship plywood to the depots each month?

<table>
<thead>
<tr>
<th>Plant</th>
<th>Depot 1</th>
<th>Depot 2</th>
<th>Depot 3</th>
<th>Depot 4</th>
<th>Capacity (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant 1</td>
<td>$4</td>
<td>$7</td>
<td>$3</td>
<td>$5</td>
<td>2500</td>
</tr>
<tr>
<td>Plant 2</td>
<td>$10</td>
<td>$9</td>
<td>$3</td>
<td>$6</td>
<td>4000</td>
</tr>
<tr>
<td>Plant 3</td>
<td>$3</td>
<td>$6</td>
<td>$4</td>
<td>$4</td>
<td>3500</td>
</tr>
<tr>
<td>Demand (units)</td>
<td>2000</td>
<td>3000</td>
<td>2500</td>
<td>1500</td>
<td></td>
</tr>
</tbody>
</table>

2This is the problem considered in Section 4.5 of Denardo

Representation of the shipping model

Plant    Depot

1 ➔ 1

2 ➔ 2

3 ➔ 4

This is the problem considered in Section 4.5 of Denardo

IOE 202: Operations Modeling, Fall 2009  Page 11
Operational decisions in the shipping problem

- What decisions do you need to make?

- What performance measure are you using to compare different decisions?

- What constraints (restrictions) must your decisions satisfy?

- What assumptions are being made?

Formulation of a mathematical model for the shipping problem

Decision variables: represent decisions by variables.

Objective function: express the performance criterion in terms of the decision variables; should it be minimized or maximized?
Formulation of a model for the shipping problem – cont.

Constraints: express all (explicit and implicit) constraints and restrictions on the values of the decision variables.

Optimal solution:

A blending model

After last week’s bakeoff, you have a few ingredients left over (see table below), and you have decided to use them to make candy to sell to spectators at a minor league ballgame this weekend. You are considering producing two types of candies: “Easy Out” and “Slugger,” both of which consist solely of sugar, nuts, and chocolate.

<table>
<thead>
<tr>
<th>Ingredient</th>
<th>Sugar</th>
<th>Nuts</th>
<th>Chocolate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount available</td>
<td>10,000 oz</td>
<td>2,000 oz</td>
<td>3,000 oz</td>
</tr>
</tbody>
</table>

The mixture used to make Easy Out must contain at least 20% nuts, while the mixture used to make Slugger must contain at least 10% nuts and 10% chocolate. Each ounce of Easy Out can be sold for $0.50, and each ounce of Slugger can be sold for $0.40 — how can you maximize revenue?

Note that there is quite a bit of flexibility in the recipes for the candy!

For a similar model, see Section 4.3 of Denardo
Operational decisions in the candy-making business

- What decisions do you need to make?

- What performance measure are you using to compare different decisions?

- What constraints (restrictions) must your decisions satisfy?

- What assumptions are being made?
Formulation of a mathematical model for candy-making

**Decision variables:** represent decisions by variables.

**Objective function:** express the performance criterion in terms of the decision variables; should it be minimized or maximized?

Formulation of a model for candy-making – cont.

**Constraints:** express all (explicit and implicit) constraints and restrictions on the values of the decision variables.

Optimal solution:
Postal employee scheduling

A post office requires different numbers of employees on different days of the week. The number of employees required is as follows:

<table>
<thead>
<tr>
<th>Day</th>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
<th>Sat</th>
<th>Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min. required</td>
<td>17</td>
<td>13</td>
<td>15</td>
<td>19</td>
<td>14</td>
<td>16</td>
<td>11</td>
</tr>
</tbody>
</table>

Union rules state that each employee must work 5 consecutive days and then receive 2 days off. For example, an employee might work Wednesday through Sunday, and be off Monday and Tuesday. The post office wants to minimize the number of employees it needs to hire while meeting the daily requirements.

Note: Each employee hired needs to be assigned to one of the seven schedules, or “tours.”

---

4Similar to the problem discussed in Section 4.8 of Denardo

Operational decisions in the scheduling problem

- What decisions do you need to make?

- What performance measure are you using to compare different decisions?

- What constraints (restrictions) must your decisions satisfy?

- What assumptions are being made?
Formulation of a mathematical model for the scheduling problem

**Decision variables:** represent decisions by variables.

**Objective function:** express the performance criterion in terms of the decision variables; should it be minimized or maximized?

---

Formulation of a model for the scheduling problem – cont.

**Constraints:** express all (explicit and implicit) constraints and restrictions on the values of the decision variables.

<table>
<thead>
<tr>
<th>Starting day of the schedule</th>
<th>Mo</th>
<th>Tu</th>
<th>We</th>
<th>Th</th>
<th>Fr</th>
<th>Sa</th>
<th>Su</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mo</td>
<td>≥1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tu</td>
<td></td>
<td>≥1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>We</td>
<td></td>
<td></td>
<td>≥1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Th</td>
<td></td>
<td></td>
<td></td>
<td>≥1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fr</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>≥1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>≥1</td>
<td></td>
</tr>
<tr>
<td>Su</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>≥1</td>
</tr>
</tbody>
</table>

**Optimal solution:**
An investment model

Presently, you have $1,000 to invest. Cash flows associated with 5 available investments are shown in the table; you can put no more than $500 in any investment. In addition to these investments, you can invest as much money as you want into 12-month CDs, which pay 6% interest. How should you invest to maximize your cash at hand at the end of year 3?

<table>
<thead>
<tr>
<th>Investment</th>
<th>Now</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-$1.00</td>
<td></td>
<td>-$1.40</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>-$1.00</td>
<td>+$1.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>-$1.00</td>
<td></td>
<td>+$1.28</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>-$1.00</td>
<td></td>
<td></td>
<td>+$1.15</td>
</tr>
<tr>
<td>E</td>
<td>-$1.00</td>
<td></td>
<td></td>
<td>+$1.32</td>
</tr>
</tbody>
</table>

Note: each year, you can only invest cash available on hand!

---

Although presented in a different setting, the model in this problem turns out to be similar to the Activity Analysis models of Section 4.10 of Denardo

Operational decisions in the investment problem

- What decisions do you need to make?

- What performance measure are you using to compare different decisions?

- What constraints (restrictions) must your decisions satisfy?

- What assumptions are being made?
Formulation of a mathematical model for the investment problem

**Decision variables:** represent decisions by variables.

**Objective function:** express the performance criterion in terms of the decision variables; should it be minimized or maximized?

**Constraints:** express all (explicit and implicit) constraints and restrictions on the values of the decision variables.

**Optimal solution:**