IOE 202: lecture 3 outline

- Announcements
- Last time...
- Optimization problems and optimization models
- Another example of a Linear Programming model: Monet picture frames
- Linear Programming (LP) optimization models — brief history
- Solving small LP models graphically and interpreting solutions

Last time

- Extension of the EOQ model for inventory management: quantity discounts
- Pet food supplier’s inventory problem
  - Does not fit EOQ framework; different model needed
  - Identified
    - Decisions that need to be made
    - Performance measure used to evaluate solutions
    - Restrictions/constraints on decisions
    - Simplifying assumptions made in the modeling process
  - Started formulating a model of the problem
Another inventory management problem

- You are the Michigan distributor of Nature’s Peak, a high-end brand of frozen dog food. You have (pre-paid) contracts with local “boutique” pet stores to deliver, in each of the next 4 months, respectively, 50, 65, 100, and 70 lb of food, and these orders must be filled on time.
- You obtain the food from the manufacturer at wholesale prices which vary month to month. In the next four months, unit prices are $5, $8, $4, and $7 per pound, respectively, and you can buy at most 80 lb each month.
- Food needs to be delivered to the stores at the end of each month. You place your order with the manufacturer in the beginning of each month, receive your order at the end of the month, and immediately deliver food to the local stores.
- If you have food remaining after the demand has been satisfied, you can keep some of it in your warehouse at a cost of $2 per pound per month until the next delivery, and donate the rest to the Humane Society.
- In 4 months, Nature’s Peak is planning to change the recipe and packaging for this food. If you have any food left at that time, you can sell it to discount pet stores in the area for $6 per pound. (Until then, the company wants to maintain the product’s high-end image.)

How should you manage your inventory for the next 4 months?

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Section 4.12 of Denardo describes a problem in which such inventory management issues are part of the decisions.

Operational decisions in the pet food distributor’s problem

- What decisions do you need to make for the coming 4 months?
  - How many pounds of food to order in each of the next 4 months
  - How many pounds of food to store in each of the next 3 months, and how much to sell after 4th month
  - How many pounds of food to donate in each of the next 4 months

- What performance measure would you use to compare different decisions?
  - Net cost (Ordering and holding costs, minus the revenue from resale at the end) — to be minimized

- What constraints (restrictions) must your decisions satisfy?
  - Ordering capacity of 80 lb each month
  - Demand must be met each month

- What assumptions are being made?
  - We assumed all parameters given are known with certainty
  - We assumed that there are no setup ordering costs, and no cost or financial benefit associated with donations
Formulation of a mathematical model for the pet food distributor’s problem

Decision variables: represent decisions by variables.
- $x_1$, $x_2$, $x_3$, $x_4$: lbs of food ordered in each of the 4 months
- $s_1$, $s_2$, $s_3$, $s_4$: lbs of food stored/sold at the end of each of the 4 months
- $y_1$, $y_2$, $y_3$, $y_4$: lbs of food donated in each of the 4 months

Constraints: express all (explicit and implicit) constraints and restrictions on the values of the decision variables.

Formulation of a model for the pet food distributor’s – cont.

Objective function: express the performance criterion in terms of the decision variables; should it be minimized of maximized?
- Minimize $5x_1 + 8x_2 + 4x_3 + 7x_4 + 2(s_1 + s_2 + s_3) - 6s_4$
Mathematical model for pet food distributor’s problem

Decision variables:
- $x_1, x_2, x_3, x_4$: lbs of food ordered in each of the 4 months
- $s_1, s_2, s_3, s_4$: lbs of food stored/sold at the end of each of the 4 months
- $y_1, y_2, y_3, y_4$: lbs of food donated in each of the 4 months

Mathematical model:

minimize $\ 5x_1 + 8x_2 + 4x_3 + 7x_4$
$\quad + 2(s_1 + s_2 + s_3) - 6s_4$  \hspace{1cm} (Net) cost objective
subject to
- $x_1 = 50 + s_1 + y_1$  \hspace{1cm} Inventory balance in month 1 constraint
- $x_2 + s_1 = 65 + s_2 + y_2$  \hspace{1cm} Inventory balance in month 2 constraint
- $x_3 + s_2 = 100 + s_3 + y_3$  \hspace{1cm} Inventory balance in month 3 constraint
- $x_4 + s_3 = 70 + s_4 + y_4$  \hspace{1cm} Inventory balance in month 4 constraint
- $x_1, x_2, x_3, x_4 \leq 80$  \hspace{1cm} Ordering capacity constraint(s)
- $s_1, s_2, s_3, s_4, y_1, y_2, y_3, y_4 \geq 0$  \hspace{1cm} Nonnegativity constraint(s)

Different, but equivalent, mathematical model for pet food problem

Decision variables:
- $x_1, x_2, x_3, x_4$: lbs of food ordered in each of the 4 months
- $s_1, s_2, s_3, s_4$: lbs of food stored/sold at the end of each of the 4 months

Mathematical model:

minimize $\ 5x_1 + 8x_2 + 4x_3 + 7x_4$
$\quad + 2(s_1 + s_2 + s_3) - 6s_4$  \hspace{1cm} (Net) cost objective
subject to
- $x_1 \geq 50 + s_1$  \hspace{1cm} Inventory balance in month 1 constraint
- $x_2 + s_1 \geq 65 + s_2$  \hspace{1cm} Inventory balance in month 2 constraint
- $x_3 + s_2 \geq 100 + s_3$  \hspace{1cm} Inventory balance in month 3 constraint
- $x_4 + s_3 \geq 70 + s_4$  \hspace{1cm} Inventory balance in month 4 constraint
- $x_1, x_2, x_3, x_4 \leq 80$  \hspace{1cm} Ordering capacity constraint(s)
- $x_1, x_2, x_3, x_4 \leq 80$  \hspace{1cm} Ordering capacity constraint(s)
- $x_1, x_2, x_3, x_4 \leq 80$  \hspace{1cm} Ordering capacity constraint(s)
- $s_1, s_2, s_3, s_4 \geq 0$  \hspace{1cm} Nonnegativity constraint(s)
Using a spreadsheet and Excel solver

We will use a spreadsheet to obtain the solution of this model. Components of such a spreadsheet:

- **Inputs** The data needed to form the objective and constraints
- **Changing cells** Use designated cells whose values will play the role of the decision variables. In particular, the component of Excel that finds the optimal (best) values of variables will change the contents of these cells.
- **Target (objective) cell** This cell will contain the formula for computing the value of objective function, referencing the changing cells.
- **Constraints** Will be specified in the Solver dialog box. Cells with expressions for left and right hand sides need to be prepared in the spreadsheet referencing the changing cells.
- **Nonnegativity constraints** Can be specified by checking a box in the Solver dialog (Also, check “Assume Linear Model”)

Suggestion: use references to input cells, rather than numbers, as much as possible in all formulas.

Optimal solution of pet food distributor’s problem

- **Optimal solution:**

- **Optimal (net) cost:**
Optimization problems

- **Optimization problems**: problems involving deciding which actions to select among all feasible ones to achieve the best (or optimal) performance, as measured by a specified performance criterion.
  - Pet food distributor’s problem
  - Deciding on the size of a production batch in TV speakers problem
  - Choose the least-cost daily diet among all those that satisfy dietary (and taste!) requirements.
  - Choose the shortest-distance route from home to school.
  - In a production facility, choose the most profitable combination of products to manufacture from the available raw materials.
  - Others?

- **Optimization models**: prescriptive mathematical models of optimization problems

Terminology of optimization models – I

- **Decision variables**: the quantities that can vary; we often call them simply variables
  - Represent quantitative decisions that need to be made
- **Objective function**: the expression that is being minimized of maximized
  - Represents the performance measure
- **Constraints**: equations and inequalities that the decision variables must satisfy
  - Represent restrictions on decisions being made
Linear functions and constraints

- Examples of linear functions:
  \[ 17x_A - 57x_B + 91x_C \text{ or } A - 3.4B + 2C. \]

  The variables of the first expression are \(x_A, x_B, x_C\), and of the second — \(A, B, C\). Both functions depend on their respective variables linearly.

- A linear constraint requires a linear function to be equal, greater-than-or-equal-to, or less-than-or-equal-to, a number:
  \[ 2A - 3B = 6 \text{ and } A - 3.4B + 2C \leq -2 \text{ and } C \geq 0. \]
  (We do not consider the inequality\[ 17x_A - 57x_B + 91x_C > 12 \]
to be a correct form of a constraint!)

- A linear programming model is an optimization model with a linear objective function and linear constraints

Terminology of optimization models – II

- **Solution**: an assignment of values to the decision variables

- **Feasible solution**: an assignment of values to the variables that satisfies all of the constraints
  - \(x_1 = x_2 = x_3 = x_4 = 80, s_1 = s_2 = s_3 = s_4 = 0\) is a solution, but it is not feasible
  - \(x_1 = x_2 = x_3 = x_4 = 80, s_1 = 30, s_2 = 45, s_3 = 25, s_4 = 35\) is a feasible solution; objective value $1,910.00

- **Optimal solution**: feasible solution with the best objective value among all feasible solutions
Example: problem of optimal resource allocation

- The Monet company produces four types of picture frames, labeled A, B, C, and D. The table below lists the unit selling price Monet charges for each type of frame.
- Each type requires a certain amount of skilled labor, metal, and glass, as shown in the table. For production during the coming week, Monet can purchase up to 4000 hours of skilled labor, 6000 ounces of metal, and 10,000 ounces of glass. The unit costs are also indicated in the table.
- Also, market constraints are such that it is impossible to sell more than 1000 type-A frames, 2000 type-B frames, 500 type-C frames, and 1000 type-D frames. How many frames of each type should Monet produce during the coming week to maximize its profit?

Data (inputs) for the Monet production problem

<table>
<thead>
<tr>
<th>Frame type</th>
<th>Skilled labor</th>
<th>Metal</th>
<th>Glass</th>
<th>Selling price</th>
<th>Maximal production</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frame A</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>$28.50</td>
<td>1000</td>
</tr>
<tr>
<td>Frame B</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>$12.50</td>
<td>2000</td>
</tr>
<tr>
<td>Frame C</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>$29.25</td>
<td>500</td>
</tr>
<tr>
<td>Frame D</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>$21.50</td>
<td>1000</td>
</tr>
<tr>
<td>Max. amount of resource</td>
<td>4000 hours</td>
<td>6000 oz</td>
<td>10,000 oz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Resource Unit prices</td>
<td>$8.00 per hour</td>
<td>$0.50 per 1 oz</td>
<td>$0.75 per 1 oz</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Operational decisions at Monet

- What decisions do managers at Monet need to make for the coming week?

- What performance measure is the company using to compare different decisions?

- What constraints (restrictions) must their decisions satisfy?

Formulation of a mathematical model for Monet

**Decision variables:** represent decisions by variables.

**Objective function:** express the performance criterion in terms of the decision variables; should it be minimized or maximized.
Formulation of a model for Monet – cont.

**Constraints:** express all (explicit and implicit) constraints and restrictions on the values of the decision variables.

Mathematical model for Monet:

**Decision variables:** \( x_A, x_B, x_C, x_D \) denote the number of frames A, B, C, and D to produce, respectively.

**Mathematical model:**

\[
\begin{align*}
\text{maximize} & \quad 6x_A + 2x_B + 4x_C + 3x_D \\
\text{subject to} & \quad 2x_A + x_B + 3x_C + 2x_D \leq 4000 \\
& \quad 4x_A + 2x_B + x_C + 2x_D \leq 6000 \\
& \quad 6x_A + 2x_B + x_C + 2x_D \leq 10000 \\
& \quad x_A \leq 1000 \\
& \quad x_B \leq 2000 \\
& \quad x_C \leq 500 \\
& \quad x_D \leq 1000 \\
& \quad x_A, x_B, x_C, x_D \geq 0
\end{align*}
\]

This model is also a *linear programming model*, since all constraints are linear, and the objective function is a linear function.
Some comments

- An alternative (but equivalent) formulation can be constructed by including variables to represent the amount of labor, metal, and glass purchased. Objective function and constraints could be expressed in terms of this variables.
- Both integer and non-integer levels of production were allowed in our formulation. (Is it a reasonable assumption?)
- Next week, we will discuss the changes in the models/solution methods when the last assumption is not reasonable; for now, let us allow variables to be non-integer.

Linear Programming (LP) history

Linear Programming (LP) models constitute a special class of optimization models.

Some history:
- Theoretical tools for solving systems of linear equations and inequalities, too calculation-intense
- 1930’s - early 1940’s — first applications of LP models to specific problems in production planning (L. Kantorovich, Soviet Union) and transportation planning (T. Koopmans, Netherlands/USA) (“programming” means “planning”)
- Mid-1940’s — development of an algorithm (the Simplex Method) capable of finding an optimal solution of any LP model by G. Dantzig coincides with development of computers, enabling broad practical applications.
- Late-1940’s - today: LP models are increasingly used in many applications; simultaneously, new algorithms for solving LPs, and software implementing these algorithms, are developed, as larger models need to be solved.
LP today

Used in all types of organizations, with applications in:
efficient resource allocation, military operations planning,
production and inventory planning, capacity expansion,
manufacturing process design, staff scheduling, location planning,
traffic routing, supply chain management, economic game theory,
airline crew and plane scheduling, telecommunication capacity
allocation and network design, medical treatment planning, image
reconstruction, publishing (typesetting), finance (asset allocation),
mathematics (as a proof technique and computational method),
data analysis, pattern classification, optimal control, mechanical
structure design, electromagnetic antennae design, etc.

Optimal solution of the Monet problem

- Optimal solution:
- Optimal profit:
- Which inequality constraints are “tight” (i.e., hold as
  equalities) at the optimal solution?

An alternative way to characterize the optimal solution:
“Do not make any frames of type D. Produce as many frames of
type A as you can sell. Use up all available labor and metal.”