IOE 202: Lecture 2 outline

- Announcements
- Last time...
- Inventory management problems and models:
  - Economic Order Quantity models: continued
  - A different inventory management situation (and a Linear Programming model)

Last time

- Problems of maintaining and replenishing *inventory*
  - One of the issues: balancing holding costs vs. ordering and shortage costs
- First example: a problem with
  - Long-term planning:
    - Known, steady demand
    - Known costs (setup and per unit ordering, holding) that do not change over time
    - Known lead time that does not change over time
  - No (planned) shortages allowed
  - Continuous review
- Approach to managing inventory:
  - Select a batch size $Q$ items
  - Order a batch of size $Q$ just as you are about to run out
- What value of $Q$ maximizes net profits?
  - What is the “Economic Order Quantity”?
Inventory level over time

Inputs and outputs of EOQ models

<table>
<thead>
<tr>
<th>Definition</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Demand per unit of time</td>
</tr>
<tr>
<td>$L$</td>
<td>Lead time</td>
</tr>
<tr>
<td>$K$</td>
<td>Setup cost for ordering one batch</td>
</tr>
<tr>
<td>$c$</td>
<td>Cost for purchasing one unit</td>
</tr>
<tr>
<td>$h$</td>
<td>Holding cost per unit</td>
</tr>
<tr>
<td>$Q$</td>
<td>Order Quantity (batch size)</td>
</tr>
<tr>
<td>$Q/a$</td>
<td>Time between orders</td>
</tr>
<tr>
<td>$T(Q)$</td>
<td>Cost per unit of time</td>
</tr>
</tbody>
</table>

Note: sales revenue does not depend on ordering policy, as long as we never run out of inventory. So, to maximize net profit, we simply need to minimize the cost of ordering and holding inventory.
One ordering cycle: details

- Cycle duration = \( \frac{Q}{a} \) units of time

- Production/ordering cost per cycle = \( K + cQ \) dollars

- Holding cost per cycle =

- Total cost per cycle =

Descriptive model: Expression of cost

Thus, the cost per unit of time is:

\[
T(Q) = \frac{\text{Total cost per unit time}}{\text{Duration of the cycle}} = \frac{aK}{Q} + ac + \frac{hQ}{2}
\]

This formula is a descriptive model: what happens when ordering quantity is \( Q \)?
Prescriptive model: EOQ

- We want a prescriptive model: what value of $Q$ is optimal, i.e., the best?
- Optimization problem: “minimize $T(Q)$ over all $Q \geq 0$"

To find the minimum, compute the derivative of $T(Q)$ and set it to 0:

$$T(Q) = \frac{aK}{Q} + ac + \frac{hQ}{2}, \text{ so } T'(Q) = -\frac{aK}{Q^2} + \frac{h}{2}$$

The value of $Q$ that minimizes the annual inventory ordering and holding cost:

$$Q^* = \sqrt{\frac{2aK}{h}}$$

(the “Economic Order Quantity”)

Time between orders:

$$t^* = \frac{Q^*}{a}$$

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Back to CubicleMin’s problem:

Parameters (with units):
- $a = 100$ cameras per month $= 1200$ cameras per year
- $L = 1$ week
- $K = $35
- $c = $100 per camera
- $h = $10 per camera per year

Hence, the optimal solution is to set up the facility to order cameras in batches of

$$Q^* = \sqrt{\frac{2aK}{h}} = \sqrt{\frac{2 \cdot 1200 \cdot 35}{10}} = 91.6$$

once every

$$t^* = \frac{Q^*}{a} = \frac{91.6}{1200} = 0.076 \text{ years} \approx 4 \text{ weeks}$$

We should place an order when the inventory level is $\approx 23$ cameras
Example: Manufacturing speakers for TV sets
A TV manufacturing company produces its own speakers, which are then used in the production of its TV sets. The TV sets are produced on a continuous production line at a rate of 8,000 per month, and each set needs one speaker.
The speakers are produced in batches, because relatively large quantities can be produced in a short time. Therefore, speakers are placed into inventory until they are needed for assembly into TV sets. The company is interested in determining when to produce a batch of speakers and how many to produce in each batch.
Several costs must be considered in making the above decision:

- Each time a batch of speakers is produced, a setup cost of $12,000 is incurred
- The unit production cost of a single speaker (excluding the setup cost) is $10, independent of the batch size produced
- The estimated holding cost of keeping a speaker in stock is $0.30 per month

Solution of the TV speaker problem
EOQ model is appropriate in this problem
Parameters:

- \( a = 8,000 \) units per month (demand rate)
- \( L = 0 \) (lag time between order and delivery)
- \( K = 12,000 \) (fixed/setup cost for producing a batch)
- \( c = 10 \) per unit (unit ordering cost)
- \( h = 0.30 \) per unit per month (unit holding cost per unit of time)

Hence, the optimal solution is to set up the facility to produce speakers in batches of

\[
Q^* = \sqrt{\frac{2aK}{h}} = \sqrt{\frac{2 \times 8000 \times 12000}{0.30}} = 25,298
\]

Once every \( t^* = \frac{25298}{8000} = 3.2 \) months.
Total production/storage cost of speakers

\[
T(Q^*) = \frac{aK}{Q^*} + ac + \frac{hQ^*}{2} = 87589.47 \text{ per month}
\]
EOQ extension: Ordering with quantity discounts

- So far, we assumed that the unit cost of an item is the same regardless of quantity in the batch. In fact, as a result, the optimal order quantity and frequency are independent of this unit cost.
  - That is not the case, for example, if we take into account the time value of money that is tied up in inventory. If the interest rate is $i\%$ per unit of time, than $Q^* = \sqrt{\frac{2aK}{h+i}}$. See homework for derivation of this formula.
  - Often, however, discounts and economies of scale are available if large batches are ordered.
  - In the TV example, suppose that the unit cost of every speaker is
    - $c_1 = $11 if fewer than 10,000 speakers are produced,
    - $c_2 = $10 if production is at least 10,000, but fewer than 80,000 speakers, and
    - $c_3 = $9 if production is 80,000 speakers or more.
What is the optimal inventory policy?

Quantity discounts, step I

- From our discussion of the EOQ model, the total cost per month if the unit cost was $c_j$ can be computed as follows:

$$T_j(Q) = \frac{aK}{Q} + ac_j + \frac{hQ}{2}, \text{ for } j = 1, 2, 3.$$

- Let’s plot these three curves!
- The value of $Q$ that minimizes $T_j(Q)$ can be found by using the basic EOQ model: $Q^* = \sqrt{\frac{2aK}{h}} = 25,298$.
- $Q^*$ is the same for all the curves.
  - This is because the cost of capital is not taken into account, and so the effect of different values of inventory is not felt. If the cost of capital is not 0, one must use the formulas

$$Q_j^* = \sqrt{\frac{2aK}{h+i c_j}} \quad \text{— in this case, the points at which each cost curve is minimized would be different!}$$
Quantity discounts, step II

- The minimizing value \( Q^* = 25,298 \) is feasible for the cost function \( T_2(Q) \), whose value at this point is \( T_2(Q^*) = $87,589 \).
- Note that for any value of \( Q \), \( T_1(Q) > T_2(Q) \), so ordering at the cost \( c_1 \) can be eliminated from consideration.
- \( T_3(Q) \) is minimized (over its feasible range \( Q \geq 80,000 \)) at \( Q_3 = 80,000 \), with \( T_3(Q_3) = $85,200 \).
- Comparing these three values, we conclude that the quantity \( Q = 80,000 \) is optimal.

If the discounts were less steep, for example, if \( c_3 = $9.5 \), then it would be optimal to stick with \( Q^* = 25,298 \)!
Finding optimal order quantity with discounts:

1. For each value of unit cost \( c_j \), use the EOQ formula for the EOQ model to calculate its optimal order quantity, \( Q_j^* \).
2. For each \( c_j \) where \( Q_j^* \) is within the feasible range of order quantities for \( c_j \), calculate the corresponding total cost per unit time, \( T_j(Q_j^*) \).
3. For each \( c_j \) where \( Q_j^* \) is not within this feasible range, determine the order quantity \( Q_j \) that is feasible for this cost and closest to \( Q_j^* \). Calculate the cost \( T_j(Q_j) \).
   - Note the difference between \( Q_j \) and \( Q_j^* \).
4. Compare the costs obtained for all \( c_j \)'s and choose the minimum, with the order quantity obtained in step 2 or 3 that gives this minimum cost.

Another inventory management problem

- You are the Michigan distributor of Nature’s Peak, a high-end brand of frozen dog food. You have (pre-paid) contracts with local “boutique” pet stores to deliver, in each of the next 4 months, respectively, 50, 65, 100, and 70 lb of food, and these orders must be filled on time.
- You obtain the food from the manufacturer at wholesale prices which vary month to month. In the next four months, unit prices are $5, $8, $4, and $7 per pound, respectively, and you can buy at most 80 lb each month.
- Food needs to be delivered to the stores at the end of each month. You place your order with the manufacturer in the beginning of each month, receive your order at the end of the month, and immediately deliver food to the local stores.
- If you have food remaining after the demand has been satisfied, you can keep some of it in your warehouse at a cost of $2 per pound per month until the next delivery, and donate the rest to the Humane Society.
- In 4 months, Nature’s Peak is planning to change the recipe and packaging for this food. If you have any food left at that time, you can sell it to discount pet stores in the area for $6 per pound. (Until then, the company wants to maintain the product’s high-end image.)
- How should you manage your inventory for the next 4 months?

1Section 4.12 of Denardo describes a problem in which such inventory management issues are part of the decisions.
Operational decisions in the pet food distributor’s problem

- Is an EOQ model appropriate for this problem?
  - We need a different type of mathematical model!
- What decisions do you need to make for the coming 4 months?

- What performance measure would you use to compare different decisions?

- What constraints (restrictions) must your decisions satisfy?

- What assumptions are being made?

Formulation of a mathematical model for the pet food distributor’s problem

**Decision variables:** represent decisions by variables.

**Objective function:** express the performance criterion in terms of the decision variables; should it be minimized or maximized?
Formulation of a model for the pet food distributor’s – cont.

Constraints: express all (explicit and implicit) constraints and restrictions on the values of the decision variables.

Mathematical model for pet food distributor’s problem

Decision variables:
- $x_1, x_2, x_3, x_4$: lbs of food ordered in each of the 4 months
- $s_1, s_2, s_3, s_4$: lbs of food stored/sold at the end of each of the 4 months
- $y_1, y_2, y_3, y_4$: lbs of food donated in each of the 4 months

Mathematical model:

minimize

\[ 5x_1 + 8x_2 + 4x_3 + 7x_4 + 2(s_1 + s_2 + s_3) - 6s_4 \]

(Net) cost objective

subject to

\[ x_1 = 50 + s_1 + y_1 \]

Inventory balance in month 1 constraint

\[ x_2 + s_1 = 65 + s_2 + y_2 \]

Inventory balance in month 2 constraint

\[ x_3 + s_2 = 100 + s_3 + y_3 \]

Inventory balance in month 3 constraint

\[ x_4 + s_3 = 70 + s_4 + y_4 \]

Inventory balance in month 4 constraint

\[ x_1, x_2, x_3, x_4 \leq 80 \]

Ordering capacity constraint(s)

\[ y_1, y_2, y_3, y_4 \geq 0 \]

Nonnegativity constraint(s)