Finding Fixed Points by Averaging with Well-Behaved Maps

Marina Epelman* Thomas L. Magnanti† Georgia Perakis‡

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Abstract

We introduce and examine a general framework for parametrically solving fixed point problems through a family of problems that are "better behaved" than our original problem. One class of examples parametrically combines a well-behaved map with the fixed point map into a composite map. Special cases include (outside) averaging the underlying map with a contractive map, or alternatively, with a nonexpansive map whose fixed points include those of the original map, averaging with the identity map, and inside averaging (i.e. averaging before applying the underlying map). We establish conditions under which the trajectory of fixed points of the parametric map converge to a fixed point of the original problem as the parameter approaches zero. To implement this solution framework, we also consider an iterative scheme that approximates the parametric fixed point trajectory. We establish a convergence result and characterize the limit points of this approximation scheme.

Key Words
Fixed Point Problems, Averaging Schemes, Nonexpansive Maps.

1 Introduction

Algorithms for solving problems in many applied settings establish a mapping $T$ whose iterative application leads to a fixed point solution that solves the original problem. The convergence of this iterative procedure to a fixed point solution often requires strong assumptions on the algorithmic map $T$ that restrict the algorithm's domain of applicability. For example, the classical function iteration $x_{k+1} = T(x_k)$ requires the map $T$ to be contractive. In many instances, various forms of averaging extend the range of applicability of the algorithmic (fixed point) map $T$.

In this paper, we introduce a general averaging framework for solving the finite dimensional fixed point problem

\[ \text{Find } x^* \in K \text{ satisfying } x^* = T(x^*) \]

*Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, Michigan.
†School of Engineering and Sloan School of Management, MIT, Cambridge, MA.
‡Sloan School of Management and Operations Research Center, MIT, Cambridge, MA.
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defined over a given ground set $K \subseteq \mathbb{R}^n$ by a given map $T : K \to K$. This problem formulation is closely related to several problem classes in mathematical programming, such as linear and nonlinear optimization, complementarity problems, and variational inequality problems (see [12] for more details). For example, we obtain an equivalent fixed point reformulation of variational inequality problems, that is,

$$\text{Find } x^* \in K : \quad f(x^*) T(x - x^*) \geq 0, \ \forall x \in K,$$

through the map $T = \text{Pr}_K(I - \rho f)$ defined by the projection operator $\text{Pr}_K$ onto the set $K$ and some positive constant $\rho$.

To solve the fixed point problem, we introduce a general averaging framework that considers at each step a parameterization of the original fixed point problem. The averaging framework thus creates a family of problems which are easier to solve than the original problem. A special case of this framework is averaging the fixed point map $T$ with maps $g$ that are “well-behaved” to counteract the “bad” properties of the map $T$. In particular, the framework includes as special cases averaging with the identity map, with nonexpansive maps whose fixed points include those of the original problem, as well as with general contractive maps.

The framework includes as special cases “outside averaging” $\lambda g(x) + (1 - \lambda) T(x)$ with an averaging parameter $0 \leq \lambda \leq 1$. Halfem [8] and Browder [3] introduced this type of averaging and Bauschke [1] and Wittmann [15] studied it further in the special case when $g(x) = \text{constant}$. Dunn [5] introduced the special case of $g(x) = x$, which Magnani and Perakis further studied for variational inequality problems (see [10] and [11]). The framework also includes “inside averaging” $T(\lambda g(x) + (1 - \lambda)x)$. It also permits averaging with the identity map (line search procedures), or outside and inside averaging with contractive maps as well as with the proximal point map. When applied to fixed point and variational inequality problems, this framework produces certain known methods as well as several new ones.

Our goals in this paper are to

(i) study averaging schemes based upon parameterizations of (that is, averaging) the original fixed point map with “well-behaved” maps such as contractive maps, the identity map or, more generally, nonexpansive maps whose fixed points include those of the original problem,

(ii) develop a general averaging framework that unifies several averaging approaches,

(iii) solve a large class of fixed point problems by considering fixed point maps $T$ satisfying properties weaker than contractiveness (including forms of nonexpansiveness that permit fixed point problems with multiple solutions), and

(iv) create as a special case averaging schemes that are more efficient than the classical function iteration even when the underlying map is contractive.
1.1 Notation and preliminaries

Throughout this paper we will be working with a convex set $K \subseteq \mathbb{R}^n$ and a map $T : K \rightarrow K$. We refer to the problem

$$\text{Find } x^* \in K \text{ satisfying } x^* = T(x^*)$$

as the fixed point problem associated with the map $T$ and the set $K$. We let $\text{FP}(T)$, which we assume to be a nonempty set, denote the set of fixed point solutions of problem (1). We permit the ground set $K$ to be unbounded and the fixed point problem to have multiple solutions. That is, $\text{FP}(T)$ can contain more than one point.

**Definition 1** A map $G : K \rightarrow K$ is called contractive (or a contraction) with a contraction constant $a \in [0, 1)$ if for any $x, y \in K$,

$$||G(x) - G(y)|| \leq a ||x - y||.$$  

A map $G : K \rightarrow K$ is called nonexpansive if for any $x, y \in K$,

$$||G(x) - G(y)|| \leq ||x - y||.$$  

A map $G : K \rightarrow K$ is called firmly nonexpansive if for any $x, y \in K$,

$$||G(x) - G(y)||^2 \leq (x - y)^T(G(x) - G(y)).$$

Equivalently, $G$ is firmly nonexpansive if for any $x, y \in K$,

$$||G(x) - G(y)||^2 \leq ||x - y||^2 - ||(x - y)\cdot(G(x) - G(y))||^2.$$  

We use the following result repeatedly in our analysis (see, for example, Dunn[5]):

**Proposition 1** Suppose $G : K \rightarrow K$ is nonexpansive, with $\text{FP}(G) \neq \emptyset$, and the sequence $\{x_k\}_{k=0}^{\infty} \subseteq K$ satisfies the condition $\lim_{k \rightarrow \infty} ||x_k - G(x_k)|| = 0$. Then every limit point of the sequence is a fixed point of $G$.

We denote the identity map by $I$. Throughout our discussion, we use the following well-known results concerning the relationship between a given map $G : K \rightarrow K$ and the map $(I - G) : K \rightarrow K$:

**Proposition 2** [4] A map $G$ is contractive with contraction constant $a \in [0, 1)$ if and only if the map $I - G$ is a strongly monotone map with monotonicity constant $\frac{1-a^2}{2}$. Even more strongly, for any $x$ and $y$,

$$(x - G(x) - y + G(y))^T(x - y) \geq \frac{1-a^2}{2} ||x - y||^2 + \frac{1}{2} ||x - G(x) - y + G(y)||^2.$$  

A map $G$ is nonexpansive if and only if the map $I - G$ is a strongly-$f$-monotone map with monotonicity constant $\frac{1}{2}$. That is, for any $x$ and $y$,

$$(x - G(x) - y + G(y))^T(x - y) \geq \frac{1}{2} ||x - G(x) - y + G(y)||^2.$$  

For any convex closed set $S \subseteq \mathbb{R}^n$ and any point $y \in \mathbb{R}^n$, we denote the Euclidean projection of $y$ onto $S$ by $\text{Pr}_S(y)$. 
2 Averaging trajectories

2.1 Generalized Averaging Map

As a first step in our analysis, we approximate the original fixed point problem with a parameterized family of fixed point problems

\[ x_\lambda = F_\lambda(x_\lambda) \]  \hspace{1cm} (2)

with parameterization parameter \( \lambda \in [0, 1] \). Our intent is to choose the family of functions \( F_\lambda(x) : K \to K \) to be a parameterization of the map \( T \) and be "better behaved" than the map \( T \). In particular, we make the following assumptions on this family of functions:

**A1** For all \( \lambda \in (0, 1] \), the parameterized problem \( x_\lambda = F_\lambda(x_\lambda) \) has at least one solution.

**A2** The function \( F_\lambda(x) = F(\lambda, x) : [0, 1] \times K \to K \) is a continuous function of \((\lambda, x)\).

**A3** \( \text{FP}(T) \supseteq \text{FP}(F_0) \neq \emptyset \).

We refer to any map \( F_\lambda(\cdot) \) satisfying conditions **A1–A3** as a generalized averaging map for \( T \).

For parameterizations satisfying these three assumptions, the following result provides a characterization of the limit points of the set \( \{x_\lambda\} \) induced by the parameterized fixed point problems (2) as \( \lambda \to 0^+ \).

**Proposition 3** Suppose \( T : K \to K \) is a map with \( \text{FP}(T) \neq \emptyset \). Suppose further that the family of maps \( F_\lambda : K \to K \) satisfies conditions **A1–A3**. Then as \( \lambda \to 0^+ \), every limit point of the sequence (or trajectory) \( \{x_\lambda\} \) is a fixed point of \( T \).

**Proof:** Condition **A1** states that the iterates induced by (2) exist. Let \( \bar{x} \in K \) be a limit point of the sequence (or trajectory) \( \{x_\lambda\} \) as \( \lambda \to 0^+ \), i.e., for some sequence \( \{\lambda_k\} \subset (0, 1] \) with \( \lim_{k \to \infty} \lambda_k = 0 \), \( \bar{x} = \lim_{k \to \infty} x_{\lambda_k} \). Then as \( k \to \infty \)

\[ x_{\lambda_k} - F_0(x_{\lambda_k}) = F_{\lambda_k}(x_{\lambda_k}) - F_0(x_{\lambda_k}) = F(\lambda_k, x_{\lambda_k}) - F(0, x_{\lambda_k}) \to 0 \]

since \( \lambda_k \to 0 \), \( x_{\lambda_k} \) is a convergent subsequence, and, by assumption **A2**, the function \( F(\lambda, x) \) is continuous. Consequently, the limit point \( \bar{x} \) is a fixed point of \( F_0 \), and therefore, from condition **A3**, a fixed point of \( T \).

The previous analysis has characterized the limit points of the trajectory of solutions of the parametric family of fixed point subproblems as solutions of the original problem. To further understand the nature of the limit points, we need to impose some additional conditions on the generalized averaging map \( F_\lambda \).

**A4** \( F_\lambda = F(G_\lambda) \), for maps \( F : K \to K \) and \( G_\lambda : K \to K \) satisfying the conditions

- \( F \) is nonexpansive and \( \text{FP}(T) \subseteq \text{FP}(F) \),
- \( G_0 \) is nonexpansive and \( \text{FP}(T) \subseteq \text{FP}(G_0) \),
\( \lim_{\lambda \to 0^+} |G_\lambda(x_\lambda - x_\lambda)|^2 = 0. \)

**A5** For all \( \lambda \in [0, 1] \), all \( x \in K \) and all \( x^* \in \text{FP}(T) \),

\[
G_\lambda(x)^T(x - x^*) \leq \lambda G_1(x)^T(x - x^*) + (1 - \lambda)G_0(x)^T(x - x^*).
\]

**A6a** \( G_1(x) \) is a nonexpansive map.

**A6b** \( G_1(x) \) is a contractive map with contraction constant \( a \in [0, 1) \).

Note that condition **A4** implies that \( \text{FP}(T) \subseteq \text{FP}(F_0) \) since if \( x^* \in \text{FP}(T) \) then \( F_0(x^*) = F(G_0(x^*)) = F(x^*) = x^* \) from the set inclusions in condition **A4**. Taken together, conditions **A3** and **A4**, imply that \( \text{FP}(T) = \text{FP}(F(G_0)). \)

It might initially appear unclear how to choose the maps \( F \) and \( G_0 \) satisfying conditions **A3** and **A4** without finding the fixed points of \( T \) beforehand. Examples in Section 2.2 show that we need not already know the fixed points of \( T \), for example, by letting \( F = T \) and \( G_0 = I \), or \( F = I \) and \( G_0 = T \). In both cases, these maps trivially satisfy the set containment conditions imposed by assumptions **A3** and **A4**.

By imposing these additional assumptions, we can further characterize the limit points of the trajectory induced by the parametric fixed point problem \( (1) \).

**Theorem 1**

I) Suppose the generalized averaging map \( F_\lambda \) satisfies conditions **A1–A6a**. Then as \( \lambda \to 0^+ \), every limit point \( \bar{x} \) of the sequence \( \{x_\lambda\} \) is a fixed point of \( T \) and satisfies the condition

\[
\bar{x} = \text{Pr} \text{FP}(T)G_1(\bar{x}).
\]

II) Suppose, alternatively, the generalized averaging map \( F_\lambda \) satisfies conditions **A1–A6b**. Then as \( \lambda \to 0^+ \), the sequence \( \{x_\lambda\} \) has at most one limit point \( x^* \) which is the unique fixed point of \( T \) defined as

\[
x^* = \text{Pr} \text{FP}(T)G_1(x^*).
\]

**Proof:** Suppose assumptions **A1–A5** and either **A6a** or **A6b** (which implies **A6a**) are valid for the family \( F_\lambda \). Let \( x \in \text{FP}(F) \) be an arbitrary fixed point of \( T \).

Condition **A4** implies that \( x \in \text{FP}(F) \) and \( F \) is nonexpansive. Proposition 2 shows that

\[
0 \leq ((G_\lambda(x_\lambda) - F(G_\lambda(x_\lambda)) - (x - F(x)))^T(G_\lambda(x_\lambda) - x)
\]

\[
= (G_\lambda(x_\lambda) - x_\lambda)^T(G_\lambda(x_\lambda) - x) = ||G_\lambda(x_\lambda) - x_\lambda||^2 + (G_\lambda(x_\lambda) - x_\lambda)^T(x_\lambda - x) + (G_\lambda(x_\lambda) - x_\lambda)^T(x_\lambda - x)
\]

\[
= ||G_\lambda(x_\lambda) - x_\lambda||^2 - (x_\lambda - G_0(x_\lambda))^T(x_\lambda - x) + (G_\lambda(x_\lambda) - G_0(x_\lambda))^T(x_\lambda - x)
\]

Condition **A5** implies that for all \( \lambda \in [0, 1] \),

\[
G_\lambda(x_\lambda)^T(x_\lambda - x) \leq \lambda G_1(x_\lambda)^T(x_\lambda - x) + (1 - \lambda)G_0(x_\lambda)^T(x_\lambda - x),
\]
or, upon rearranging terms,

\[(G_\lambda(x_\lambda) - G_0(x_\lambda))^t(x_\lambda - x) \leq \lambda (G_1(x_\lambda) - G_0(x_\lambda))^t(x_\lambda - x).\]

Substituting this inequality into (3), we conclude that for all \(\lambda \in [0, 1]\),

\[0 \leq \|G_\lambda(x_\lambda) - x_\lambda\|^2 - (x_\lambda - G_0(x_\lambda))^t(x_\lambda - x) + \lambda (G_1(x_\lambda) - G_0(x_\lambda))^t(x_\lambda - x)\]

\[= \|G_\lambda(x_\lambda) - x_\lambda\|^2 - (1 - \lambda)(x_\lambda - G_0(x_\lambda))^t(x_\lambda - x) + \lambda (G_1(x_\lambda) - x_\lambda)^t(x_\lambda - x).\]

Since \(x \in FP(T) \subseteq FP(G_0)\) and \(G_0\) is a nonexpansive map (from condition A4),

\[(x_\lambda - G_0(x_\lambda))^t(x_\lambda - x) \geq 0.\]

Therefore, for all \(\lambda \in (0, 1]\),

\[\frac{\|G_\lambda(x_\lambda) - x_\lambda\|^2}{\lambda} \geq (x_\lambda - G_1(x_\lambda))^t(x_\lambda - x).\]

Condition A4 implies that \(\limsup_{\lambda \to 0^+}(x_\lambda - G_1(x_\lambda))^t(x_\lambda - x) \leq 0\). Therefore as \(\lambda \to 0^+\), any limit point \(\bar{x}\) (which, if it exists, is also a fixed point solution of the map \(T\) as we have shown in Theorem 3) satisfies the condition

\[(\bar{x} - G_1(\bar{x}))^t(\bar{x} - x) \leq 0 \quad \forall x \in FP(T). \tag{4}\]

Condition (4) can alternatively be interpreted as \(\bar{x} = Pr_{FP(T)}G_1(\bar{x})\), thus establishing the first claim of the theorem.

To establish the second claim of the theorem, suppose assumption A6b is valid. Since the set \(FP(T)\) is nonempty, the map \(Pr_{FP(T)}G_1 : FP(T) \to FP(T)\) is well-defined. Under assumption A6b, this map is a contraction, and therefore has a unique fixed point \(x^* \in FP(T)\). Therefore, \(\bar{x} = x^*\), i.e., as \(\lambda \to 0^+\), the sequence \(\{x_\lambda\}\) has at most one limit point \(x^*\), establishing the second part of the theorem.

Note that if the generalized averaging map \(F_\lambda\) satisfies conditions A1–A6a or A1–A6b, and the induced trajectory \(\{x_\lambda\}\) is bounded, then the trajectory possesses limit points as \(\lambda \to 0\), each of them a fixed point of \(T\). Moreover, when \(G_1\) is a contractive mapping (i.e., condition A6b applies), the entire trajectory will converge.

**Example:** To illustrate the limiting conditions, consider the fixed point problem with \(K = \mathbb{R}^2\) and the map \(T(x_1, x_2) = (2 - x_1, x_2)\). The set of fixed point solutions is \(FP(T) = \{(1, x_2) : x_2 \in \mathbb{R}\}\).

The classical function iteration cycles by reflecting about the line of fixed points \(FP(T)\). To remedy this cycling behavior we consider special cases of the parameterization scheme (1). The first parameterization considers the map

\[F_\lambda^1(x) = \frac{\lambda x}{2} + (1 - \lambda)T(x).\]

In this case, \(F_\lambda^1(x) = F^1(G_\lambda^1(x))\) with \(F^1 = I\) and \(G_\lambda^1 = \frac{\lambda}{2}I + (1 - \lambda)T\). This averaging map satisfies conditions A1–A6b and the point \(x^* = (1, 0)\) satisfies the condition
$x^* = \text{Pr}_{FP(T)}G_1^1(x^*)$. The map $F^1_\lambda$ creates the averaging trajectory $x^1_\lambda = \lambda \frac{x^1_0}{2} + (1-\lambda)T(x^1_0)$. Algebraic manipulations yield the explicit expression for $x^1_\lambda$ to be $x^1_\lambda = \left(\frac{4-4\lambda}{4-3\lambda}, 0\right)$, converging to $x^*$ as $\lambda \to 0$.

As another example, let $F^2_\lambda(x) = T\left(\frac{\lambda x}{2} + (1-\lambda)x\right) = T((1-\frac{1}{2}\lambda)x)$. In this case, $F^2_\lambda(x) = F^2(G^2_\lambda(x))$ with $F^2 = T$ and $G^2_\lambda = (1-\frac{1}{2}\lambda)I$. Again, the point $x^* = (1, 0)$ is the unique point satisfying the condition $x^* = \text{Pr}_{FP(T)}G^2_1(x^*)$. This map produces the trajectory $x^2_\lambda = T((1-\frac{1}{2}\lambda)x^2_0)$ that can be expressed explicitly as $x^2_\lambda = (\frac{4}{4-3\lambda}, 0)$. This trajectory also converges to $x^*$ as $\lambda \to 0$. Figure 1 illustrates the averaging trajectories for this example.

![Figure 1: Averaging Trajectories](image)

Figure 1: Averaging Trajectories

Observe that these averaging schemes satisfy conditions A1–A6b. Moreover, although the trajectories $\{x^1_\lambda\}$ and $\{x^2_\lambda\}$ are quite different, they both approach the same limit point. As Theorem 1 illustrates, this result is not a coincidence, since the point $x^* = \text{Pr}_{FP(T)}G_1(x^*)$ is the same in both cases.

As yet another example, suppose $F = I$ and

$$G^2_\lambda = \lambda x + (1 - \lambda)(I + c(I - T))^{-1}(x), \text{ with } c > 0.$$
This averaging map satisfies conditions A1–A6a, and therefore, the first (weaker) part of
Theorem 1 applies\(^1\).

In the next section we show that these types of averaging trajectories are special cases
of the averaging framework we have considered in this section.

2.2 Examples of generalized averaging maps

We next introduce several examples of generalized averaging maps \(F_\lambda(\cdot)\) and show that
they are special cases of the averaging framework we have introduced (i.e., satisfy conditions
A1 – A6). We assume in these examples that the ground set \(K\) is a closed convex set, the
map \(T : K \to K\) is nonexpansive, and the map \(g : K \to K\) is either (i) contractive, or (ii)
nonexpansive with fixed points containing those of the original problem (e.g., \(g = I\)). The
special cases to follow typically use various forms of inside and outside averaging.

The fact that the fixed point map \(T\) is nonexpansive allows us to apply the proximal
point map \(J_{(I-T)} \hat{=} (I+c(I-T))^{-1}\) for some constant \(c > 0\) to the map \((I-T)\) in some
of these examples\(^2\). For a discussion of the proximal point map and its relationship to fixed
point problems, see Eckstein and Bertsekas [6] and Rockafellar [14].

1. Outside averaging

\[
F_\lambda(x) = F_\lambda^{\text{out}}(x) \hat{=} \lambda g(x) + (1 - \lambda)T(x).
\]

We refer to this map as “outside averaging” since the values it generates are convex
combinations of the values of the original map \(T\) and values of the map \(g\). The prior
literature has examined special cases of outside averaging. Halpern [8] and Browder [3]
introduced, and Bauschke [1] and Wittmann [15] further studied, a special case of
this map with the constant map \(g(x) \equiv x_0\) for some constant \(x_0 \in K\). Dunn [5]
and Magnanti and Perakis [10, 11] study this type of averaging for the choice of map
\(g(x) = x\). The developments in this paper allow us to extend the results to a richer
class of maps \(g(x)\).

Observe that \(F_\lambda = F(G_\lambda)\) with \(F = I\) and \(G_\lambda = \lambda g + (1 - \lambda)T\). As demonstrated
next, this parametric scheme satisfies conditions A1 – A6.

A1 When \(g\) is contractive, \(F_\lambda(x)\) is also contractive for all \(\lambda \in (0,1]\) and, therefore,
has a fixed point. When \(g\) is nonexpansive with nonempty set of fixed points
containing those of \(T\), \(F_\lambda(x)\) is also nonexpansive with fixed points containing
those of \(T\).

A2 \(F(\lambda, x) = F_\lambda(x)\) is continuous in \((\lambda, x)\) since \(T(x)\) and \(g(x)\) are nonexpansive,
and hence continuous.

A3 \(F_0 = T\), therefore, condition holds trivially.

A4 Holds trivially

\(^1\)In fact, it is not hard to see that \(FP(F_\lambda) = FP(T)\) for any \(\lambda \in [0,1]\).

\(^2\)For example, whenever \(\text{Image}(2I - T) = \mathbb{R}^n\), the map \(J_{(I-T)}\) is defined over the entire space \(\mathbb{R}^n\).
A5 Holds trivially, since
\[ G_\lambda = \lambda G_1 + (1 - \lambda)G_0. \]

A6 Condition A6a holds whenever \( g(x) \) is nonexpansive; condition A6b holds whenever \( g(x) \) is a contraction.

It is not hard to see that when \( g \) is a contraction, the fixed point \( x_\lambda \) is unique for each \( \lambda > 0 \), and the trajectory \( \{x_\lambda\} \) is bounded for \( \lambda \in (0, 1] \). Indeed, let \( x^* \in \text{FP}(T) \) be an arbitrary fixed point, and suppose the contraction constant of \( g \) is \( \alpha \in [0, 1) \). Then
\[
\|x_\lambda - x^*\| \leq \lambda \|g(x_\lambda) - x^*\| + (1 - \lambda)\|T(x_\lambda) - T(x^*)\|
\]
\[
\leq \lambda \|g(x_\lambda) - g(x^*)\| + \lambda \|g(x^*) - x^*\| + (1 - \lambda)\|x_\lambda - x^*\| \leq (\alpha + 1 - \lambda)\|x_\lambda - x^*\| + \lambda \|g(x^*) - x^*\|
\]
or, after rearranging terms and cancelling \( \lambda > 0 \),
\[
\|x_\lambda - x^*\| \leq \frac{\|g(x^*) - x^*\|}{1 - \alpha}.
\]

We conclude that the trajectory \( \{x_\lambda\} \) is bounded and, therefore, converges to the unique limit point \( x^* \) as described in Theorem 1.

2. Inside averaging:
\[
F_\lambda(x) = F_\lambda^{\text{in}}(x) = T(\lambda g(x) + (1 - \lambda)x).
\]

We refer to this map as “inside averaging” since it generates values by applying the map \( T \) to the convex combination of the identity map and map \( g(x) \), which are, therefore, averaged “inside,” or within the argument, of \( T \).

Observe that \( F_\lambda = F(G_\lambda) \) for \( F = T \) and \( G_\lambda = \lambda g + (1 - \lambda)I \). As demonstrated next, with one provision, this parametric scheme satisfies conditions A1 – A6. The provision is that when \( g(x) \) is nonexpansive, we need to select the sequence \( \{x_\lambda\} \) from a bounded set. In Section 3, we discuss a procedure that permits us to eliminate the boundedness condition.

A1 When \( g \) is contractive, \( F_\lambda(x) \) is also contractive for all \( \lambda \in (0, 1] \) and, therefore, has a fixed point. When \( g \) is nonexpansive with a nonempty set of fixed points containing those of \( T \), \( F_\lambda(x) \) is also nonexpansive with fixed points containing those of \( T \).

A2 \( F(\lambda, x) = F_\lambda(x) \) is continuous in \( (\lambda, x) \) since \( T(x) \) and \( g(x) \) are nonexpansive, and hence continuous.

A3 \( F_0 = T \), therefore, the condition holds trivially.

A4 The first two statements hold trivially. The last statement of condition A4 is equivalent to
\[
\lim_{\lambda \to 0} \lambda \|g(x_\lambda) - x_\lambda\|^2 = 0.
\]

(5)
When $g(x)$ is a contraction, the fixed point $x_\lambda$ is unique for each $\lambda > 0$, and the trajectory $\{x_\lambda\}$ is bounded for $\lambda \in (0, 1]$, which implies (5). When $g(x)$ is nonexpansive with $FP(T) \subseteq FP(g)$, each map $F_\lambda(x)$ might have more than one fixed point (e.g., any point $x^* \in FP(T)$ is a fixed point of $F_\lambda$). However, as long as we select the sequence $\{x_\lambda\}$ from a bounded set, it would satisfy condition (5).

**A5** Holds trivially, since

$$G_\lambda \equiv \lambda G_1 + (1 - \lambda)G_0.$$

**A6** Condition **A6a** holds whenever $g(x)$ is nonexpansive; condition **A6b** holds whenever $g(x)$ is a contraction.

When $g$ is a contractive map, then an argument similar to the one used in our discussion of outside averaging, shows that the trajectory $\{x_\lambda\}$ is bounded for $\lambda \in (0, 1]$ and, therefore, converges to the unique fixed point $x^*$ described in Theorem 1.

3. **Outside averaging with the proximal point map:** Whenever $0 \in K$ and the proximal point map $J_{c(I - T)}(x) = (I + c(I - T))^{-1}(x)$ is defined on $K$, we can consider the map

$$F_\lambda(x) = (1 - \lambda)J_{c(I - T)}(x) \equiv (1 - \lambda)(I + c(I - T))^{-1}(x).$$

In this case, $F = I$ and $G_\lambda = \lambda(1 - \lambda)J_{c(I - T)}$, and we can establish the validity of conditions **A1 − A6b** as in our discussion in Example 1, since the map $J_{c(I - T)}$ is nonexpansive with fixed points coinciding with those of $T$. Moreover, we can replace the constant map $g(x) \equiv 0$ by an arbitrary map $g : K \to K$ satisfying the properties as discussed in Example 1.

4. **Inside averaging with the proximal point map:** Again, assuming $0 \in K$ and the proximal point map $J_{c(I - T)}(x)$ is well defined on $K$, we consider the map

$$F_\lambda(x) = J_{c(I - T)}((1 - \lambda)x) \equiv (I + c(I - T))^{-1}((1 - \lambda)x).$$

In this case, $F = J_{c(I - T)}$ and $G_\lambda = (1 - \lambda)I$.

**Remarks:**

1. Note that Examples 3 and 4 are analogous to outside and inside averaging of the map $J_{c(I - T)}$ with $g \equiv 0$. Furthermore, as in Examples 1 and 2, we can apply averaging for a more general contractive map $g$ (or nonexpansive map $g$ whose fixed points contain those of $T$), or a family of maps $g_\lambda$ as in Example 6 to follow.

2. It is interesting to compare the trajectories generated by the averaging schemes of Example 1 (with $g(x) \equiv 0$) and Example 4. Let us denote the points on these trajectories by $x^A_\lambda$ and $x^I_\lambda$, respectively. We compare these trajectories by noting that

$$(1 - \lambda)x^A_\lambda = (I + c(I - T))(x^A_\lambda) = (c + 1)x^A_\lambda - cT(x^A_\lambda).$$

Therefore, $x^A_\lambda = \frac{c}{c + 1}T(x^A_\lambda)$, implying that

$$x^A_\lambda = x^I_\lambda \frac{c}{c + 1}.$$
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Notice that in this case the trajectory of the fixed points generated by inside averaging with the proximal point mapping (that is, the trajectory \( \{x^n\} \)) is the same as the trajectory of the fixed points generated by outside averaging (that is, the sequence \( \{x^n\} \)). Observe that whenever \( c \geq 1 \), the trajectory \( \{x^n\} \) converges faster than the trajectory \( \{x^\lambda\} \).

5. Convex combinations of any \( F_\lambda \) maps that satisfy the conditions A1-A6a or A1-A6b

6. Averaging with a family of maps \( g_\lambda \): In Examples 1-5, we can replace the map \( g \) with a family of maps \( g_\lambda : K \to K \) if for some constants \( \alpha \in [0,1) \) and \( \gamma \geq 0 \):

\[
\|g_\lambda(x) - g_\lambda(y)\| \leq \alpha \|x - y\| \quad \forall x, y \in K, \forall \lambda \in [0,1]
\]

and

\[
\|g_\lambda(x) - g_\lambda^2(x)\| \leq \gamma |\lambda^n - \lambda^n|, \quad \forall x \in K, \forall \lambda, \lambda^2 \in [0,1].
\]

The resulting family \( F_\lambda \) satisfies conditions A1-A3, and therefore Proposition 3 applies. The resulting parametric scheme might violate condition A5 and, therefore, Theorem 1 might not be directly applicable to the limit points of the trajectory of fixed points induced by this family. However, it is not hard to show that, as \( \lambda \) approaches 0, this trajectory approaches the trajectory of fixed points induced by (inside or outside) averaging with the map \( g_0 \), and consequently converges to \( x^* = \text{Pr}_{FP(T)}(g_0(x^*)) \). This extension allows us to incorporate inexact computations on the early stages of the averaging (i.e., for larger values of \( \lambda \)), as long as the magnitude of errors decreases as \( \lambda \) approaches 0.

3 Approximate averaging trajectories — a first approach

The averaging framework described in the previous section provides an intuitive way of approximating fixed point solutions of a map \( T \) with the trajectory of fixed points solutions induced from a class of parameterized subproblems. Following this trajectory exactly and computing the fixed points of the parameterized subproblems typically will be very expensive computationally. As an extreme example, note that the trivial averaging scheme \( F_\lambda \equiv T \) satisfies conditions A1-A6. Therefore, we would like to solve the original fixed point problem by choosing the parametric families \( F_\lambda \) more judiciously, i.e., by imposing stronger assumptions. These stronger assumptions will also allow us to compute points that lie close to the trajectory of fixed points for the parameterized subproblems, rather than solving the subproblems exactly, while still guaranteeing convergence to the desired fixed points of \( T \).

To motivate the general approach in this section, consider the iterative scheme that approximates the trajectory of fixed points induced by outside averaging of Example 1:

\[
x_0 \in K, \quad x_{k+1} = \lambda_{k+1} g(x_k) + (1 - \lambda_{k+1}) T(x_k), \quad k > 0.
\]

Halpern [8] first considered this iterative scheme and Wittmann [15] and Bauschke [7] further analyzed it, all for the case of constant map \( g \). Averaging with the identity map
(see [5], [10], [11]) is another special case of this scheme. The framework we introduce in
this paper, however, extends well beyond these very special cases and includes as special
cases, among many others, inside and outside averaging with contractive maps \( g \). As a first
approach, in this section we consider the following solution scheme that approximates the
trajectory of fixed points for the parameterized subproblems:

\[
x_0 \in K, \ x_{k+1} = F_{\lambda_{k+1}}(x_k), \ k > 0, \tag{6}
\]

with the sequence of parameter values \( \lambda_k \in (0,1] \) and the family of functions \( F_{\lambda} : K \to K \)
satisfying the following conditions:

\begin{itemize}
  \item \textbf{B1} \( \text{FP}(T) \supseteq \text{FP}(F_0) \neq \emptyset \)
  \item \textbf{B2} For some constant \( \alpha \in (0,1] \),
    \[ ||F_{\lambda}(x) - F_{\lambda}(y)|| \leq (1 - \alpha \lambda)||x - y||, \]
    for any \( \lambda \in [0,1] \) and any \( x, y \in K \).
  \item \textbf{B3} For some constant \( L > 0 \),
    \[ ||F_{\lambda}(x) - F_{\mu}(x)|| \leq L|\lambda - \mu| \cdot ||x||, \]
    for any \( \lambda, \mu \in [0,1] \) and any \( x \in K \).
  \item \textbf{B4} \( \lambda_k \to 0^+, \ \sum_{k=1}^{\infty} \lambda_k = +\infty, \ \sum_{k=1}^{\infty} |\lambda_k - \lambda_{k+1}| < +\infty \).
\end{itemize}

Observe that conditions \textbf{B1–B3} imply conditions \textbf{A1–A3}.

Before analyzing the behavior of the general averaging map \( F_{\lambda}(x) \), we examine several
special cases. In particular, we show that several iterative schemes which are the algorithmic
extensions of the examples of the previous section satisfy properties \textbf{B1–B4}.

In the following examples, we assume that the map \( T : K \to K \) is nonexpansive with
\( \text{FP}(T) \neq \emptyset \) and the “step sizes” \( \{\lambda_k\} \) satisfy condition \textbf{B4}.

\textbf{Examples of the iterative scheme:}

1. \textbf{Outside averaging:}

\[
x_{k+1} = \lambda_{k+1}g(x_k) + (1 - \lambda_{k+1})T(x_k),
\]

for a contractive map \( g : K \to K \) with bounded norm and a contraction constant \( a \in [0,1] \). In this case,

\[
F_{\lambda}(x) = \lambda g(x) + (1 - \lambda)T(x),
\]

and

\textbf{B1} \( F_0(x) = T(x) \), therefore, \( \text{FP}(T) = \text{FP}(F_0) \).
For any $\lambda$, $x$, $y$,

$$||F_\lambda(x) - F_\lambda(y)|| \leq \alpha ||x - y|| + (1 - \lambda)||x - y|| = (1 - \alpha \lambda)||x - y||$$

with $\alpha = (1 - \alpha) \in (0, 1]$.

For any $\lambda$, $\mu$, $x$,

$$||F_\lambda(x) - F_\mu(x)|| \leq |\lambda - \mu||g(x)|| + |\lambda - \mu||T(x)|| = |\lambda - \mu||(|g(x)|| + ||T(x)||),$$

which implies the condition.

2. Inside averaging:

$$x_{k+1} = T(\lambda_{k+1}g(x_k) + (1 - \lambda_{k+1})x_k),$$

for a contractive map $g : K \to K$ with bounded norm and a contraction constant $\alpha \in [0, 1]$. In this case,

$$F_\lambda(x) = T(\lambda g(x) + (1 - \lambda)x),$$

and

B1 $F_0(x) = T(x)$, therefore, $FP(T) = FP(F_0)$.

B2 Since $T$ is nonexpansive, for any $\lambda$, $x$, $y$,

$$||F_\lambda(x) - F_\lambda(y)|| \leq ||\lambda g(x) + (1 - \lambda)x - \lambda g(y) - (1 - \lambda)y||$$

$$\leq \lambda||x - y|| + (1 - \lambda)||x - y|| = (1 - \alpha \lambda)||x - y||$$

with $\alpha = 1 - \alpha \in (0, 1]$.

B3 For any $\lambda$, $\mu$, $x$,

$$||F_\lambda(x) - F_\mu(x)|| \leq ||\lambda g(x) + (1 - \lambda)x - \mu g(x) - (1 - \mu)x||$$

$$\leq |\lambda - \mu||g(x)|| + |\lambda - \mu||x|| = |\lambda - \mu||(|g(x)|| + ||x||),$$

which implies the condition.

3. Outside averaging with the proximal point mapping: Assume $0 \in K$ and the map $J_{\alpha(I-T)}$ is defined on $K$. Then we can define

$$x_{k+1} = (1 - \lambda_{k+1})J_{\alpha(I-T)}(x_k) \equiv (1 - \lambda_{k+1})(I + c(I - T))^{-1}(x_k).$$

In this case,

$$F_\lambda(x) = (1 - \lambda)J_{\alpha(I-T)}(x),$$

and

B1 $F_0(x) = (I + c(I - T))^{-1}(x)$, therefore, $FP(F_0) = FP(T)$.
B2 Since $J_{c(I-T)}$ is a nonexpansive map, for any $\lambda, x, y$,

$$||F_\lambda(x) - F_\lambda(y)|| \leq (1 - \lambda)||x - y||,$$

satisfying the condition with $\alpha = 1$.

B3 For any $\lambda, \mu, x$,

$$||F_\lambda(x) - F_\mu(x)|| = ||(1-\lambda)J_{c(I-T)}(x) - (1-\mu)J_{c(I-T)}(x)|| = |\lambda - \mu| \cdot ||J_{c(I-T)}(x)||,$$

which implies the condition.

4. **Inside averaging with the proximal point mapping:** Again, assume that $0 \in K$ and $J_{c(I-T)}$ is well-defined on $K$. Let

$$x_{k+1} = (I + c(I - T))^{-1}((1 - \lambda_{k+1})x_k).$$

In this case,

$$F_\lambda(x) = J_{c(I-T)}((1 - \lambda)x),$$

and

B1 $F_0(x) = (I + c(I - T))^{-1}(x) \text{ and, therefore, } FP(F_0) = FP(T).

B2 Since $J_{c(I-T)}$ is a nonexpansive map, for any $\lambda, x, y$,

$$||F_\lambda(x) - F_\lambda(y)|| \leq (1 - \lambda)||x - y||,$$

satisfying the condition with $\alpha = 1$.

B3 Since $J_{c(I-T)}$ is nonexpansive, for any $\lambda, \mu, x$,

$$||F_\lambda(x) - F_\mu(x)|| \leq |\lambda - \mu| \cdot ||x||,$$

which implies the condition.

5. **Convex combinations:** Similar to the previous cases.

The following results (Theorems 2 and 3) extend to the general iterative scheme (6) the results of Wittmann [15] and Bauschke [1], which, essentially, concerned the case of outside averaging with a constant map. Some of the proof techniques used in establishing these theorems are similar to those used by Bauschke.

**Theorem 2** Suppose $T : K \to K$ is a nonexpansive map with $FP(T) \neq 0$. Suppose further that the sequence $\{\lambda_k\}$ and the family of functions $F_\lambda$ satisfy conditions B1 \text{ -- B4}. Then every limit point of the sequence $\{x_k\}$ induced by the averaging scheme (6) is a fixed point of the map $T$. 
Proof: We begin by showing that the sequence \( \{x_k\} \) induced by the averaging scheme (6) is bounded. Let \( x^* \) be an arbitrary point in FP(\( T \)). Define

\[
D = \max \left\{ \|x_0 - x^*\|, \frac{|x^*|L}{\alpha} \right\},
\]

with the constants \( L > 0 \) and \( \alpha \in (0, 1] \) specified as in conditions B2 and B3. We will show by induction that \( \|x_k - x^*\| \leq D \) for any \( k \geq 0 \). This inequality is valid by definition for \( k = 0 \). Suppose the inequality is valid for some \( k \geq 0 \). Then

\[
\|x_{k+1} - x^*\| = \|F_{\lambda_{k+1}}(x_k) - F_0(x^*)\| \leq \|F_{\lambda_{k+1}}(x_k) - F_{\lambda_{k+1}}(x^*)\| + \|F_{\lambda_{k+1}}(x^*) - F_0(x^*)\|
\]

\[
\leq (1 - \alpha \lambda_{k+1})\|x_k - x^*\| + \lambda_{k+1}L\|x^*\|
\]

establishing the desired relationship.

Since the sequence \( \{x_k\} \) is bounded, for some constant \( C > 0 \), \( L\|x_k\| \leq C \) and \( \|x_k - x_{k+1}\| \leq C \) for all \( k \geq 0 \). Then for any \( k \geq 0 \),

\[
\|x_{k+1} - x_k\| = \|F_{\lambda_{k+1}}(x_k) - F_{\lambda_k}(x_{k-1})\|
\]

\[
\leq \|F_{\lambda_{k+1}}(x_k) - F_{\lambda_{k+1}}(x_{k-1})\| + \|F_{\lambda_{k+1}}(x_{k-1}) - F_{\lambda_k}(x_{k-1})\|
\]

\[
\leq (1 - \alpha \lambda_{k+1})\|x_k - x_{k-1}\| + \lambda_{k+1} - \lambda_k\|x_{k-1}\|
\]

\[
\leq (1 - \alpha \lambda_{k+1})\|x_k - x_{k-1}\| + |\lambda_{k+1} - \lambda_k|C.
\]

Applying this relationship inductively, and using the fact that \( 1 - \alpha \lambda_k \leq 1 \) for all \( k \), shows that for all \( k \geq m > 0 \)

\[
\|x_{k+1} - x_k\| \leq \|x_m - x_{m-1}\| \prod_{i=m}^k (1 - \alpha \lambda_{i+1}) + C \sum_{i=m}^k |\lambda_{i+1} - \lambda_i|.
\]

Thus,

\[
\lim_{k \to \infty} \|x_{k+1} - x_k\| \leq C \prod_{i=m}^\infty (1 - \alpha \lambda_{i+1}) + C \sum_{i=m}^\infty |\lambda_{i+1} - \lambda_i|. \tag{7}
\]

The assumptions \( \lim_{k \to \infty} \lambda_k = 0 \) and \( \sum_{k=1}^{\infty} \lambda_k = +\infty \) from B4 imply that \( \prod_{i=m}^\infty (1 - \alpha \lambda_{i+1}) = 0 \), (see, for example, Proposition 2.1 of [1]). Therefore, \( \lim_{m \to \infty} \prod_{i=m}^\infty (1 - \alpha \lambda_{i+1}) = 0 \). Additionally, from assumption B4, \( \lim_{m \to \infty} \sum_{i=m}^\infty |\lambda_{i+1} - \lambda_i| = 0 \). Letting \( m \to \infty \) in (7), we obtain \( \lim_{k \to \infty} \|x_{k+1} - x_k\| = 0 \). Also,

\[
\|x_k - F_0(x_k)\| \leq \|x_k - x_{k+1}\| + \|x_{k+1} - F_0(x_k)\|
\]

\[
= \|x_k - x_{k+1}\| + \|F_{\lambda_{k+1}}(x_k) - F_0(x_k)\|
\]

\[
\leq \|x_k - x_{k+1}\| + \lambda_{k+1}L\|x_k\| \leq \|x_k - x_{k+1}\| + \lambda_{k+1}C \to_{k \to \infty} 0.
\]
Therefore, every limit point of the sequence \( \{x_k\} \) is a fixed point solution of the map \( F_0 \). Property B1 implies that any limit point is also a fixed point solution of \( T \). In addition, since the sequence \( \{x_k\} \) is bounded, it has at least one limit point. []

The previous theorem states that the limit points of the sequence induced by the iterative scheme (6) are, indeed, fixed points of \( T \). Can we, as before, characterize these limiting fixed points? To do so, we consider a stronger version of assumptions B1–B4, obtained by replacing conditions B2 and B3 with the following stronger conditions.

**B2’** \( F_\lambda = F(G_\lambda) \), with \( F : K \to K \) and \( G_\lambda : K \to K \) satisfying the assumptions:

- \( F \) is nonexpansive and \( \text{FP}(T) \subseteq \text{FP}(F) \).
- \( G_0 \) is nonexpansive, \( \text{FP}(T) \subseteq \text{FP}(G_0) \), and there exists \( \alpha \in (0, 1) \) such that
  \[
  |G_\lambda(x) - G_\lambda(y)| \leq (1 - \alpha\lambda)\|x - y\|, \quad \forall x, y \in K
  \]
- For all \( \lambda \in [0, 1] \), all \( x \in K \) and all \( x^* \in \text{FP}(T) \),
  \[
  G_\lambda(x^*)^t(x - x^*) \leq \lambda G_1(x^*)^t(x - x^*) + (1 - \lambda)G_0(x^*)^t(x - x^*).
  \]

**B3’** For some constant \( L > 0 \), \( |G_\lambda(x) - G_\mu(x)| \leq L|\lambda - \mu|\|x\| \) for any \( \lambda, \mu \in [0, 1] \) and any \( x \in K \).

As demonstrated in the following proposition, the last assumption of condition B2’ is essentially a weaker version of condition A5 of Section 2.

**Proposition 4** Suppose the family of maps \( G_\lambda : K \to K \), \( \lambda \in [0, 1] \) satisfies condition A5, i.e., for all \( \lambda \in [0, 1] \), all \( x \in K \) and all \( x^* \in \text{FP}(T) \),

\[
G_\lambda(x^*)^t(x - x^*) \leq \lambda G_1(x^*)^t(x - x^*) + (1 - \lambda)G_0(x^*)^t(x - x^*).
\]

Suppose further that for all \( \lambda \in [0, 1] \), the map \( G_\lambda \) is continuous. Then

\[
G_\lambda(x^*)^t(x - x^*) \leq \lambda G_1(x^*)^t(x - x^*) + (1 - \lambda)G_0(x^*)^t(x - x^*)
\]

for all \( \lambda \in [0, 1] \), all \( x \in K \) and all \( x^* \in \text{FP}(T) \).

**Proof:** Let \( \lambda \in [0, 1] \), \( x^* \in \text{FP}(T) \) and \( y \in K \) be arbitrary. For \( t \in (0, 1) \), let \( x = ty + (1 - t)x^* \in K \). Then by supposition,

\[
tG_\lambda(ty + (1 - t)x^*)^t(y - x^*) \leq t[\lambda G_1(ty + (1 - t)x^*) + (1 - \lambda)G_0(ty + (1 - t)x^*)]^t(y - x^*),
\]

and since \( t > 0 \),

\[
G_\lambda(ty + (1 - t)x^*)^t(y - x^*) \leq [\lambda G_1(ty + (1 - t)x^*) + (1 - \lambda)G_0(ty + (1 - t)x^*)]^t(y - x^*)
\]

Letting \( t \to 0^+ \), this inequality implies that

\[
G_\lambda(x^*)^t(y - x^*) \leq \lambda G_1(x^*)^t(y - x^*) + (1 - \lambda)G_0(x^*)^t(y - x^*). \]

Note that the Examples 1 through 4 of generalized averaging mappings satisfy these new conditions with \( F \) and \( G_\lambda \) defined as follows:
1. $F = I$ and $G_{\lambda} = \lambda g(x) + (1 - \lambda)T(x)$.
2. $F = T$ and $G_{\lambda} = \lambda g(x) + (1 - \lambda)x$.
3. $F = I$ and $G_{\lambda} = (1 - \lambda)J_{c(I-T)}$.
4. $F = J_{c(I-T)}$ and $G_{\lambda} = (1 - \lambda)x$.

The next result establishes convergence of the sequence $\{x_k\}$ induced by the iterative scheme (6) if it satisfies conditions (B1, B2, B3' and B4).

**Theorem 3** Suppose $T : K \to K$ is a nonexpansive map with $FP(T) \neq \emptyset$. Suppose further that the sequence $\{\lambda_k\}$ and the family of functions $F_{\lambda}$ satisfy conditions (B1, B2, B3' and B4). Then the sequence $\{x_k\}$ induced by the iterative scheme (6) converges to the fixed point solution $x^* = \Pr_{FP(T)} G_1(x^*) \in FP(T)$.

**Proof:** First observe that condition B2' implies that the map $G_1(x)$ is a contraction. Therefore, the map $\Pr_{FP(T)} G_1 : FP(T) \to FP(T)$ is also a contraction, and so it has a unique fixed point $x^* \in FP(T) \subseteq FP(G_0)$. The definition of the Euclidean projection implies that

$$\langle \bar{x} - x^* \rangle (G_1(x^*) - x^*) \leq 0 \quad \text{for all } \bar{x} \in FP(T). \quad (8)$$

From Theorem 2, any limit point of the sequence $\{x_k\}$ induced by the iterates (6) is a fixed point of $T$ and therefore, of $G_0$. Consequently,

$$\limsup_{k \to \infty} \langle G_0(x_k) - x^* \rangle (G_1(x^*) - x^*) \leq \limsup_{k \to \infty} \langle x_k - x^* \rangle (G_1(x^*) - x^*) \leq 0,$$

with the inequality following from expression (8). Therefore, for any $\epsilon > 0$,

$$\langle G_0(x_k) - x^* \rangle (G_1(x^*) - x^*) \leq \epsilon \quad (9)$$

for $k$ sufficiently large.

Define $C = \max\{L||x||, \sup_{k \geq 0} L||x_k||\}$ (recall from the proof of Theorem 2 that the supremum is finite). For any $\epsilon > 0$,

$$2C^2 \lambda_k \leq \epsilon \quad (10)$$

for $k$ sufficiently large.

Let $\epsilon > 0$ be arbitrary, and suppose $k$ is sufficiently large as required by conditions (9) and (10). Then if we let $\lambda = \lambda_{k+1}$,

$$||x_{k+1} - x^*||^2 = ||F(G_{\lambda}(x_k)) - F(G_0(x^*))||^2 \leq ||G_{\lambda}(x_k) - G_0(x^*)||^2$$

$$= ||G_{\lambda}(x_k) - G(x^*)||^2 + ||G(x^*) - G_0(x^*)||^2 + 2\langle G_{\lambda}(x_k) - G(x^*) \rangle (G(x^*) - G_0(x^*))$$

$$= ||G_{\lambda}(x_k) - G(x^*)||^2 + ||G(x^*) - G_0(x^*)||^2 + 2\langle G_{\lambda}(x_k) - G_0(x_k) \rangle (G(x^*) - G_0(x^*))$$

$$+ 2\langle G_0(x_k) - G_0(x^*) \rangle (G_{\lambda}(x^*) - G_0(x^*)) - 2||G_{\lambda}(x^*) - G_0(x^*)||^2.$$
\[
\leq (1 - \alpha \lambda)^2 \|x_k - x^*\|^2 + 2C^2 \lambda^2 + 2(G_0(x_k) - G_0(x^*))'(G_\lambda(x^*) - G_0(x^*)).
\]

To obtain the previous inequality, we have made use of the Cauchy-Schwartz inequality and conditions B2' and B3'. Applying condition B2' with \(x = G_0(x_k)\) to the last term in this expression and using the fact that \(G_0(x^*) = x^*\) gives
\[
\|x_{k+1} - x^*\|^2 \leq (1 - \alpha \lambda)^2 \|x_k - x^*\|^2 + 2C^2 \lambda^2 + 2\lambda(G_0(x_k) - x^*)'(G_1(x^*) - G_0(x^*))
\]
\[
\leq (1 - \alpha \lambda)^2 \|x_k - x^*\|^2 + 2C^2 \lambda^2 + 2\lambda \leq (1 - \alpha \lambda)^2 \|x_k - x^*\|^2 + 3\lambda \varepsilon.
\]

Recall that \(\lambda = \lambda_{k+1}\). Since \(1 - \alpha \lambda_{k+1} \leq 1\),
\[
\|x_{k+1} - x^*\|^2 \leq (1 - \alpha \lambda_{k+1}^2 \|x_k - x^*\|^2 + 3\varepsilon \lambda_{k+1}.
\]

(11)

Let
\[
\alpha_m = 1 - \prod_{\ell = m}^k (1 - \alpha \lambda_{\ell+1}), \ m \leq k.
\]

Observe that \(\alpha_m \in [0, 1]\) for any \(m\). We will show that
\[
\|x_{k+1} - x^*\|^2 \leq \frac{3}{\alpha} \varepsilon \alpha_m + (1 - \alpha_m) \|x_m - x^*\|^2
\]

(12)

for any \(m \leq k\), with \(m\) chosen sufficiently large to satisfy conditions (9) and (10).

For \(m = k\), \(\alpha_k = \alpha \lambda_{k+1}\), and we rewrite relationship (11) as
\[
\|x_{k+1} - x^*\|^2 \leq (1 - \alpha_k) \|x_k - x^*\| + \frac{3\varepsilon}{\alpha} \alpha_k,
\]

establishing (12) for \(m = k\).

Suppose (12) holds for some \(m \leq k\). By definition,
\[
\alpha_{m-1} = 1 - (1 - \alpha \lambda_m)(1 - \alpha_m) = \alpha_m + \alpha \lambda_m (1 - \alpha_m).
\]

Applying (11) to the term \(\|x_m - x^*\|^2\), we obtain
\[
\|x_{k+1} - x^*\|^2 \leq (1 - \alpha_m) \|x_m - x^*\|^2 + \frac{3}{\alpha} \varepsilon \alpha_m
\]
\[
\leq (1 - \alpha_m) ((1 - \alpha \lambda_m) \|x_{m-1} - x^*\|^2 + 3\varepsilon \lambda_m) + \frac{3}{\alpha} \varepsilon \alpha_m
\]
\[
= (1 - \alpha_{m-1}) \|x_{m-1} - x^*\|^2 + \frac{3}{\alpha} \varepsilon (\alpha_m + (1 - \alpha_m) \alpha \lambda_m)
\]
\[
= (1 - \alpha_{m-1}) \|x_{m-1} - x^*\|^2 + \frac{3}{\alpha} \varepsilon \alpha_{m-1},
\]

establishing (12).

We conclude that for \(m\) sufficiently large,
\[
\|x_{k+1} - x^*\|^2 \leq \frac{3}{\alpha} \varepsilon \alpha_m + \|x_m - x^*\|^2 \prod_{\ell = m}^k (1 - \alpha \lambda_{\ell+1})
\]
for any \( k \geq m \). Note that the sequence \( \{\alpha \lambda_k\} \in [0, 1) \) converges to 0 and satisfies

\[
\sum_{i=1}^{\infty} (\alpha \lambda_i) = +\infty.
\]

Therefore, \( \prod_{i=1}^{\infty} (1 - \alpha \lambda_i) = 0 \) (see, for example, Proposition 2.1 of Bauschke [1]), and so

\[
\lim_k ||x_{k+1} - x^*||^2 \leq \frac{3}{\alpha} \epsilon.
\]

Therefore, since \( \epsilon \) is arbitrarily small,

\[
||x_{k+1} - x^*|| \rightarrow k \rightarrow \infty 0.
\]

We conclude that the sequence \( \{x_k\} \) converges to the fixed point solution \( x^* \) of \( T \) satisfying the property \( x^* = \text{Pr}_{FP}(T)(G_1(x^*)) \), completing the proof. □

**Remark:**

Following Halpern [8], as an alternative of the approximate averaging framework we have introduced, we could consider multiple applications of the averaging map \( F_{\lambda_k+1} \) rather than one. As a result we can impose less restrictive assumptions on the sequence of step sizes \( \{\lambda_k\} \). By applying the averaging map \( F_{\lambda_k+1} \) several times, we approximate the trajectory of the parameterized fixed point subproblems more accurately.

To provide some motivation, consider the sequence \( x_\lambda \) induced by the parameterized fixed point subproblem \( x_\lambda = F_\lambda(x) \) as in (2). If conditions B1–B3 are valid, then the trajectory \( \{x_\lambda\} \) is bounded, and as \( \lambda \rightarrow 0^+ \), the limit points of \( x_\lambda \) are fixed point solutions of the original problem. If, in addition, \( F_\lambda \) is a contractive map for every \( \lambda > 0 \), then we can compute the fixed point \( x_\lambda \) as

\[
x_\lambda = \lim_{k \rightarrow \infty} F^k_\lambda(X_0)
\]

(13)

for any starting point \( X_0 \in K \). We therefore consider an iterative scheme that “follows” the path of fixed points \( \{x_\lambda\} \). That is, given a sequence \( \lambda_k \rightarrow 0 \), \( \lambda_k \in (0, 1] \) and a sequence of nonnegative integers \( \{m_k\} \), for an arbitrary \( X_0 \in K \), we can approximate the original fixed point problem through

\[
X_{k+1} = F^{m_{k+1}}_{\lambda_k+1}(X_k)
\]

(14)

by ensuring that the iterates \( X_k \) stay “close” to the fixed point solutions of the corresponding subproblems \( x_\lambda_k = F_\lambda(x_{\lambda_k}) \).

The following result formalizes this iterative scheme.

**Proposition 5** Suppose \( F_\lambda : K \rightarrow K \) satisfies conditions B1–B3. Assume the sequences \( \{\lambda_k\} \) and \( \{m_k\} \) satisfy the conditions \( \lambda_k \rightarrow 0 \) and \( (1 - \alpha \lambda_k)^m_k \rightarrow 0 \) as \( k \rightarrow \infty \). Then the limit points of the sequence \( \{X_k\} \) induced by relation (14) are fixed points of \( T \).

**Proof:** It is not hard to see that the trajectory \( \{x_\lambda\}_{\lambda \in (0,1]} \) is bounded. Moreover, following an argument similar to the one employed in the proof of Theorem 2, we can show that the
sequence \( \{X_k\} \) is also bounded. Consequently, for some constant \( L \), \( \|X_k - x_\lambda\| \leq L \) for all \( k \geq 0 \) and all \( \lambda \in (0, 1] \). Furthermore, since \( x_{\lambda_k} \) is a fixed point solution of the map \( F_{\lambda_k} \) and, therefore, of the map \( F^{m_{\lambda_k}}_{\lambda_k} \),
\[
\|X_{k+1} - x_{\lambda_{k+1}}\| = \|F^{m_{\lambda_{k+1}}}_{\lambda_{k+1}}(X_k) - F^{m_{\lambda_{k+1}}}_{\lambda_{k+1}}(x_{\lambda_{k+1}})\| \\
\leq (1 - \alpha \lambda_{k+1})^{m_{\lambda_{k+1}}} \|X_k - x_{\lambda_{k+1}}\| \leq (1 - \alpha \lambda_{k+1})^{m_{\lambda_{k+1}}} L \to_{k \to \infty} 0.
\]
Therefore, the set of limit points of the sequence \( \{X_k\} \) are contained in the set of limit points of the trajectory \( \{x_\lambda\}_{\lambda \in (0, 1]} \), which, by Theorem 2, are fixed points of \( T \). \( \blacksquare \)

4 Approximate averaging trajectories — a second approach

In the previous section, we introduced a general averaging framework that induced points lying close to the trajectory of fixed points of a class of parameterized subproblems. The averaging maps \( F_\lambda \) for this analysis were contractive for each fixed value of \( \lambda \in (0, 1) \). We next study a similar general averaging framework when the map \( F_\lambda \) is nonexpansive rather than contractive. To compensate for this weaker property on map \( F_\lambda \), we need the additional assumption that its fixed points contain those of the original map \( T \). The points generated in this case approximate the trajectory of fixed points of the parameterized subproblems \( \text{FP}(F_\lambda) \) satisfying assumptions A1-A6b. Averaging with the identity map (line search procedures) is a type of averaging that is a special case. This averaging scheme has received much attention in the literature (see for example among others [5], [10], [11]). For this type of averaging, the averaging map \( F_\lambda \) is nonexpansive rather than contractive and its fixed point solutions include those of the original map \( T \). Our averaging framework in this section incorporates this type of averaging.

In particular, as before, we will consider the sequence of iterates
\[
x_0 \in K, \ x_{k+1} = F_{\lambda_{k+1}}(x_k), \ k > 0,
\]
with the sequence of parameter values \( \lambda_k \in (0, 1] \) and the family of functions \( F_\lambda : K \to K \) satisfying the following conditions:

C1 \( \text{FP}(F_0) = \text{FP}(T) \neq \emptyset \).

C2 \( F_\lambda = F(G_\lambda) \), with \( F : K \to K \) and \( G_\lambda : K \to K \) satisfying the assumptions
- \( F \) is nonexpansive and \( \text{FP}(T) \subseteq \text{FP}(F) \).
- \( G_0 \) is nonexpansive and \( \text{FP}(T) = \text{FP}(G_0) \).
- \( G_1 \) is nonexpansive and \( \text{FP}(T) \subseteq \text{FP}(G_1) \).

C3 For all \( \lambda \in [0, 1] \), all \( x^* \in \text{FP}(T) \) and all \( x \in K \),
\[
G_\lambda(x^*)^T (x - x^*) \leq \lambda G_1(x^*)^T (x - x^*) + (1 - \lambda) G_0(x^*)^T (x - x^*),
\]
which in light of C2 implies
\[
(G_\lambda(x^*) - G_0(x^*))^T (x - x^*) \leq 0.
\]
C4 For all $\lambda \in [0, 1]$, all $x^* \in \text{FP}(T)$ and all $x \in K$,

$$||G_\lambda(x) - G_\lambda(x^*)||^2 \leq ||x - x^*||^2 - \gamma(\lambda)||x - G_0(x)||^2,$$

for some function $\gamma : [0, 1] \to \mathbb{R}_+.$

C5 The step sizes $\lambda_k$ satisfy the conditions $\sum_{k=1}^{\infty} \gamma(\lambda_k) = +\infty$.

Once again, we begin by examining several special cases:

1. **Averaging with the identity map:** We begin by considering a classical averaging scheme and describing how it fits into this generalized averaging framework:

$$F_\lambda(x) = \lambda x + (1 - \lambda)T(x).$$

This averaging scheme (see Dunn [5] and Magnanti and Perakis [10], [11]) is applicable for a map $T$ that is nonexpansive.

C1 $F_0 = T$, so the condition holds trivially.

C2 $F = I$, $G_\lambda = \lambda I + (1 - \lambda)T$, i.e., $G_0 = T$ and $G_1 = I$, so the condition is satisfied.

C3 Holds trivially since $G_1 = \lambda G_1 + (1 - \lambda)G_0$.

C4 For all $\lambda \in [0, 1]$, all $x^* \in \text{FP}(T)$ and all $x \in K$, since $x^* = G_\lambda(x^*)$, we have (see Dunn [5]):

$$||G_\lambda(x) - G_\lambda(x^*)||^2 + \lambda(1 - \lambda)||T(x) - x||^2 = ||\lambda x + (1 - \lambda)T(x) - x^*||^2 + \lambda(1 - \lambda)||T(x) - x||^2$$

$$= ||(1 - \lambda)(T(x) - x) + (x - x^*)||^2 + \lambda(1 - \lambda)||T(x) - x||^2$$

$$= 2(1 - \lambda)(T(x) - x)^t(x - x^*) + ||x - x^*||^2 + (1 - \lambda)||T(x) - x||^2$$

$$= (1 - \lambda) ||T(x) - x^*||^2 + ||x - x^*||^2$$

$$\leq ||x - x^*||^2,$$

i.e.,

$$||G_\lambda(x) - G_\lambda(x^*)||^2 \leq ||x - x^*||^2 - \lambda(1 - \lambda)||T(x) - x||^2 = ||x - x^*||^2 - \lambda(1 - \lambda)||G_0(x) - x||^2.$$

Therefore, the condition is satisfied with $\gamma(\lambda) = \lambda(1 - \lambda)$.

C5 The condition can be satisfied by selecting the sequence $\{\lambda_k\} \subset [0, 1]$ to satisfy $\sum_{k=1}^{\infty} \lambda_k = \infty$, just as in the cited papers.

2. **Outside averaging:**

$$F_\lambda^{\text{out}}(x) = \lambda g(x) + (1 - \lambda)T(x)$$

We consider the case when the map $T$ is firmly nonexpansive and the map $g$ is nonexpansive with $\text{FP}(g) \supseteq \text{FP}(T)$, (an example of a map $g$ is the identity map).

C1 $F_0 = T$, so the condition holds trivially.
\[ C_2 \quad F = I, \quad G_\lambda = \lambda g + (1 - \lambda)T, \] 
\[ \text{i.e., } G_0 = T \text{ and } G_1 = g \] 
by the above assumptions, the condition is satisfied.

\[ C_3 \quad \text{Holds at equality, since } G_\lambda = \lambda G_1 + (1 - \lambda)G_0. \]

\[ C_4 \quad \text{Observe that } G_\lambda(x^*) = x^*. \] 
Therefore,

\[ ||G_\lambda(x) - G_\lambda(x^*)||^2 = ||\lambda g(x) + (1 - \lambda)T(x) - x^*||^2 = \]

\[ = \lambda^2||g(x) - x^*||^2 + (1 - \lambda)^2||T(x) - x^*||^2 + 2\lambda(1 - \lambda)(g(x) - x^*)' (T(x) - x^*) \leq \]

\[ \lambda^2||g(x) - x^*||^2 + (1 - \lambda)^2||T(x) - x^*||^2 + 2\lambda(1 - \lambda)||g(x) - x^*||^2 = \]

\[ = \lambda||g(x) - x^*||^2 + (1 - \lambda)||T(x) - x^*||^2 \leq \lambda||g(x) - x^*||^2 + (1 - \lambda)||T(x) - x^*||^2 \]

The second inequality follows since \( T \) is firmly nonexpansive. Therefore, the condition is satisfied with \( \gamma(\lambda) = (1 - \lambda). \)

\[ C_5 \quad \text{This condition can be satisfied by selecting the sequence } \{\lambda_k\} \subseteq (0, 1] \text{ to satisfy } \sum_{k=1}^{\infty} (1 - \lambda_k) = \infty. \]

Observe that in this method, \( g \) could be the identity map. In this case, the method reduces to averaging with the identity map. In fact, for the case of averaging with the identity map, it is sufficient to require that that map \( T \) is nonexpansive rather than firmly nonexpansive (see for example [5], [10], [11]).

3. Outside averaging with the proximal point mapping:

\[ F_\lambda(x) = \lambda g(x) + (1 - \lambda)J_{c(I-T)}(x) = \lambda g(x) + (1 - \lambda)(I + c(I - T))^{-1}(x). \]

We consider the case when the map \( T \) is nonexpansive and the map \( g \) is nonexpansive with \( \text{FP}(g) \subseteq \text{FP}(T). \)

This is essentially an extension of the previous case, since the proximal point operator \( J_{c(I-T)} \) is firmly nonexpansive whenever \( T \) is nonexpansive.

**Theorem 4** Suppose \( T : K \to K \) is a nonexpansive map with \( \text{FP}(T) \neq \emptyset. \) Suppose further that the sequence \( \{\lambda_k\} \) and the family of functions \( F_\lambda \) satisfy conditions \( C_1-C_6. \) Then the sequence \( \{x_k\} \) induced by the averaging scheme (15) converges to a fixed point of the map \( T. \)

**Proof:** Let \( x^* \in \text{FP}(T). \) Then

\[ ||G_{\lambda_{k+1}}(x_k) - x^*||^2 = ||G_{\lambda_{k+1}}(x_k) - G_0(x^*)||^2 \]

\[ = ||G_{\lambda_{k+1}}(x_k) - G_{\lambda_{k+1}}(x^*)||^2 - ||G_{\lambda_{k+1}}(x^*) - G_0(x^*)||^2 + 2(G_{\lambda_{k+1}}(x_k) - G_0(x^*))'(G_{\lambda_{k+1}}(x_k) - G_0(x^*)) \]

\[ \leq ||G_{\lambda_{k+1}}(x_k) - G_{\lambda_{k+1}}(x^*)||^2, \]

with the inequality obtained by applying condition \( C_3 \) with \( x = G_{\lambda_{k+1}}(x_k). \)
Conditions C2 and C4 and the fact that $x^* = F(G_{\lambda_k+1}(x^*))$ imply that

$$
||x_{k+1} - x^*||^2 = ||F(G_{\lambda_k+1}(x_k)) - x^*||^2 \leq ||G_{\lambda_k+1}(x_k) - G_{\lambda_k+1}(x^*)||^2 \\
\leq ||x_k - x^*||^2 - \gamma(\lambda_{k+1})||x_k - G_0(x_k)||^2
$$

(16)

Relation (16) implies that the sequence $\{||x_k - x^*||^2\}$ is nonincreasing, and hence converges for any fixed point $x^* \in FP(T)$. Moreover, it implies that the sequence $\{x_k\}$ is bounded and, therefore, has at least one limit point.

We claim that for some limit point $\bar{x}$ of the sequence $\{x_k\}$, $||\bar{x} - G_0(\bar{x})||^2 = 0$. Suppose this is not the case. Then for some sufficiently large $K$ and some constant $B > 0$, $||x_k - G_0(x_k)||^2 \geq B$ for all $k \geq K$. Since the sequence $\{\lambda_k\}$ satisfies condition C5,

$$
0 \leq \lim_{k \to \infty} ||x_k - x^*||^2 \leq ||x_K - x^*||^2 - B \sum_{k=K}^{+\infty} \gamma(\lambda_k+1) = -\infty
$$

which is a contradiction.

We conclude that the sequence $\{x_k\}$ has a limit point $\bar{x}$ satisfying the conditions $||\bar{x} - G_0(\bar{x})||^2 = 0$, implying that $\bar{x} \in FP(G_0) = FP(T)$. Therefore, the sequence $\{||x_k - \bar{x}||^2\}$ is convergent. Since $\bar{x}$ is a limit point of $\{x_k\}$, we conclude that $\lim_{k \to \infty} ||x_k - \bar{x}||^2 = 0$. □

Future research related to this work might fruitfully examine a number of issues, for example:

1. Test these methods computationally to determine how they perform in practice.
2. Examine how this approach applies to the solution of variational inequality, complementarity, linear and nonlinear optimization problems.
3. By making necessary refinements, improve the convergence behavior of averaging schemes. Moreover, develop results by imposing weaker assumptions on the original problem map $T$.
4. Develop results on the rate of convergence of averaging schemes, perhaps by considering additional refinements on the schemes.

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References


