Linear Optimization Problems

and their role in IMRT beamlet weight optimization

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Outline

- Optimization problems and models; terminology
- Linear optimization problems (LPs)
- Example: an LP formulation of an IMRT planning problem
- A two-variable LP, to demonstrate
  - Nice properties of LPs
  - Solving LPs
  - Analysis of solution sensitivity
An optimization problem is a decision problem in which we are choosing among several decisions.
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Note: it does not matter how good the infeasible decisions are; we don’t consider them, since they cannot (or should not) be implemented.
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To construct a mathematical model of an optimization problem, we need to quantify the above concepts. Decisions need to be expressed as numerical quantities, i.e., *decisions variables*; a particular value of a variable corresponds to a particular decision, or a component of a decision; every decision can be represented by assigning certain values to the variables.
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Fidelity in the constraints: any assignment of values to the variables that satisfies all the constrains corresponds to a decision that is feasible, and vice versa — every feasible decision is represented by values of the decision variables that satisfy all the constraints.
To construct a mathematical model of an optimization problem, we need to quantify the above concepts. To compare feasible decisions, need to express our “quality” criterion as an **objective function** of the variables. We are searching for a decision with the objective function value either as large, or as small, as possible, depending on the context.
To construct a mathematical model of an optimization problem, we need to quantify the above concepts:

- Decision variables
- Constraints
- Objective function, minimize or maximize
- A feasible solution — assignment of values to the variables satisfying all the constraints
- An optimal solution, or a solution — the best feasible solution, as measured by the objective function
Brain tumor $\approx 50,000$ voxels; 5-beam arrangement of 371 beamlets

Protocol:
1. Chiasm dose: $< 55$ Gy to $1/2$ volume
2. Mean PTV1 dose: $90$ Gy $\pm 5\%$
3. Min PTV3-PTV2 annulus dose: $60$ Gy
4. Min PTV2-PTV1 annulus dose: $70$ Gy
5. Outer annuli as homogeneous as possible
6. Normal brain outside PTV3 (Top): as low as possible
Intensity of each beamlet: $w_b \geq 0$ for each beamlet $b$
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Most aspects of future discussions will involve doses delivered; for convenience, let’s introduce additional variables:

Dose delivered to each voxel: $d_v \geq 0$ for each voxel $v$
Modelling: decision variables

- Intensity of each beamlet: $w_b \geq 0$ for each beamlet $b$

- Dose delivered to each voxel: $d_v \geq 0$ for each voxel $v$

- $d_v$’s are variables (unknowns), but their values depend on the $w_b$’s:

$$d_v = \sum_{all \ b} a_{bv} w_b \quad \text{for each voxel } v.$$

Here, $a_{bv}$’s are elements of the dose matrices.
Modelling: some constraints

Connecting doses and intensities:

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- Item 3 of the protocol:

\[ d_v \geq 60 \quad \text{for each } v \text{ in PTV3-PTV2} \]
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- Item 4 of the protocol:

\[ d_v \geq 70 \text{ for each } v \text{ in PTV2-PTV1} \]
Modelling: some constraints

- Item 2 of the protocol:
  \[ \sum_{\text{all } v \text{ in PTV1}} d_v \geq 85.5 \cdot (\# \text{ in PTV1}) \]
  and
  \[ \sum_{\text{all } v \text{ in PTV1}} d_v \leq 94.5 \cdot (\# \text{ in PTV1}) \]
Modelling: some constraints

- Item 2 of the protocol:
  \[
  \sum_{v \in PTV1} d_v \geq 85.5 \cdot (\# \text{ in PTV1})
  \]
  and
  \[
  \sum_{v \in PTV1} d_v \leq 94.5 \cdot (\# \text{ in PTV1})
  \]

- More restricted than Item 1 of the protocol:
  \[d_v \leq 55 \text{ for each } v \text{ in Chiasm}\]
Modelling: objective function

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Objective function:

$$\min \sum_{\text{all } v \text{ in Top}} d_v$$
Putting it all together

\[\begin{align*}
\text{min} & \quad \sum_{\text{all } v \text{ in Top}} d_v \\
\text{subject to} & \quad d_v = \sum_{\text{all } b} a_{bv} w_b \text{ for each voxel } v \\
& \quad d_v \geq 60 \text{ for each } v \text{ in PTV3-PTV2} \\
& \quad d_v \geq 70 \text{ for each } v \text{ in PTV2-PTV1} \\
& \quad \sum_{\text{all } v \text{ in PTV1}} d_v \geq 85.5 \cdot (\# \text{ in PTV1}) \\
& \quad \sum_{\text{all } v \text{ in PTV1}} d_v \leq 94.5 \cdot (\# \text{ in PTV1}) \\
& \quad d_v \leq 55 \text{ for each } v \text{ in Chiasm} \\
& \quad w_b \geq 0 \text{ for each beam } b
\end{align*}\]
Note that in the above model all variables were continuous.

The objective was a linear function (i.e., a weighted sum of the variables, with some pre-specified weights).

All constraints were written as linear equations or inequalities.

Such optimization problems are called Linear Problems, or Linear Programs (LPs).
Why focus on LPs?
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- Flexible modelling tool to represent a lot of performance measures and restrictions on decisions
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- Flexible modelling tool to represent a lot of performance measures and restrictions on decisions
- Easy to understand
- “Easy” to solve — that is, there are software packages, a.k.a. solvers, that solve very large LPs in default setting, and some humongous LPs with some intelligent tweaking, fast
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- Easy to understand
- “Easy” to solve
- Can be analyzed
A (very) small example – 2 beamlets

\[
\begin{align*}
\text{min} & \quad 1.5x + y \\
\text{s.t.} & \quad x + 2y \leq 6 \\
& \quad 2x + y \leq 6 \\
& \quad x + 2y \geq 2 \\
& \quad 2x + y \geq 2 \\
& \quad 1.5x + 1.5y \geq 2.5 \\
& \quad x \leq 1.5 \\
& \quad x, y \geq 0
\end{align*}
\]

Think of having 4 voxels with \( d_1 = x + 2y, \ d_2 = 2x + y, \ d_3 = x, \ d_4 = 1.5x + y \).
Feasible solutions

By definition, a feasible solution is any assignment of values to \((x, y)\) that satisfies all the above constraints. For example, \((1.5, 2.25)\) is feasible. So are \(\left(\frac{1}{3}, \frac{4}{3}\right)\) and \((1, 1.75)\) \((0.75, 0.75)\) is infeasible: violates the fifth constraint.
All feasible solutions
Feasible region
Unique optimal solution: \( x = 0.3333, \)
\( y = 1.3333. \) Optimal objective value

\[ 1.5x + y = 1.8333 \]
Unique optimal solution: \( x = 0.3333, \ y = 1.3333 \). Optimal objective value \( 1.5x + y = 1.8333 \)

How do I know? Fundamental property of LPs: if you found a corner of the feasible region that is better than (as measured by the objective function) as all its “neighboring” corners, your corner is the optimal solution.
Unique optimal solution: $x = 0.3333, y = 1.3333$. Optimal objective value $1.5x + y = 1.8333$

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Why is that true? Look at the solutions that have the same objective function values, i.e., $1.5x + y = \text{value}$...
Optimal solution
Multiple optimal solutions

Suppose the objective function has been replaced with $\min 0.5x + 0.5y$
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Now there are many optimal solutions (see picture)
Multiple optimal solutions

Suppose the objective function has been replaced with $\min 0.5x + 0.5y$

Now there are many optimal solutions (see picture)

But any one of them is as good as any other because they all have the same objective function value
Optimal solution

Minimize $0.5x + 0.5y$

Opt. solutions
Sensitivity analysis

Back to the original objective function
**Sensitivity analysis**

- In the above problem, the constants on the right-hand sides of the constraints might be subject to revisions.

- Intuition gained from the solution can be used to foresee the impact on the solution of some of such revisions, and answer several other questions.

- Reminder: the optimal solution we found had the objective value of 1.8333. At that solution, we had $1.5x + 1.5y = 2.5$ and $2x + y = 2$, and the rest of the constrains were not binding.
Q Can I construct a plan in which $1.5x + y$ is at most 2? 1.5?

A1 Yes. In fact, we already found a plan in which it happens.

A2 No — the best you can hope for is 1.8333
Q What would happen if I tighten the upper bound on $2x + y$ and $x + 2y$?

A If you don’t change them drastically, nothing — that is, you would still have a feasible solution of the same quality.
Sensitivity analysis

Q What would happen if I tighten the lower bound on $2x + y$ and $x + 2y$?

A You will still be able to find feasible solutions, but since $2x + y$ is currently at its lower bound, solution quality is going to suffer. In particular, if constraints become $2x + y \geq 2 + \theta$ and $x + 2y \geq 2 + \theta$, the best value of the objective you can get is $1.5x + y = 1.8333 + 0.5\theta$ (and even worse if $\theta > \frac{1}{2}$).

By the way, the answer to the above question is essentially generated by the solver software in the process of solving the problem automatically; just need to know where to look.
Illustration of the last Q

Sensitivity analysis

old solution

new solution
Advantages of LP

- Easy to formulate and solve (existing solvers, solution process makes use of geometry)
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- Only allowed one objective — all other considerations are constraints. But can explore different decisions with the above analysis!
One potential drawback of LP

- To formulate any constraints on the DVH, need to break the assumption that variables are continuous
- I.e., need to introduce a slew of variables that take on values 1 or 0 (interpret: “true” or “false”)
- MUCH harder to solve
- Most sensitivity analysis (at least as precise as above) becomes impossible.
- BTW, these are the same considerations that make the “sum-of-costlest” functions discontinuous, and hence difficult to optimize