

For notational purposes, I take e_i as it is in the draft, and define f_i as below:

$$f_i = \frac{e_{i-1} \cdot e_i}{|e_{i-1}| |e_i|}$$

The regularized energy I use is below (regularization to give differentiability where $\theta = 0$):

$$\begin{aligned} E &= \sum_{i=0}^{n-1} \sqrt{\theta_i^2 + \epsilon} \\ &= \sum_{i=0}^{n-1} \sqrt{\cos^{-1} \left(\frac{e_{i-1} \cdot e_i}{|e_{i-1}| |e_i|} \right)^2 + \epsilon} \\ &= \sum_{i=0}^{n-1} \sqrt{\cos^{-1} (f_i)^2 + \epsilon} \end{aligned}$$

Each vertex (x_i, y_i) is dependent on only three angles: θ_{i-1} , θ_i , and θ_{i+1} . Thus:

$$\begin{aligned} \frac{\partial E}{\partial x_i} &= \sum_{j=i-1}^{i+1} \frac{\partial(\sqrt{\theta_j^2 + \epsilon})}{\partial x_i} \\ &= \sum_{j=i-1}^{i+1} \frac{\cos^{-1}(f_j)}{\sqrt{\cos^{-1}(f_j)^2 + \epsilon}} \cdot \frac{-1}{\sqrt{1 - f_j^2}} \cdot \frac{\partial f_j}{\partial x_i} \end{aligned}$$

The last part of that expression is given by:

$$\frac{\partial f_j}{\partial x_i} = \begin{cases} \frac{1}{|e_{i-2}| |e_{i-1}|} \left[x_{i-1} - x_{i-2} - f_{i-1} \left(\frac{|e_{i-2}|}{|e_{i-1}|} (x_i - x_{i-1}) \right) \right], & j = i - 1 \\ \frac{1}{|e_{i-1}| |e_i|} \left[x_{i+1} - 2x_i + x_{i-1} - f_i \left(\frac{|e_i|}{|e_{i-1}|} (x_i - x_{i-1}) + \frac{|e_{i-1}|}{|e_i|} (x_i - x_{i+1}) \right) \right], & j = i \\ \frac{1}{|e_i| |e_{i+1}|} \left[x_{i+1} - x_{i+2} - f_{i+1} \left(\frac{|e_{i+1}|}{|e_i|} (x_i - x_{i+1}) \right) \right], & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

The above expressions are also valid for y-components of the vertex, simply by replacing “x” with “y” everywhere in the expression.

It may also be necessary to note that in my code, I try to update the position of a vertex only in the direction ‘normal’ to the vertex, as described below, to prevent points from crossing each other.

I define n_i to be the unit vector perpendicular to the vector $(x_{i+1} - x_{i-1}, y_{i+1} - y_{i-1})$, then the update is given by:

$$x_i^{t+1} = x_i^t - \Delta t \left(\frac{\partial E}{\partial x_i} \cdot n_i \right) n_i$$

Superscripts indicate the time-step, and the right hand side is the projection of the standard update position vector onto the normal vector described above (where $x_i = (x_i, y_i)$). The negative sign provides for gradient descent (rather than ascent).