1. Using your favorite numerical software for matrices, construct the modularity matrix for this small network:

![Network Diagram]

Find the eigenvector of the modularity matrix corresponding to the largest eigenvalue and hence divide the network into two communities. For full credit, give the eigenvector you found and draw a figure showing how it divides the network into communities.

2. Consider the following network model. Each of \( n \) nodes is assigned a non-negative real parameter \( \theta_i \) and then undirected edges are created such that the number of edges between nodes \( i \) and \( j \) is a Poisson-distributed independent random number with mean \( \theta_i \theta_j \), except for self-edges, which have mean \( \frac{1}{2} \theta_i^2 \). The goal is to find the values of the parameters \( \theta_i \) that best fit a given observed network.

   (a) Derive an expression for the likelihood (i.e., the probability) that a network with adjacency matrix \( A \) was generated by this model, given the values of the parameters \( \theta_i \), and hence show that the log-likelihood

   \[
   \mathcal{L} = \log P(A | \theta) = \frac{1}{2} \sum_{ij} [A_{ij} \log(\theta_i \theta_j) - \theta_i \theta_j].
   \]

   (b) By maximizing the log-likelihood with respect to the \( \theta \) parameters show that for the best fit of this model to an observed network, the mean number of edges between distinct nodes \( i \) and \( j \) is \( k_i k_j / 2m \), where \( k_i \) is the observed degree of node \( i \) and \( m \) is the number of edges in the network. (In other words, this model is basically the configuration model.)

3. Consider a configuration model network that has nodes of degree 1, 2, and 3 only, in fractions \( p_1 \), \( p_2 \), and \( p_3 \), respectively.

   (a) Find the value of the critical node occupation probability \( \phi_c \) at which site percolation takes place on the network.

   (b) Show that there is no giant cluster for any value of the occupation probability \( \phi \) if \( p_1 > 3p_3 \). In terms of the structure of the network, why is this? And why does the result not depend on \( p_2 \)?

   (c) Find an expression for the size of the giant cluster as a function of \( \phi \). (Hint: you may find it useful to remember that \( u = 1 \) is always a solution of the equation \( u = 1 - \phi + \phi g_1(u) \).)
4. Consider the problem of nonuniform percolation on a configuration model network, as discussed in Section 15.3 of the course pack, where the occupation probability \( \phi_k \) for a node is a function of degree \( k \).

(a) Show, by a graphical argument or otherwise, that a giant component can exist in the network only if \( f_1'(1) > 1 \), where \( f_1(z) \) is the function defined in Eq. (15.34).

(b) A configuration model network has a (properly normalized) geometric degree distribution \( p_k = (1 - a) a^k \) with \( a < 1 \) and an occupation probability \( \phi_k = b^k \) with \( b < 1 \), so that high-degree nodes are more likely to be removed than low-degree ones. Show that the system has a giant cluster if \( 2ab^2(1 - a)^2 > (1 - ab)^3 \).

5. Extra credit: Write a program in the computer language of your choice to do the following:

(a) Generate a Poisson random graph of the type \( G(n, p) \) with \( n = 10000 \) nodes and mean degree \( c = 4 \).

(b) Carry out the percolation algorithm of Section 15.5 on this network, occupying the nodes one by one and updating cluster labels after each one. Keep a running tally of the size of the largest cluster.

(c) Repeat the entire calculation 100 times for 100 different random graphs with the same value of \( c \) and produce a plot showing the mean (over the 100 runs) of the size of the largest cluster as a function of the number of occupied nodes.

(d) Based on this plot, at about what value would you say the percolation threshold falls?

For full extra credit, turn in a printout of your complete program, a printout of the plot you produced, and your answer to the question in part (d).