Physics 390: The quantum simple harmonic oscillator

The energy eigenstates of the simple harmonic oscillator satisfy
\[
-h^2 \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi = E \psi. \quad (1)
\]
Consider the new function \( \psi_- \) defined by
\[
\psi_- = \frac{h}{m \omega} \frac{d \psi}{dx} + x \psi. \quad (2)
\]
Multiplying by \(-\frac{1}{2} \hbar \omega\) and differentiating we find that
\[
-\frac{1}{2} \hbar \omega \frac{d \psi_-}{dx} = -\frac{h^2}{2m} \frac{d^2 \psi_-}{dx^2} - \frac{1}{2} \hbar \omega \psi_- - \frac{1}{2} \hbar \omega x \frac{d \psi_-}{dx} = (E - \frac{1}{2} \hbar \omega) \psi_- \quad (3)
\]
where we have made use of Eqs. (1) and (2) in the second and fourth lines respectively. Now multiplying by \( \hbar / m \omega \) and differentiating again we get
\[
-\frac{h^2}{2m} \frac{d^2 \psi_-}{dx^2} = \frac{h}{m \omega} (E - \frac{1}{2} \hbar \omega) \frac{d \psi}{dx} - \frac{1}{2} \hbar \omega \psi_- - \frac{1}{2} \hbar \omega x \frac{d \psi_-}{dx} = \frac{h}{m \omega} (E - \frac{1}{2} \hbar \omega) \frac{d \psi}{dx} - \frac{1}{2} \hbar \omega \psi_- + (E - \frac{1}{2} \hbar \omega) x \psi - \frac{1}{2} m \omega^2 x^2 \psi_- \quad (4)
\]
where we have used Eq. (3) to eliminate \( d \psi_- / dx \). Rearranging this expression and collecting terms, we find that
\[
-\frac{h^2}{2m} \frac{d^2 \psi_-}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi_- = (E - \frac{1}{2} \hbar \omega) \psi_- \quad (5)
\]
As noted, the quantity in square brackets is none other than \( \psi_- \) and hence we find that
\[
-\frac{h^2}{2m} \frac{d^2 \psi_-}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi_- = (E - \hbar \omega) \psi_- \quad (6)
\]
But this is simply the Schrödinger equation again. It tells us that if the original wavefunction \( \psi \) was a solution of the Schrödinger equation, then so too is \( \psi_- \) defined by Eq. (2), but with a different energy \( E - \hbar \omega \), exactly \( \hbar \omega \) lower than the original energy.

Thus if we can find one solution to the Schrödinger equation, with any energy \( E \), then we can immediately find another one with energy \( E - \hbar \omega \). But now we can repeat the process and hence find a state with an energy \( \hbar \omega \) lower still, and so forth. In this way we can get a whole “ladder” of solutions to the Schrödinger equation with energies \( \hbar \omega \) apart. We simply keep applying Eq. (2). (We can also go in the opposite direction and increase energy by \( \hbar \omega \), extending the ladder in the upward direction as well—see this week’s homework set.)

There is, however, a catch. The energy cannot go on decreasing forever: the energy of the simple harmonic oscillator is a sum of nonnegative quantities, so it is itself nonnegative. But how can this be, when we have just shown that we can go on decreasing the energy by steps of \( \hbar \omega \) for as long as we like? The answer is that at some point there must be a solution \( \psi \) of the equation for which the trick above does not work and Eq. (2) does not generate another solution with lower energy. How could this happen mathematically? Well, it happens if
\[
\frac{h}{m \omega} \frac{d \psi}{dx} + x \psi = 0. \quad (7)
\]
In this case, \( \psi_- \) still technically satisfies Eq. (6) (because both sides will be zero), but we don’t get a new solution with lower energy.

If we multiply Eq. (7) by \( \hbar / m \omega \) and differentiate, we get
\[
\frac{h^2}{m^2 \omega^2} \frac{d^2 \psi}{dx^2} + \frac{h}{m \omega} \psi = \frac{h}{m \omega} \frac{d \psi}{dx} = \frac{h^2}{m^2 \omega^2} \frac{d^2 \psi}{dx^2} + \frac{h}{m \omega} \psi - x^2 \psi = 0, \quad (8)
\]
where we have used Eq. (7) to eliminate $d\psi/dx$. Now we multiply this equation by $-\frac{1}{2}ma^2$ and rearrange to get

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}ma^2x^2\psi = \frac{1}{2}\hbar\omega\psi.$$  

(9)

This is just the Schrödinger equation again, and so we see that this wavefunction, the wavefunction of the state at the bottom of the ladder, which is the ground state of the simple harmonic oscillator, has energy $E = \frac{1}{2}\hbar\omega$.

Thus, by a process of deduction, we conclude that the energy levels of the simple harmonic oscillator start at $\frac{1}{2}\hbar\omega$ and go up in steps of $\hbar\omega$, so that the $n$th energy level has energy

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$  

(10)

As a final trick, we can also solve for the actual wave function of the ground state, by solving the differential equation (7). Separating the variables and integrating gives

$$\int \frac{d\psi}{\psi} = -\frac{ma}{\hbar} \int x \, dx,$$  

(11)

or

$$\psi(x) = A \exp(-ma^2 x^2 / 2\hbar),$$  

(12)

where $A$ is a normalization constant. You can check that this agrees with the answer claimed in the book (Eq. (6-58) in Tipler & Llewellyn).