Physics 411: Midterm 3

This is a take-home exam. You have 2 days to complete it. Your answers must be handed in at or before the end of class, 11:30am on Thursday, April 3 to receive a grade. **Answers turned in late will not receive a grade** (unless you've made arrangements with the professor to take your exam at a different time).

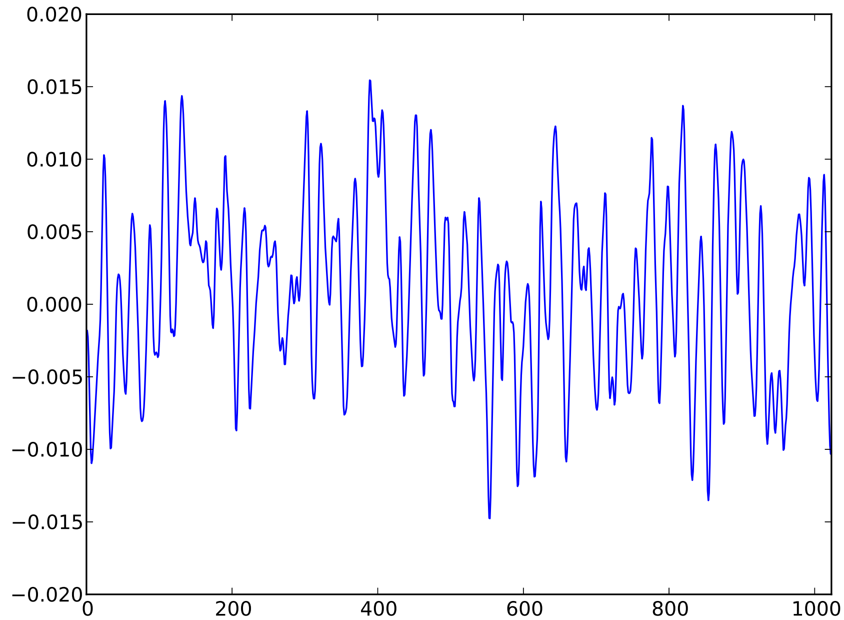
There are four questions, arranged roughly in order of difficulty—the later ones are harder, and also carry more points. Partial credit will be given for all parts of a question that are correctly completed.

The rules are the same as for the previous midterms. You may consult the textbook and you can use any of the on-line resources that accompany the textbook; you may use programs and functions from the on-line resources page as a starting point if you want, as well as programs you yourself have written for the homework assignments. But collaboration is not allowed. You may not discuss the exam with anyone else and you may not look up answers or use material from any other source, including friends or classmates, the internet, or other books. Everything you turn in must be your own original work and no one else's.

Paragraphs marked with a check-mark thus ☑ indicate what is to be handed in for each question.

Good luck!

1. **[6 points]** On the course web site you'll find a file called `signal.txt`. Download a copy of this file onto your computer. It contains a set of 1024 measurements of a certain signal. When plotted the data in the file look like this:



It is not immediately apparently from this plot, but there is, in fact, a clear, periodic signal buried in the noise within these data. Use a Fourier transform to find this signal. If the data points in the file correspond to measurements made at a rate of 10 kHz, what is the frequency of the buried signal in Hertz?

☑ **For full credit** turn in a printout of the program you wrote, along with your complete working and answer to the question.

2. **[8 points]** The gravitational orbit of any two massive bodies around one another in space is periodic and stable (as long as we ignore energy loss due to tides, atmospheric friction, gravity waves, and so forth). The motion of three or more bodies, on the other hand, is usually aperiodic and often unstable. There are, however, exceptions. About twenty years ago, the physicist Cristopher Moore discovered a remarkable stable, periodic orbit of three bodies around one another. He made the discovery using numerical methods, but it was later confirmed analytically by Alain Chenciner and Richard Montgomery.

   (a) Consider three stars of equal mass m in empty space. Show that the equation of motion governing the position $\mathbf{r}_1$ of the first star is

   $$\frac{\mathrm{d}^2\mathbf{r}_1}{\mathrm{d}t^2} = Gm\left(\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3}\right),$$

   where G is Newton's gravitational constant. Derive two similar equations for the positions $\mathbf{r}_2$ and $\mathbf{r}_3$ of the other two stars. Then convert the three second-order equations into six equivalent first-order equations, using the techniques you have learned.

   (b) Working in units where $Gm = 1$, write a program to solve your equations using the fourth-order Runge–Kutta method with fixed step size, and hence calculate the motion of the stars from $t = 0$ to $t = 100$ for the case where the motion of all three stars lies in the $z = 0$ plane and their initial positions and velocities are

	Position (x, y)	Velocity (v_x, v_y)
Star 1:	$(0, 0)$	$(0.93240737, 0.86473146)$
Star 2:	$(0.97000436, -0.24308753)$	$(-0.46620369, -0.43236573)$
Star 3:	$(-0.97000436, 0.24308753)$	$(-0.46620369, -0.43236573)$

   This is the Moore–Chenciner–Montgomery orbit. Make a plot showing the trails of all three stars (i.e., a graph of y against x for each star).

   (c) Modify your program to make an animation of the motion on the screen.

☑ **For full credit** turn in a printout of your final program from part (c), your derivations from part (a), your plot from part (b), and a snapshot of your animation in action.

3. **[10 points]** The Belousov–Zhabotinsky reaction is a chemical oscillator, a cocktail of chemicals which, when heated, undergoes a series of reactions that cause the chemical concentrations in the mixture to oscillate between two extremes. You can add an indicator dye to the reaction which changes color depending on the concentrations and watch the mixture switch back and forth between two different colors for as long as you go on heating it.

   Physicist Ilya Prigogine formulated a mathematical model of this type of chemical oscillator, which he called the "Brusselator" after his home town of Brussels. The equations for the Brusselator are

$$\frac{dx}{dt} = 1 - (b+1)x + ax^2y, \qquad \frac{dy}{dt} = bx - ax^2y.$$

   Here x and y represent concentrations of chemicals and a and b are positive constants.

   Write a program to solve these equations for the case $a = 1$, $b = 3$ with initial conditions $x = y = 0$, to an accuracy of at least $\delta = 10^{-10}$ per unit time in both x and y, using the adaptive fourth-order Runge–Kutta method discussed in Section 8.4 of the book. Calculate a solution from $t = 0$ to $t = 20$ and make a plot of x and y as a function of time, both on the same graph.

   ☑ **For full credit** turn in a printout of your program and the plot it produces.

4. **[12 points]** Many elementary mechanics problems deal with the physics of objects moving or flying through the air, but they almost always ignore friction and air resistance to make the equations solvable. If we're using a computer, however, we don't need solvable equations.

   Consider a spherical cannonball shot from a cannon standing on level ground. The air resistance on a moving sphere is a force in the opposite direction to the motion with magnitude

   $$F = \tfrac{1}{2}\pi R^2 \rho C v^2,$$

   where R is the sphere's radius, ρ is the density of air, v is the velocity, and C is the so-called *coefficient of drag* (a property of the shape of the moving object, in this case a sphere).

   (a) Starting from Newton's second law, $F = ma$, show that the equations of motion for the position x, y of the cannonball are

   $$\ddot{x} = -\frac{\pi R^2 \rho C}{2m}\,\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}, \qquad \ddot{y} = -g - \frac{\pi R^2 \rho C}{2m}\,\dot{y}\sqrt{\dot{x}^2+\dot{y}^2},$$

   where m is the mass of the cannonball, g is the acceleration due to gravity, and $\dot{x}$ and $\ddot{x}$ are the first and second derivatives of x with respect to time.

   (b) Change these two second-order equations into four first-order equations using the methods you have learned, then write a program that solves the equations using (ordinary, non-adaptive) fourth-order Runge–Kutta for a cannonball of mass 1 kg and radius 8 cm, shot at 30° to the horizontal with initial velocity $100\,\mathrm{ms}^{-1}$. The density of air is $\rho = 1.22\,\mathrm{kg\,m}^{-3}$ and the coefficient of drag for a sphere is $C = 0.47$. You can assume the altitude of the cannonball is zero at the moment it leaves the cannon. Make a plot of the trajectory of the cannonball (i.e., a graph of y as a function of x).

   (c) Now modify your program to use the shooting method to find the angle we would have to shoot the cannonball at to make it travel 200 m, plus or minus 10 cm, assuming the same initial velocity of $100\,\mathrm{ms}^{-1}$. Hint: There will probably be no time-step (other than $t = 0$) at which the altitude of the flying cannonball is exactly zero, so you'll have to find some way to calculate the distance the ball travels from the points you have. Linear interpolation is probably the simplest way to go (page 202 in the book).

☑ **For full credit** turn in a printout of your final program along with the results it gives, your derivation from part (a), and your plot from part (b).